WHICH GRAPHS ARE PLUPERFECT?

Frank Harary
Department of Computer Science, New Mexico State University,
Las Cruces, NM 88003, U.S.A.

Martin Lewinter
Department of Mathematics, State University of New York,
Purchase, NY 10577, U.S.A.

Dedicated to Gary Chartrand on his 55th Birthday

Abstract

A graph G is pluperfect if one can add new multiple edges to obtain a multigraph M satisfying:

(1) G spans M,

(2) the degrees of the nodes of M are consecutive integers,

(3) G and M have the same minimum degree.

Our purpose is to display several families of pluperfect graphs in order to stimulate research on this topic.

1. Introduction

My good friends Mehdi Behzad and Gary Chartrand wrote a note [1] titled "No graph is perfect". They did this in jest in order to parody the last line, "Nobody's perfect" of the celebrated film "Some Like It Hot". They defined a perfect graph as one whose n nodes have distinct degrees and noted that there are none! Such a multigraph is now called *irregular*.

Theorem A. No nontrivial graph is irregular.

Proof. Assume there exists such a graph G with $n \geq 2$ nodes. Then its degree sequence is $(n - 1, n - 2, \ldots, 3, 2, 1, 0)$. The node of degree 0 is isolated but the node of degree $n - 1$ is adjacent to all the other nodes, a contradiction. □

The *underlying graph* G(M) of a multigraph M is the spanning subgraph of M having a single edge wherever M has either a single edge or multiple edges. In general we follow the notation and terminology of [2] and [4].

Recently Chartrand and et al. proved in [3] that for each connected graph G with $n \geq 3$ nodes, there exists an irregular multigraph M having G as its underlying graph.

2. Pluperfect graphs

We now call a connected graph G with $n \geq 3$ nodes and minimum degree δ *pluperfect* if there exists a multigraph M with underlying graph G such that the degree sequence σ of M is consecutive and as small as possible:

$$\sigma = (n + \delta - 1, n + \delta - 2, \dots, \delta + 2, \delta + 1, \delta) \quad (1)$$

Our observations are elementary but novel. Ou⁻ purpose is to present several pluperfect graphs in order to stimulate research on the question in the title of this note.

In [6], we defined the *irregularity cost* ic(G) as the minimum number of new multiple edges in an irregular multigraph M with underlying graph G. Let q(G) and q(M) be the number of edges of graph G and of multigraph M, respectively. Similarly let $\delta(G)$ and $\delta(M)$ be their minimum degrees.

Theorem 1. For a pluperfect graph G with spanning supermultigraph M satisfying (1), ic(G) = q(M) - q(G).

Proof. As $\delta(M) = \delta(G)$, a multigraph satisfying (1) must necessarily have the smallest possible number of new multiple edges among all irregular multigraphs M with G = G(M). □

Question 1. If G is pluperfect does there exist a unique multigraph M satisfying (1)?

Although each pluperfect graph of order 4 has a unique irregular multigraph in which (1) holds, not all pluperfect graphs do.

In providing the next theorems which specify those paths and cycles that are pluperfect, it is convenient to use the following more general result which holds for all graphs.

Lemma 2a. Some necessary conditions for a graph to be pluperfect are:

(i) No graph with $n \equiv 2 \pmod 4$ is pluperfect.

(ii) No graph with $n \equiv 1 \pmod 4$ and odd minimum degree is pluperfect.

(iii) No graph with $n \equiv 3 \pmod 4$ and even minimum degree is pluperfect.

Proof. The sum S of the degrees of the multigraph $M(G)$ of a pluperfect graph G of order n and minimum degree δ is given by

$$S = n\delta + \binom{n}{2} \tag{2}$$

To prove (i), observe that when $n = 4k + 2$ is inserted into (2), we obtain

$$S = 4k\delta + 2\delta + (2k + 1)(4k + 1)$$

which is odd. The proofs of (ii) and (iii) are similar and hence are omitted. □

The next result specifies the pluperfect paths.

Theorem 2. The pluperfect paths P_n for $n \geq 3$ are precisely those with $n \equiv 0$ or $3 \pmod 4$.

Proof. The necessity of the theorem follows at once from Lemma 2a, (i) and (ii), as $\delta = 1$ for paths. To show sufficiency we consider two cases. Edge multiplicities shall be denoted by positive integer weights.

Case 1. $n \equiv 3 \pmod 4$.

Assign the following weights to the edges of P_n consecutively:

$(1, 2, 3, \ldots, \lfloor n/2 \rfloor, \lceil n/2 \rceil, \lceil n/2 \rceil - 2, \lceil n/2 \rceil - 2, \lceil n/2 \rceil - 4, \lceil n/2 \rceil - 4, \ldots, 4, 4, 2, 2)$.

The resulting degree sequence listed consecutively is

$$(1, 3, 5, \ldots, n, n-1, n-3, \ldots, 6, 4, 2).$$

Case 2. $n \equiv 0 \pmod 4$.

The (consecutive) edge weights are now

$(1, 2, 3, \ldots, n/2, n/2, (n/2) - 2, (n/2) - 2, (n/2) - 4, (n/2) - 4, \ldots, 4, 4, 2, 2)$

with resulting degree sequence

$$(1, 3, 5, \ldots, n-1, n, n-2, \ldots, 6, 4, 2). \quad \square$$

Figure 1 illustrates these sequences for P_7 and P_8.

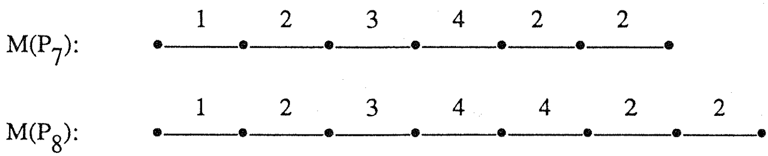

Figure 1. The paths P_7 and P_8 are pluperfect.

Using Theorem 2, it is easy to decide which cycles are pluperfect.

Corollary 2a. The pluperfect cycles C_n are precisely those with $n \equiv 0$ or $1 \pmod 4$.

Proof. The necessity follows readily from Lemma 2a, (i) and (iii), as $\delta = 2$ for cycles. We prove the sufficiency by the following construction. A pluperfect cycle C_{4t} is obtained by increasing each edge weight of a pluperfect path P_{4t-1} (as obtained in the proof of Theorem 2) by one, and then joining a new node v to the endnodes of the path using edges of unit weight. The resulting C_{4t} is easily seen to be pluperfect. The pluperfect cycle C_{4t-1} is obtained from the pluperfect path P_{4t} in exactly the same way. $\square$

Figure 2 exhibits the pluperfect cycles C_8 and C_9 obtained from the paths of

Figure 1.

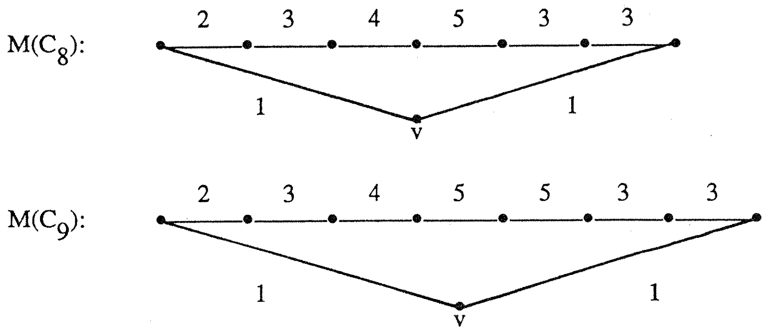

Figure 2. The pluperfect cycles C_8 and C_9 obtained from the pluperfect paths P_7 and P_8.

Remark. The construction of Corollary 2a as illustrated in Figure 2 are reminiscent of the pioneering work of Hayes [7] who introduced the graph theoretic model for node fault tolerance in computers. He mentioned that given the path P_n, the smallest number of edges in a supergraph G with $n + 1$ nodes such that for each node $u \in V(G)$, the subgraph $G - u \supset P_n$ is realized by the cycle, $G = C_{n+1}$. In fact, each $G - u = P_n$, an observation which led us to the concept mentioned in [5] of "exact fault tolerance".

Theorem 3. No star $K_{1,r}$ with $r \geq 3$ is pluperfect.

Proof. The unique multigraph $M(K_{1,r})$ has edge-multiplicities $1, 2, \ldots, r$ so that the irregularity cost of the star is:

$$ic(K_{1,r}) = \left[\begin{matrix} r \\ 2 \end{matrix}\right] \quad (3)$$

Theorem 4. The five connected graphs of order $n = 4$ other than the star are pluperfect.

Proof. By Theorems 2 and 3, the path P_4 is pluperfect and the star $K_{1,3}$ is not. The remaining four connected graphs of order 4 are the cycle C_4, the graph $K_3 \cdot K_2$ obtained by coalescing a node of a triangle with a node of a single edge, the so-called [4] random graph $K_4 - e$ and the complete graph K_4. They are all pluperfect as Figure 3 demonstrates. □

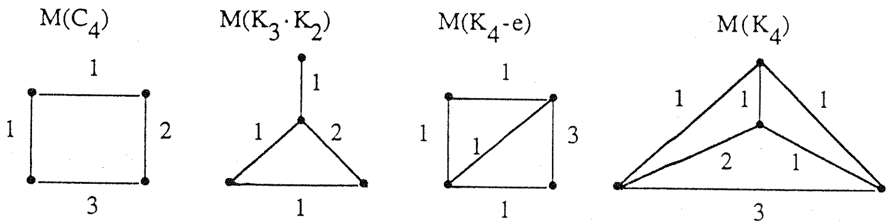

Figure 3. Pluperfect graphs of order 4.

Acknowledgement. The authors thank each other for this most enjoyable collaboration. The second author is grateful for being invited to join coauthorship of a note already accepted for publication while the first author is happy to find that his colleague has fine insight and impressive speed in extending the results.

References

[1] M. Behzad, G. Chartrand, No graph is perfect. *Amer. Math. Monthly* 74 (1967) 962-963.

[2] G. Chartrand, L. Lesniak, *Graphs and Digraphs*, 2nd Edition. Wadsworth, Monterey (1986).

[3] G. Chartrand, M. Jacobson, J. Lehel, O. Oellermann, S. Ruiz, F. Saba, Irregular networks. *Congressus Numerantium* 64 (1988) 197-210.

[4] F. Harary, *Graph Theory*. Addison-Wesley, Reading (1969).

[5] F. Harary and J.P. Hayes, Survey of fault tolerance for graphs. *Proceedings of the Seventh International Kalamazoo Conference on Graph Theory.* Wiley, New York. To appear.

[6] F. Harary and O. Oellermann. The irregularity cost of a graph. To appear.

[7] J.P. Hayes, A graph model for fault tolerant computer systems. *IEEE Trans. Computers* C-25 (1976) 876-884.

(Received 30/4/92; revised 16/9/92)