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A new presentation for the inner Tutte group of a matroid

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Abstract

The inner Tutte group of a matroid is a finitely generated abelian group introduced as an algebraic counterpart of Tutte's homotopy theory of matroids. The aim of this work is to provide a new presentation for this group with a set of generators that is smaller than those previously known.

1 Introduction

Tutte groups of matroids provide an algebraic setting for the study of problems in matroid theory. They were first introduced and studied by Dress and Wenzel in [6] and subsequent papers [11] and [7]. In particular, in [8] and [12] Dress and Wenzel discussed how *matroids with coefficients* arise from the construction of Tutte groups.

This algebraic approach has been successfully exploited in describing reorientation spaces of different types of "decorated" matroids. For instance, an oriented matroid can be seen as a decoration of an "underlying" matroid by means of the set of "real signs" $\{0, +, -\}$ (see [3, Chapter 3]). The space of reorientation classes of oriented matroids with given underlying matroid is described in [9] as an algebraically defined subset of homomorphisms from the inner Tutte group to the multiplicative group $\{+, -\}$. With similar techniques, Delucchi and the author gave in [5] an analogous characterization for spaces of phased matroids with given underlying matroid in terms of homomorphisms from the inner Tutte group to the multiplicative group S^1. In fact, making a parallel with oriented matroid theory, a phased matroid can be seen as a decoration of a "underlying" matroid by means of elements of the set of "phases" $\{0\} \cup S^1$ (see [1, Section 2] and [5, Section 1]). Moreover, the recent work of Baker [2] suggests that this framework can be fruitfully exploited to study rescaling spaces of matroids over hyperfields (reference work in progress).

The *inner Tutte group* of a matroid M was first defined in [6] as the kernel of a natural homomorphism from the *Tutte group* of M to a free abelian group (compare Definition 2.1). Explicit formulas for the rank of these groups were obtained in [4, Theorem 3.3] and [5, Theorem 8.1]. These results, together with [11, Theorem 5.4], imply a full description of the inner Tutte group of all matroids with ground set of up to 7 elements.

In [9, Theorem 3] and [9, Theorem 4] two presentations of the inner Tutte group are described. However, both these presentations have very large sets of generators.

In this work we provide a presentation of the inner Tutte group of a matroid with a smaller set of generators (Definition 3.1). Together with [5, Theorem 8.1], this affords a relatively efficient computation of the rank of the inner Tutte group of a matroid. The central idea of our proof (Theorem 3.1) is to get rid of redundancies exploiting a geometric interpretation of the generators of the inner Tutte group in terms of "cross-ratios" (see [9, Corollary 2] and [5, Theorem 6.4] for further details).

As an illustration of the size of our improvements, in Table 1 we compare the number of generators of the presentation of the inner Tutte group given in [9, Theorem 4] with the number of generators of our presentation for some remarkable matroids.

Overview. Section 2 provides some basic definitions on matroids and Tutte groups. In Section 3, to each matroid we associate our "modified" abelian group, presented by generators and relations, and we prove that it is isomorphic to the inner Tutte group of the given matroid.

2 Matroids and Tutte groups

In this section we quickly provide some basic notions about matroids and Tutte groups. We point to the book [10] for a detailed treatment of matroid theory and we refer to [6] and subsequent papers of the same authors for a general theory of Tutte groups.

2.1 Matroids

A *matroid* M is a pair $(E, \mathfrak{I})$, where E is a finite *ground set* and $\mathfrak{I} \subseteq 2^E$ is a family of subsets of E that fulfill the following axioms:

(I1) $\emptyset \in \mathfrak{I}$;

(I2) If $I \in \mathfrak{I}$ and $J \subseteq I$, then $J \in \mathfrak{I}$;

(I3) If I and J are in $\mathfrak{I}$ and $|I| < |J|$, then there exists $e \in J \setminus I$ with the property that $I \cup \{e\} \in \mathfrak{I}$.

The subsets of E that belong to $\mathfrak{I}$ are the *independent sets* of M. Maximal independent sets (with respect to set inclusion) are named *bases* and we write $\mathfrak{B}$ for the collection of basis of M. A *dependent* set of M is a subset of E that is not in $\mathfrak{I}$. The subsets of E that are minimal with respect to set inclusion are called *circuits* and $\mathfrak{C}$ stands for the family of circuits of M.

For a subset $S \subseteq E$, we define its *rank* by the formula

$$\mathrm{rk}(S) = \max \{|S \cap B| \mid B \in \mathfrak{B}\}$$

and we set the *rank of the matroid* M by $\mathrm{rk}(M) := \mathrm{rk}(E)$. If $\mathrm{rk}(S) = \mathrm{rk}(M)$ we say that S is a *spanning* set of M.

The collection of complements of spanning sets of M does fulfills axioms (I1), (I2) and (I3). Hence, it is the family of independent sets of a matroid called *dual* to M and denoted by M^*. Its rank function rk^* is related to that of M by the identity

$$\mathrm{rk}^*(A) = \mathrm{rk}(E \setminus A) + |A| - \mathrm{rk}(E)$$

We say *cocircuits* and *cobases* of M for the of circuits and bases of M^*. The collections of cocircuits and cobases of M will be denoted by $\mathfrak{C}^*$ and $\mathfrak{B}^*$, respectively.

Now, let us consider a subset T of the ground set E of the given matroid M. It is not hard to see that the family of subsets of T that are independent sets of M fulfills axioms (I1), (I2) and (I3). Thus, this is the collection of independent sets of a matroid named *restriction* of M to T. We write $M[T]$ for the restriction of M to T. On the other hand, the *contraction* of T in M is defined as the matroid $(M^*[E \setminus T])^*$. A matroid that can be obtained from M with a sequence of restrictions and contractios will be called a *minor* of M.

Example 2.1 (The Fano matroid). The Fano matroid is the matroid denoted by F_7 and defined on the ground set $E = \{1, 2, \dots, 7\}$ by the collection of hyperplanes

$$\mathfrak{H} = \{\{1,2,3\}, \{2,5,7\}, \{1,4,7\}, \{1,5,6\}, \{3,4,5\}, \{3,6,7\}, \{2,4,6\}\}.$$

Often in this paper we will work with matroids “without minors of Fano or dual-Fano type”, that are matroids for which neither F_7 nor its dual can appear as minors.

Given matroids M_1 and M_2 with disjoint ground sets E_1 and E_2 and independent sets $\mathfrak{I}_1$ and $\mathfrak{I}_2$, the *direct sum* of M_1 and M_2 is the matroid $M_1 \oplus M_2$ with ground set $E_1 \cup E_2$ and independent sets

$$\{I_1 \cup I_2 \mid I_1 \in \mathfrak{I}_1 \text{ and } I_2 \in \mathfrak{I}_2\}$$

We say that M is *disconnected* if there exists a proper non-empty subset T of the ground set E such that $M = M[T] \oplus M[E \setminus T]$. M will be called *connected* otherwise. A *connected component* of M is a maximal inclusion subset T of E such that $M[T]$ is connected. From [10, Corollary 4.2.13] there is a unique (up to permutations) decomposition of M as direct sum of connected matroids. In particular, from this it follows that the number c_M of connected components of M is well defined.

2.2 Tutte groups

We consider sets of the form $F = C_1 \cup \cdots \cup C_k$ with $C_i \in \mathfrak{C}$. If $\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_d = F$ is a maximal chain of such sets, then d depends only on F and is called the *dimension* of F. We denote it by $\dim(F)$. Notice that

$$\dim(F) = |F| - \mathrm{rk}(F) - 1$$

The group $\mathbb{T}_M^{\mathfrak{C},\mathfrak{C}^*}$ is defined to be the multiplicative abelian group with formal generators given by the symbols

- ϵ_M;
- $C(x)$ for $C \in \mathfrak{C}$ and $x \in C$;
- $D(y)$ for $D \in \mathfrak{C}^*$ and $y \in D$;

with relations

- $\epsilon_M^2 = 1$;
- $C(x)D(x) = \epsilon_M C(y)D(y)$ for $C \in \mathfrak{C}$, $D \in \mathfrak{C}^*$ with $\{x, y\} = C \cap D$.

The *Tutte group* $\mathbb{T}_M$ is then the subgroup of $\mathbb{T}_M^{\mathfrak{C},\mathfrak{C}^*}$ generated by

- ϵ_M;
- $C(x)C(y)^{-1}$ for $C \in \mathfrak{C}$, $x, y \in C$;
- $D(x)D(y)^{-1}$ for $D \in \mathfrak{C}^*$, $x, y \in D$.

Definition 2.1 (Inner Tutte group)**.** Let us consider the group homomorphism $\Lambda : \mathbb{T}_M^{\mathfrak{C},\mathfrak{C}^*} \longrightarrow \mathbb{Z}^{|E|} \times \mathbb{Z}^{|\mathfrak{C}|} \times \mathbb{Z}^{|\mathfrak{C}^*|}$, defined by

- $\epsilon_M \mapsto 0$;
- $C(x) \mapsto (\mathbb{1}_x, \mathbb{1}_C, 0)$;
- $D(y) \mapsto (-\mathbb{1}_y, 0, \mathbb{1}_D)$;

where $\mathbb{1}$ is the indicator function. The *inner Tutte group* $\mathbb{T}_M^{(0)}$ of M is the kernel of the homomorphism Λ.

One has $\mathbb{T}_M^{(0)} \lhd \mathbb{T}_M \lhd \mathbb{T}_M^{\mathfrak{C},\mathfrak{C}^*}$. Moreover, if c_M is the number of connected components of M, with [6, Theorem 1.5] we have

$$\mathbb{T}_M \cong \mathbb{T}_M^{(0)} \times \mathbb{Z}^{|E|-c_M} \quad \mathbb{T}_M^{\mathfrak{C},\mathfrak{C}^*} \cong \mathbb{T}_M^{(0)} \times \mathbb{Z}^{|E|-c_M} \times \mathbb{Z}^{|\mathfrak{C}|} \times \mathbb{Z}^{|\mathfrak{C}^*|}$$

Thus, any of these groups is known as soon as we know $\mathbb{T}_M^{(0)}$. In order to prove Theorem 3.1, following [9] we provide the subsequent definition.

Definition 2.2. Given a matroid M the group $\mathbb{T}_M^{(2)}$ is the finitely generated abelian group with formal generators given by the symbols

(G1) ξ_M;

(G2) $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}]$, where C_{i_1}, C_{i_2}, C_{i_3}, $C_{i_4} \in \mathfrak{C}$ are circuits of M such that $L = C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4} = C_{i_k} \cup C_{i_l}$ for $k = 1, 2$, $l = 3, 4$, $\dim(L) = 1$;

and relations

(R1) $\xi_M^2 = 1_{\mathbb{T}_M^{(2)}}$;

(R2) $\xi_M = 1_{\mathbb{T}_M^{(2)}}$ if M has minors of Fano or dual-Fano type;

(R3) $[C_{i_1}C_{i_2}|C_{i_3}C_{i_3}] = 1_{\mathbb{T}_M^{(2)}}$;

(R4) $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}] = [C_{i_3}C_{i_4}|C_{i_1}C_{i_2}]$;

(R5) $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}][C_{i_1}C_{i_2}|C_{i_4}C_{i_5}][C_{i_1}C_{i_2}|C_{i_5}C_{i_3}] = 1_{\mathbb{T}_M^{(2)}}$;

(R6) $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}][C_{i_1}C_{i_4}|C_{i_2}C_{i_3}][C_{i_1}C_{i_3}|C_{i_4}C_{i_2}] = \xi_M$;

(R7) $[C_{i_1}C_{i_2}|C_{i_6}C_{i_9}][C_{i_2}C_{i_3}|C_{i_4}C_{i_7}][C_{i_3}C_{i_1}|C_{i_5}C_{i_8}] = 1_{\mathbb{T}_M^{(2)}}$ for any family of circuits $\{C_{i_1}, \ldots, C_{i_9}\} \subseteq \mathfrak{C}$ such that:

- $\dim(L_{i_p}) = 1$ for $L_{i_p} = C_{i_q} \cup C_{i_r}$, where $\{p, q, r\} = \{1, 2, 3\}$;
- $\dim(P) = 2$ where $P = C_{i_1} \cup C_{i_2} \cup C_{i_3}$;
- $C_{i_{s+3}}, C_{i_{s+6}} \subseteq L_{i_s}$ for $s = 1, 2, 3$;
- $\dim(L_{i_h}) = 1$ for $L_{i_h} = C_{i_{3+h}} \cup C_{i_{4+h}} \cup C_{i_{5+h}}$, $h \in \{1, 4\}$;
- $\{C_{i_1}, C_{i_2}, C_{i_3}\} \cap \{C_{i_4}, \ldots, C_{i_9}\} = \emptyset$.

We write $\mathbb{W}_M^{(2)}$ for the set of generators of $\mathbb{T}_M^{(2)}$ and we denote by G_M the number of generators of $\mathbb{T}_M^{(2)}$, that is, $G_M = \left|\mathbb{W}_M^{(2)}\right|$. Recall that the groups $\mathbb{T}_M^{(2)}$ and $\mathbb{T}_M^{(0)}$ are isomorphic (see [9, Theorem 4] for more details).

Note 2.1. Relations (R3) and (R5) imply that $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}] = [C_{i_1}C_{i_2}|C_{i_4}C_{i_3}]^{-1}$.

3 Results

Throughout this section we assume that, for a matroid M with set of circuits $\mathfrak{C}$, an enumeration $\{C_j\}_{j \in J}$ of $\mathfrak{C}$ and an arbitrary total order $<_J$ on J are given.

Our aim is to define a group $\mathcal{T}_{M,<_J}^{(0)}$ that is isomorphic to the inner Tutte group of M, but it has a presentation with less generators than those described in [9]. To show this isomorphism result, we exploit some techniques developed in the proof of [5, Theorem 6.4] in order to define bijective homomorphisms from $\mathbb{T}_M^{(0)}$ to $\mathcal{T}_{M,<_J}^{(0)}$ and from $\mathcal{T}_{M,<_J}^{(0)}$ to $\mathbb{T}_M^{(0)}$.

Definition 3.1. We denote by $\mathcal{T}_{M,<_J}^{(0)}$ the multiplicative abelian group with formal generators given by the symbols

(Q1) $\eta_{M,<_J}$;

(Q2) $(C_{j_1}C_{j_2}|C_{j_3}C_{j_4})$ where $C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4} \in \mathfrak{C}$ are pairwise distinct circuits of M such that $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1$ and $j_1 < j_2$, $j_3 < j_4$, $j_1 < j_3$;

and relations

(S1) $\eta^2_{M,<J} = 1_{\mathcal{T}^{(0)}_{M,<J}}$;

(S2) $\eta_{M,<J} = 1_{\mathcal{T}^{(0)}_{M,<J}}$ if M has minors of Fano or dual-Fano type;

(S3) $(C_{j_1}C_{j_2}|C_{j_3}C_{j_4})(C_{j_1}C_{j_4}|C_{j_2}C_{j_3})(C_{j_1}C_{j_3}|C_{j_2}C_{j_4})^{-1} = \eta_{M,<J}$ for any family of circuits $\{C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}\} \subseteq \mathfrak{C}$ such that $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4}) = 1$ and $j_1 < j_2 < j_3 < j_4$;

(S4)

$$\begin{cases} (C_{j_1}C_{j_2}|C_{j_3}C_{j_4})(C_{j_1}C_{j_2}|C_{j_4}C_{j_5})(C_{j_1}C_{j_2}|C_{j_3}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_3}|C_{j_2}C_{j_4})(C_{j_1}C_{j_3}|C_{j_4}C_{j_5})(C_{j_1}C_{j_3}|C_{j_2}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_4}|C_{j_2}C_{j_3})(C_{j_1}C_{j_4}|C_{j_3}C_{j_5})(C_{j_1}C_{j_4}|C_{j_2}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_5}|C_{j_2}C_{j_3})(C_{j_1}C_{j_5}|C_{j_3}C_{j_4})(C_{j_1}C_{j_5}|C_{j_2}C_{j_4})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_4}|C_{j_2}C_{j_3})(C_{j_2}C_{j_3}|C_{j_4}C_{j_5})(C_{j_1}C_{j_5}|C_{j_2}C_{j_3})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_3}|C_{j_2}C_{j_4})(C_{j_2}C_{j_4}|C_{j_3}C_{j_5})(C_{j_1}C_{j_5}|C_{j_2}C_{j_4})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_3}|C_{j_2}C_{j_5})(C_{j_2}C_{j_5}|C_{j_3}C_{j_4})(C_{j_1}C_{j_4}|C_{j_2}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_2}|C_{j_3}C_{j_4})(C_{j_2}C_{j_5}|C_{j_3}C_{j_4})(C_{j_1}C_{j_5}|C_{j_3}C_{j_4})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_2}|C_{j_3}C_{j_5})(C_{j_2}C_{j_4}|C_{j_3}C_{j_5})(C_{j_1}C_{j_4}|C_{j_3}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \\ (C_{j_1}C_{j_2}|C_{j_4}C_{j_5})(C_{j_2}C_{j_3}|C_{j_4}C_{j_5})(C_{j_1}C_{j_3}|C_{j_4}C_{j_5})^{-1} = 1_{\mathcal{T}^{(0)}_{M,<J}} \end{cases}$$

where C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}, $C_{j_5} \in \mathfrak{C}$ are circuits of M with the properties that $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4} \cup C_{j_5}) = 1$ and $j_1 < j_2 < j_3 < j_4 < j_5$;

(S5) $\langle C_{j_1}C_{j_2}|C_{j_6}C_{j_9}\rangle\langle C_{j_2}C_{j_3}|C_{j_4}C_{j_7}\rangle\langle C_{j_3}C_{j_1}|C_{j_5}C_{j_8}\rangle = 1_{\mathcal{T}^{(0)}_{M,<J}}$ for any family of circuits $\{C_{i_1}, \ldots, C_{i_9}\} \subseteq \mathfrak{C}$ as in (R7) with the extra conditions:

(O1) $j_1 < j_2 < j_3$;

(O2) $j_4 \geq j_7$, $j_5 \geq j_8$ and $j_6 \geq j_9$ do not all hold at the same time.

Here $\langle C_{d_1}C_{d_2}|C_{d_3}C_{d_4}\rangle$ are the symbols given by the formula

$$\langle C_{d_1}C_{d_2}|C_{d_3}C_{d_4}\rangle = \begin{cases} 1_{\mathcal{T}^{(0)}_{M,<J}} & \text{if } d_1 = d_2 \quad \text{or} \quad d_3 = d_4 \\ (C_{d_1}C_{d_2}|C_{d_3}C_{d_4}) & \text{if } d_1 < d_2 \quad d_3 < d_4 \quad d_1 < d_3 \\ (C_{d_3}C_{d_4}|C_{d_1}C_{d_2}) & \text{if } d_1 < d_2 \quad d_3 < d_4 \quad d_3 < d_1 \\ (C_{d_1}C_{d_2}|C_{d_4}C_{d_3})^{-1} & \text{if } d_1 < d_2 \quad d_4 < d_3 \quad d_1 < d_4 \\ (C_{d_4}C_{d_3}|C_{d_1}C_{d_2})^{-1} & \text{if } d_1 < d_2 \quad d_4 < d_3 \quad d_4 < d_1 \\ (C_{d_2}C_{d_1}|C_{d_3}C_{d_4})^{-1} & \text{if } d_2 < d_1 \quad d_3 < d_4 \quad d_2 < d_3 \\ (C_{d_3}C_{d_4}|C_{d_2}C_{d_1})^{-1} & \text{if } d_2 < d_1 \quad d_3 < d_4 \quad d_3 < d_2 \\ (C_{d_2}C_{d_1}|C_{d_4}C_{d_3}) & \text{if } d_2 < d_1 \quad d_4 < d_3 \quad d_2 < d_4 \\ (C_{d_4}C_{d_3}|C_{d_2}C_{d_1}) & \text{if } d_2 < d_1 \quad d_4 < d_3 \quad d_4 < d_2 \end{cases} \qquad (*)$$

Matroid M	G_M	g_M
$U_2(4)$	85	4
$U_2(5)$	421	16
$U_3(5)$	261	16
$M(K_4)$	109	1
$\mathcal{W}^3$	307	10
Q_6	615	28
P_6	1033	55
$U_3(6)$	1561	91
R_6	505	19
F_7	379	1
F_7^*	127	1
F_7^-	775	19
$(F_7^-)^*$	325	10
P_7	1171	37

Table 1: Comparison between G_M and g_M for some remarkable cases.

and defined for any family of circuits $\{C_{d_1}, C_{d_2}, C_{d_3}, C_{d_1}\} \subseteq \mathfrak{C}$ satisfying $\dim(C_{d_1} \cup C_{d_2} \cup C_{d_3} \cup C_{d_4}) = 1$ and $\{C_{d_1}, C_{d_2}\} \cap \{C_{d_3}, C_{d_4}\} = \emptyset$.

We write $\mathcal{W}^{(0)}_{M,<_J}$ for the set of generators of $\mathcal{T}^{(0)}_{M,<_J}$ and we denote by g_M the number of generators of $\mathcal{T}^{(0)}_{M,<_H}$, that is, $g_M = \left|\mathcal{W}^{(0)}_{M,<_J}\right|$.

Note 3.1. If $\{C_h\}_{h\in H}$ is another enumeration of $\mathfrak{C}$ with total order $<_H$ on H, then the groups $\mathcal{T}^{(0)}_{M,<_H}$ and $\mathcal{T}^{(0)}_{M,<_J}$ are isomorphic. To see this it suffices to consider a suitable relabeling of the circuits of M. As a consequence of this, the quantity g_M is well defined, neither depending on the choice of the enumeration of circuits of M nor the total ordering of such enumeration.

Theorem 3.1. *The groups $\mathcal{T}^{(0)}_{M,<_J}$ and $\mathbb{T}^{(0)}_M$ are isomorphic.*

In Table 1 we compare the number G_M of generators of the presentation of the inner Tutte group of [9, Theorem 4] (see Definition 2.2) with the number g_M of generators of our presentation (see Definition 3.1) for some remarkable matroids. These results are obtained with SAGE on a standard laptop.

Remark 3.1. Using again relation (S3) it is even possible to reduce the number of generators of our presentation. However, it seems quite hard to describe how to modify relations (S4) and (S5) with this further reduced presentation.

Proof of Theorem 3.1. With [9, Theorem 4] it is enough to prove that the groups $\mathbb{T}_M^{(2)}$ and $\mathcal{T}_{M,<J}^{(0)}$ are isomorphic. To see this, let $\mathbb{W}_M^{(2)}$ and $\mathcal{W}_{M,<J}^{(0)}$ be the set of generators of $\mathbb{T}_M^{(2)}$ and $\mathcal{T}_{M,<J}^{(0)}$ as in Definition 2.2 and Definition 3.1. Let us consider the map $\phi : \mathbb{W}_M^{(2)} \longrightarrow \mathcal{T}_{M,<J}^{(0)}$ given by

(D1) $\phi(\xi_M) = \eta_{M,<J}$;

(D2) $\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_4}]) = \langle C_{i_1}C_{i_2}|C_{i_3}C_{i_4}\rangle$ where $\langle C_{i_1}C_{i_2}|C_{i_3}C_{i_4}\rangle$ are the symbols defined in $(*)$.

The map $\phi : \mathbb{W}_M^{(2)} \longrightarrow \mathcal{T}_{M,<J}^{(0)}$ fulfills the subsequent properties:

(VR1) $\phi(\xi_M)^2 = 1_{\mathcal{T}_{M,<J}^{(0)}}$. This follows directly from (D1) and (S1).

(VR2) $\phi(\xi_M) = 1_{\mathcal{T}_{M,<J}^{(0)}}$ if M has minors of Fano or dual-Fano type. This follows from (D1) and (S2).

(VR3) $\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_3}]) = 1_{\mathcal{T}_{M,<J}^{(0)}}$. This follows immediately from $(*)$.

(VR4) $\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_4}]) = \phi([C_{i_3}C_{i_4}|C_{i_1}C_{i_2}])$. This follows from a straightforward check of $(*)$.

(VR5) $\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_4}])\phi([C_{i_1}C_{i_2}|C_{i_4}C_{i_5}])\phi([C_{i_1}C_{i_2}|C_{i_5}C_{i_3}]) = 1_{\mathcal{T}_{M,<J}^{(0)}}$. To see this, we need to distinguish between two cases:

- If $i_1 = i_2$ or $i_3 = i_4$ or $i_4 = i_5$ or $i_3 = i_5$ (VR5) easly follows from $(*)$.
- Assume $i_1 \neq i_2$, $i_3 \neq i_4$, $i_4 \neq i_5$, $i_3 \neq i_5$. From $(*)$, it is not hard to see that (VR5) is equivalent to exaclty one among the equations of the family (S4). Thus, (VR5) fulfills since all the equations of the family (S4) hold.

(VR6) $\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_4}])\phi([C_{i_1}C_{i_4}|C_{i_2}C_{i_3}])\phi([C_{i_1}C_{i_3}|C_{i_4}C_{i_2}]) = \eta_{M,<J}$. To check this, let C_{i_1}, C_{i_2}, C_{i_3}, C_{i_4} be pairwise distinct circuits of M such that $\dim(C_{i_1} \cup C_{i_2} \cup C_{i_3} \cup C_{i_4}) = 1$ and let $\sigma \in \mathcal{S}_4$ be a permutation such that $i_{\sigma(1)} < i_{\sigma(2)} < i_{\sigma(3)} < i_{\sigma(4)}$. From (VR3), (VR4) and (VR5), together with Note 2.1, the symbols

$$\phi([C_{i_1}C_{i_2}|C_{i_3}C_{i_4}])$$

fulfill the hypothesis of Lemma 3.1 below. Thus, (VR6) is equivalent to

$$\begin{aligned} \phi([C_{i_{\sigma(1)}}C_{i_{\sigma(2)}}|C_{i_{\sigma(3)}}C_{i_{\sigma(4)}}]) \cdot \phi([C_{i_{\sigma(1)}}C_{i_{\sigma(4)}}|C_{i_{\sigma(2)}}C_{i_{\sigma(3)}}]) & \\ \cdot\phi([C_{i_{\sigma(1)}}C_{i_{\sigma(3)}}|C_{i_{\sigma(4)}}C_{i_{\sigma(2)}}]) &= \eta_{M,<J}. \end{aligned}$$

From $(*)$, this is the same as

$$\begin{aligned} (C_{i_{\sigma(1)}}C_{i_{\sigma(2)}}|C_{i_{\sigma(3)}}C_{i_{\sigma(4)}}) \cdot (C_{i_{\sigma(1)}}C_{i_{\sigma(4)}}|C_{i_{\sigma(2)}}C_{i_{\sigma(3)}}) & \\ \cdot(C_{i_{\sigma(1)}}C_{i_{\sigma(3)}}|C_{i_{\sigma(2)}}C_{i_{\sigma(4)}})^{-1} &= \eta_{M,<J} \end{aligned}$$

and this holds by (S3).

(VR7) $\phi([C_{i_1}C_{i_2}|C_{i_6}C_{i_9}])\phi([C_{i_2}C_{i_3}|C_{i_4}C_{i_7}])\phi([C_{i_3}C_{i_1}|C_{i_5}C_{i_8}]) = 1_{\mathcal{T}^{(0)}_{M,<J}}$ for any family of circuits $C_{i_1},\dots,C_{i_9}$ as in (R7). To check this, we have to distinguish between two subcases.

- If $i_6 = i_9$, $i_4 = i_7$ and $i_5 = i_8$, (VR7) is obviously verified since (VR3) holds.
- Conversely, there is a permutation $\sigma \in \mathcal{S}_9$ such that conditions (O1) and (O2) of Definition 3.1 fulfill with $j_d = i_{\sigma(d)}$, $1 \le d \le 9$. With the same arguments of the proof of (VR6), we can apply Lemma 3.2 below. Hence, (VR7) is equivalent to

$$\begin{aligned}&\phi([C_{i_{\sigma(1)}}C_{i_{\sigma(2)}}|C_{i_{\sigma(\sigma^{-1}(3)+3)}}C_{i_{\sigma(\sigma^{-1}(3)+6)}}])\\ \cdot&\phi([C_{i_{\sigma(2)}}C_{i_{\sigma(3)}}|C_{i_{\sigma(\sigma^{-1}(1)+3)}}C_{i_{\sigma(\sigma^{-1}(1)+6)}}])\\ \cdot&\phi([C_{i_{\sigma(3)}}C_{i_{\sigma(1)}}|C_{i_{\sigma(\sigma^{-1}(2)+3)}}C_{j_{\sigma(\sigma^{-1}(2)+6)}}] \;=\; 1_{\mathcal{T}^{(0)}_{M,<J}}.\end{aligned}$$

From $(*)$, this is the same as

$$\langle C_{i_{\sigma(1)}}C_{i_{\sigma(2)}}|C_{i_{\sigma(\sigma^{-1}(3)+3)}}C_{i_{\sigma(\sigma^{-1}(3)+6)}}\rangle\cdot\langle C_{i_{\sigma(2)}}C_{i_{\sigma(3)}}|C_{i_{\sigma(\sigma^{-1}(1)+3)}}C_{i_{\sigma(\sigma^{-1}(1)+6)}}\rangle$$
$$\cdot\langle C_{i_{\sigma(3)}}C_{i_{\sigma(1)}}|C_{i_{\sigma(\sigma^{-1}(2)+3)}}C_{j_{\sigma(\sigma^{-1}(2)+6)}}\rangle \;=\; 1_{\mathcal{T}^{(0)}_{M,<J}}.$$

and this holds by (S5).

So there exists a unique group homomorphism $\Phi : \mathbb{T}^{(2)}_M \longrightarrow \mathcal{T}^{(0)}_{M,<J}$ with $\Phi|_{\mathbb{W}^{(2)}_M} = \phi$. Similarly, let us consider the map $\psi : \mathcal{W}^{(0)}_{M,<J} \longrightarrow \mathbb{T}^{(2)}_M$ defined by

(T1) $\psi(\eta_{M,<J}) = \xi_M$;

(T2) $\psi((C_{i_1}C_{i_2}|C_{i_3}C_{i_4})) = [C_{i_1}C_{i_2}|C_{i_3}C_{i_4}]$.

The map $\psi : \mathcal{W}^{(0)}_{M,<J} \longrightarrow \mathbb{T}^{(2)}_M$ satisfies the subsequent properties:

(VS1) $\psi(\eta_{M,<J})^2 = 1_{\mathbb{T}^{(2)}_M}$. This follows from (T1) and (R1).

(VS2) $\psi(\eta_{M,<J}) = 1_{\mathbb{T}^{(2)}_M}$ if M has minors of Fano or dual-Fano type. This follows from (T1) together with (R2).

(VS3) $\psi((C_{i_1}C_{i_2}|C_{i_3}C_{i_4}))\psi((C_{i_1}C_{i_4}|C_{i_2}C_{i_3}))\psi((C_{i_1}C_{i_3}|C_{i_2}C_{i_4}))^{-1} = \xi_M$ for any family $\{C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}\} \subseteq \mathfrak{C}$ with $\dim(C_{j_1}\cup C_{j_2}\cup C_{j_3}\cup C_{j_4}) = 1$ and $j_1 < j_2 < j_3 < j_4$. To see this, notice that by (T2), (VS3) is equivalent to

$$[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}][C_{i_1}C_{i_4}|C_{i_2}C_{i_3}][C_{i_1}C_{i_3}|C_{i_2}C_{i_4}]^{-1} = \xi_M$$

By Note 2.1 this is the same as

$$[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}][C_{i_1}C_{i_4}|C_{i_2}C_{i_3}][C_{i_1}C_{i_3}|C_{i_4}C_{i_2}] = \xi_M$$

and this holds by (R6).

(VS4)

$$\begin{cases}
\psi((C_{j_1}C_{j_2}|C_{j_3}C_{j_4}))\psi((C_{j_1}C_{j_2}|C_{j_4}C_{j_5}))\psi((C_{j_1}C_{j_2}|C_{j_3}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_3}|C_{j_2}C_{j_4}))\psi((C_{j_1}C_{j_3}|C_{j_4}C_{j_5}))\psi((C_{j_1}C_{j_3}|C_{j_2}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_4}|C_{j_2}C_{j_3}))\psi((C_{j_1}C_{j_4}|C_{j_3}C_{j_5}))\psi((C_{j_1}C_{j_4}|C_{j_2}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_5}|C_{j_2}C_{j_3}))\psi((C_{j_1}C_{j_5}|C_{j_3}C_{j_4}))\psi((C_{j_1}C_{j_5}|C_{j_2}C_{j_4}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_4}|C_{j_2}C_{j_3}))\psi((C_{j_2}C_{j_3}|C_{j_4}C_{j_5}))\psi((C_{j_1}C_{j_5}|C_{j_2}C_{j_3}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_3}|C_{j_2}C_{j_4}))\psi((C_{j_2}C_{j_4}|C_{j_3}C_{j_5}))\psi((C_{j_1}C_{j_5}|C_{j_2}C_{j_4}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_3}|C_{j_2}C_{j_5}))\psi((C_{j_2}C_{j_5}|C_{j_3}C_{j_4}))\psi((C_{j_1}C_{j_4}|C_{j_2}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_2}|C_{j_3}C_{j_4}))\psi((C_{j_2}C_{j_5}|C_{j_3}C_{j_4}))\psi((C_{j_1}C_{j_5}|C_{j_3}C_{j_4}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_2}|C_{j_3}C_{j_5}))\psi((C_{j_2}C_{j_4}|C_{j_3}C_{j_5}))\psi((C_{j_1}C_{j_4}|C_{j_3}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}} \\
\psi((C_{j_1}C_{j_2}|C_{j_4}C_{j_5}))\psi((C_{j_2}C_{j_3}|C_{j_4}C_{j_5}))\psi((C_{j_1}C_{j_3}|C_{j_4}C_{j_5}))^{-1} = 1_{\mathbb{T}_M^{(2)}}
\end{cases}$$

where C_{j_1}, C_{j_2}, C_{j_3}, C_{j_4}, $C_{j_5} \in \mathfrak{C}$ are circuits of M with the properties that $\dim(C_{j_1} \cup C_{j_2} \cup C_{j_3} \cup C_{j_4} \cup C_{j_5}) = 1$ and $j_1 < j_2 < j_3 < j_4 < j_5$. To see this, notice that by (T2), together with (R4) and Note 2.1, the given family of equations is equivalent to

$$\begin{cases}
[C_{j_1}C_{j_2}|C_{j_3}C_{j_4}][C_{j_1}C_{j_2}|C_{j_4}C_{j_5}][C_{j_1}C_{j_2}|C_{j_5}C_{j_3}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_1}C_{j_3}|C_{j_2}C_{j_4}][C_{j_1}C_{j_3}|C_{j_4}C_{j_5}][C_{j_1}C_{j_3}|C_{j_5}C_{j_2}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_1}C_{j_4}|C_{j_2}C_{j_3}][C_{j_1}C_{j_4}|C_{j_3}C_{j_5}][C_{j_1}C_{j_4}|C_{j_5}C_{j_2}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_1}C_{j_5}|C_{j_2}C_{j_3}][C_{j_1}C_{j_5}|C_{j_3}C_{j_4}][C_{j_1}C_{j_5}|C_{j_4}C_{j_2}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_2}C_{j_3}|C_{j_1}C_{j_4}][C_{j_2}C_{j_3}|C_{j_4}C_{j_5}][C_{j_2}C_{j_3}|C_{j_5}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_2}C_{j_4}|C_{j_1}C_{j_3}][C_{j_2}C_{j_4}|C_{j_3}C_{j_5}][C_{j_2}C_{j_4}|C_{j_5}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_2}C_{j_5}|C_{j_1}C_{j_3}][C_{j_2}C_{j_5}|C_{j_3}C_{j_4}][C_{j_2}C_{j_5}|C_{j_4}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_3}C_{j_4}|C_{j_1}C_{j_2}][C_{j_3}C_{j_4}|C_{j_2}C_{j_5}][C_{j_3}C_{j_4}|C_{j_5}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_3}C_{j_5}|C_{j_1}C_{j_2}][C_{j_3}C_{j_5}|C_{j_2}C_{j_4}][C_{j_3}C_{j_5}|C_{j_4}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}} \\
[C_{j_4}C_{j_5}|C_{j_1}C_{j_2}][C_{j_4}C_{j_5}|C_{j_2}C_{j_3}][C_{j_4}C_{j_5}|C_{j_3}C_{j_1}] = 1_{\mathbb{T}_M^{(2)}}
\end{cases}$$

and all of these identities hold by (R5).

(VS5) $\psi(\langle C_{j_1}C_{j_2}|C_{j_6}C_{j_9}\rangle)\psi(\langle C_{j_2}C_{j_3}|C_{j_4}C_{j_7}\rangle)\psi(\langle C_{j_3}C_{j_1}|C_{j_5}C_{j_8}\rangle) = 1_{\mathbb{T}_M^{(2)}}$ for any family of circuits $\{C_{i_1}, \ldots, C_{i_9}\} \subseteq \mathfrak{C}$ as in (S5). To check this, notice that (T2) and $(*)$, together with (R4) and Note 2.1, imply that

$$\psi(\langle C_{d_1}C_{d_2}|C_{d_3}C_{d_4}\rangle) = [C_{d_1}C_{d_2}|C_{d_3}C_{d_4}].$$

Thus (VS5) is equivalent to

$$[C_{j_1}C_{j_2}|C_{j_6}C_{j_9}][C_{j_2}C_{j_3}|C_{j_4}C_{j_7}][C_{j_3}C_{j_1}|C_{j_5}C_{j_8}] = 1_{\mathbb{T}_M^{(2)}}$$

and this holds by (R7).

Hence there is a unique homomorphism $\Psi : \mathcal{T}^{(0)}_{M,<_J} \longrightarrow \mathbb{T}^{(2)}_M$ with $\Psi|_{\mathcal{W}^{(0)}_{M,<_J}} = \psi$. To complete our proof it is enough to show that Ψ is bijective. Exploiting relations (R3) and (R4), together with Note 2.1, we can see that $\mathbb{T}^{(2)}_M$ is actually generated by those symbols $[C_{i_1}C_{i_2}|C_{i_3}C_{i_4}]$ where $i_1 < i_2$, $i_3 < i_4$ and $i_1 < i_3$. In particular, this implies that Ψ is surjective. On the other hand, by definition of the maps ϕ and ψ we have $\phi \circ \psi = \mathrm{Id}_{\mathcal{W}^{(0)}_{M,<_J}}$. So $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{T}^{(0)}_{M,<_J}}$, and from this it follows that Ψ is injective as well. □

Remark 3.2. Several presentations of the inner Tutte group of a matroid M have appeared in the literature (compare [9, Theorem 3], [11, Proposition 2.9 (i)], [6], [7] and [8]). The one provided by Gel'fand, Rybnikov and Stone in [9, Theorem 4] has a geometric interpretation in terms of the $U_2(4)$-minors of the matroid M. Starting from this point, if we compute the number g_M of generators of our presentation, we can see that

$$\frac{g_M - 1}{3} = \text{ the number of } U_2(4)\text{-minors of } M.$$

Lemma 3.1. *Given matroid M with circuit set $\mathfrak{C}$, let us consider $\mathbb{T}^{(2)}_M$-values*

$$\lceil C_{q_1}C_{q_2}|C_{q_3}C_{q_4}\rceil$$

defined for circuits C_{q_1}, C_{q_2}, C_{q_3}, $C_{q_4} \in \mathfrak{C}$ such that $\dim(C_{q_1} \cup C_{q_2} \cup C_{q_3} \cup C_{q_4}) = 1$ *and $\{C_{q_1}, C_{q_2}\} \cap \{C_{q_3}, C_{q_4}\} = \emptyset$ and fulfilling:*

(P1) $\lceil C_{q_1}C_{q_2}|C_{q_3}C_{q_3}\rceil = 1_{\mathbb{T}^{(2)}_M}$;

(P2) $\lceil C_{q_1}C_{q_2}|C_{q_3}C_{q_4}\rceil = \lceil C_{q_3}C_{q_4}|C_{q_1}C_{q_2}\rceil$;

(P3) $\lceil C_{q_1}C_{q_2}|C_{q_3}C_{q_4}\rceil = \lceil C_{q_1}C_{q_2}|C_{q_4}C_{q_3}\rceil^{-1}$.

Let us consider a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix} \in \mathcal{S}_4.$$

Then

$$\lceil C_{q_1}C_{q_2}|C_{q_3}C_{q_4}\rceil\lceil C_{q_1}C_{q_4}|C_{q_2}C_{q_3}\rceil\lceil C_{q_1}C_{q_3}|C_{q_4}C_{q_2}\rceil = \xi_M$$

is equivalent to

$$\lceil C_{q_{\sigma(1)}}C_{q_{\sigma(2)}}|C_{q_{\sigma(3)}}C_{q_{\sigma(4)}}\rceil \cdot \lceil C_{q_{\sigma(1)}}C_{q_{\sigma(4)}}|C_{q_{\sigma(2)}}C_{q_{\sigma(3)}}\rceil \cdot \lceil C_{q_{\sigma(1)}}C_{q_{\sigma(3)}}|C_{q_{\sigma(4)}}C_{q_{\sigma(2)}}\rceil = \xi_M.$$

Proof. Since $\mathcal{S}_4$ is generated by transpositions (12), (23), (34) it is enough to verify our statement for σ being such transposition, which can be done by a direct computation using (P1), (P2) and (P3). □

Lemma 3.2. *Let us consider* $\mathbb{T}_M^{(2)}$*-values*

$$\lceil C_{q_1} C_{q_2} | C_{q_3} C_{q_4} \rceil$$

as in the statement of Lemma 3.1. Let $\{C_{q_1}, \ldots, C_{q_9}\} \subseteq \mathfrak{C}$ *be circuits as in (R7), let* ι, μ, $\nu \in \mathcal{S}_9$ *be the permutations*

$$\iota = (12) \quad \mu = (23) \quad \nu = (47)(58)(69),$$

and let $\sigma \in \langle \iota, \mu, \nu \rangle$. *Then,*

$$\lceil C_{q_1} C_{q_2} | C_{q_6} C_{q_9} \rceil \lceil C_{q_2} C_{q_3} | C_{q_4} C_{q_7} \rceil \lceil C_{q_3} C_{q_1} | C_{q_5} C_{q_8} \rceil \ = \ 1_{\mathbb{T}_M^{(2)}}$$

is equivalent to

$$\begin{aligned} &\lceil C_{q_{\sigma(1)}} C_{q_{\sigma(2)}} | C_{q_{\sigma(\sigma^{-1}(3)+3)}} C_{q_{\sigma(\sigma^{-1}(3)+6)}} \rceil \\ \cdot &\lceil C_{q_{\sigma(2)}} C_{q_{\sigma(3)}} | C_{q_{\sigma(\sigma^{-1}(1)+3)}} C_{q_{\sigma(\sigma^{-1}(1)+6)}} \rceil \\ \cdot &\lceil C_{q_{\sigma(3)}} C_{q_{\sigma(1)}} | C_{q_{\sigma(\sigma^{-1}(2)+3)}} C_{q_{\sigma(\sigma^{-1}(2)+6)}} \rceil \ = \ 1_{\mathbb{T}_M^{(2)}}. \end{aligned}$$

Proof. The subgroup $\{\iota, \nu\} < \mathcal{S}_9$ is generated by (12), (23) and ν. Hence, it suffices to check the statement for these permutations. With (P2) and (P3), this can be easily done by direct computation. □

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