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Orientable $\mathbb{Z}_n$-distance magic graphs via products

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Abstract

Let Γ be an additive abelian group and $\overrightarrow{G} = (V, A)$ a directed graph, both of order n. A Γ labeling of $\overrightarrow{G}$ is a bijection $\ell\colon V \to \Gamma$. Given such a labeling ℓ, for each x in V, define $w(x)$ to be the sum of the labels on the vertices of tails of arcs with head x minus the sum of the labels on the vertices that are heads of arcs with tail x. If ℓ is a constant function, then ℓ is said to be a directed Γ-distance magic labeling for $\overrightarrow{G}$. A graph G is said to be orientable Γ-distance magic if there exists a directed graph $\overrightarrow{G}$ with underlying graph G and a directed Γ-distance magic labeling for $\overrightarrow{G}$. It has been conjectured that every $2r$-regular graph G of order n is orientable $\mathbb{Z}_n$-distance magic. In this paper we find orientable $\mathbb{Z}_n$-distance magic labelings of some products of graphs, namely the strong and lexicographic products. We provide orientable $\mathbb{Z}_n$-distance magic labelings for some classes of regular and non-regular graphs which arise via these products, and we identify some graphs which are not orientable $\mathbb{Z}_n$-distance magic.

1 Introduction

In this paper we study a generalization of distance magic graphs introduced recently in [2]. Let G be a simple, undirected graph on n vertices, and let f be a bijection $f : V(G) \to \{1, 2, \ldots, n\}$. For every vertex $x \in V(G)$, define the *weight* of x to be $w(x) = \sum_{y \in N(x)} f(y)$, where $N(x)$ is the set of vertices adjacent to x. If the weight of every vertex is equal to the same number k, then k is called the *magic constant*, and we say that f is a *distance magic labeling* of G. If such a labeling can be found, we say that G is *distance magic*. For a survey of distance magic graphs, see [1]. If one uses group elements as labels, the following generalization of distance magic labeling is possible.

Let Γ be an additive abelian group of order n, and let g be a bijection, $g : V(G) \to \Gamma$. If there exists $\mu \in \Gamma$ such that $w(x) = \sum_{y \in N(x)} g(y) = \mu$, for all vertices $x \in V(G)$, then we say G is Γ-*distance magic*. Clearly if G is distance magic, then it is also $\mathbb{Z}_n$-distance magic, but the converse is not necessarily true.

An analogous labeling in the setting of directed graphs was first introduced in [2]. Let $\overrightarrow{G} = (V, A)$ be a directed graph with underlying graph G. For a vertex x, let $N^+(x) = \{y \in V : \overrightarrow{yx} \in A\}$ and $N^-(x) = \{z \in V : \overrightarrow{xz} \in A\}$. Let $\text{indeg}(x) = |N^+(x)|$ and $\text{outdeg}(x) = |N^-(x)|$. Let ℓ be a bijection $\ell \colon V \to \Gamma$. For all $x \in V$, define the weight of x by

$$w(x) = \sum_{y \in N^+(x)} \ell(y) - \sum_{y \in N^-(x)} \ell(y),$$

where arithmetic takes place in Γ. If ℓ is a constant function, we say ℓ is a *directed* Γ-*distance magic labeling* of $\overrightarrow{G}$. A graph G is said to be *orientable* Γ-*distance magic* if there exists a directed graph $\overrightarrow{G}$ with underlying graph G and a directed Γ-distance magic labeling for $\overrightarrow{G}$.

The existence of orientable $\mathbb{Z}_n$-distance magic labelings for complete graphs, complete bipartite graphs (in combination with Theorem 5 from this paper), complete tripartite graphs, circulant graphs, and some products of graphs is established in [2]. They also showed that some graphs are not orientable $\mathbb{Z}_n$-distance magic and made a conjecture motivating our work.

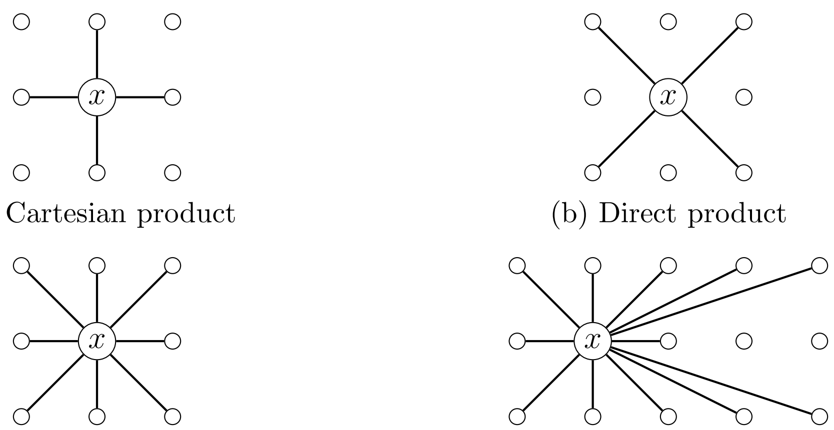
Theorem 1. [2] *Let G have order $n \equiv 2 \pmod 4$ and all vertices of odd degree. Then G is not orientable $\mathbb{Z}_n$-distance magic.*

Conjecture 2. [2] *If G is a $2r$-regular graph of order n, then G is orientable $\mathbb{Z}_n$-distance magic.*

Determining whether an arbitrary graph is orientable $\mathbb{Z}_n$-distance magic is not practical. One strategy for building classes of graphs that are more fruitful to study is to combine common families of graphs via graph products. The four graph products used in this paper are recalled in [3]. Each of the *Cartesian product* $G \square H$, the *direct* *product* $G \times H$, the *strong product* $G \boxtimes H$, and the *lexicographic product* $G \circ H$, is a graph with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \square H$ if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G;
- $G \times H$ if g is adjacent to g' in G and h is adjacent to h' in H;
- $G \boxtimes H$ if either $g = g'$ and h is adjacent with h' in H, or $h = h'$ and g is adjacent with g' in G, or g is adjacent with g' in G and h is adjacent with h' in H;
- $G \circ H$ if and only if either g is adjacent with g' in G or $g = g'$ and h is adjacent with h' in H.

If $V(G) = \{g_0, g_1, \ldots, g_{m-1}\}$ and $V(H) = \{h_0, h_1, \ldots, h_{n-1}\}$ for some m and n, respectively, we use the notation (i, j) to denote the vertex (g_i, h_j) in any of the above products. We also use (i, j) to refer to both the vertex and the label of the vertex. Since the labelings considered in this paper are bijections, this should cause no ambiguity. Observe that of the products defined above, only the lexicographic product is not necessarily commutative. The lexicographic product $G \circ H$ is sometimes called graph composition and denoted $G[H]$. The notation $G \times H$ is also commonly used to denote the Cartesian product, but we have reserved this notation for the direct product. The local structure of the product of two cycles motivates our choice of notation, as seen in Figure 1.



(a) Cartesian product

(b) Direct product

(c) Strong product

(d) Lexicographic product

Figure 1: Local structure of the product of two cycles

2 Lexicographic product

Our work stems from results in [2] on complete multipartite graphs.

Theorem 3. [2] *The complete graph K_n is orientable $\mathbb{Z}_n$-distance magic if and only if n is odd.*

Theorem 4. [2] *Let $G = K_{n_1,n_2}$ and $n_1 + n_2 = n$. If $n \not\equiv 2 \pmod 4$, then G is orientable $\mathbb{Z}_n$-distance magic.*

Combined with Theorem 4, the following theorem characterizes orientable $\mathbb{Z}_n$-distance magic complete bipartite graphs and more.

Theorem 5. *Let $n_1 + n_2 + \cdots + n_p = n$. If $n \equiv 2 \pmod 4$ and $p = 1$ or p is even, then $K_{n_1,n_2,\ldots,n_p}$ is not orientable $\mathbb{Z}_n$-distance magic.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_p}$. If $p = 1$, then $G \cong K_n$ is an odd regular graph on $n \equiv 2 \pmod 4$ vertices, so it is not orientable $\mathbb{Z}_n$-distance magic by Theorem 1. So assume p is even. For the sake of contradiction, suppose $n \equiv 2 \pmod 4$ and G is orientable $\mathbb{Z}_n$-distance magic with associated directed graph $\overrightarrow{G}$ and directed $\mathbb{Z}_n$-distance magic labeling $\ell : V(G) \to \mathbb{Z}_n$. Observe that $\mathbb{Z}_n \cong \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ by the Fundamental Theorem of Finite Abelian Groups since $\gcd\left(2, \frac{n}{2}\right) = 1$. Therefore, there exists a labeling $f : V(G) \to \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ and an element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ such that $w(v) = (a, b)$ for all $v \in V(G)$. For all $v \in V(G)$, let $f_2(v)$ represent the $\mathbb{Z}_2$ component of $f(v)$. Therefore

$$a = \sum_{y \in N^+(v)} f_2(y) - \sum_{y \in N^-(v)} f_2(y) = \sum_{y \in N(v)} f_2(y).$$

Since f is bijective, there exists an odd number of vertices $v \in V(G)$ such that $f_2(v) = 1$. Since p is even, it follows that there exists an odd number of partite sets A such that $\sum_{v \in A} f_2(v) = 1$ and an odd number of partite sets B such that $\sum_{v \in B} f_2(v) = 0$. But this leads to a contradiction since, for $x \in A$ and $y \in B$, $w(x) = (0, b) \neq (1, b) = w(y)$. □

Corollary 6. *Let $G = K_{n_1,n_2}$ be a complete bipartite graph such that $n_1 + n_2 = n$. Then G is orientable $\mathbb{Z}_n$-distance magic if and only if $n \not\equiv 2 \pmod 4$.*

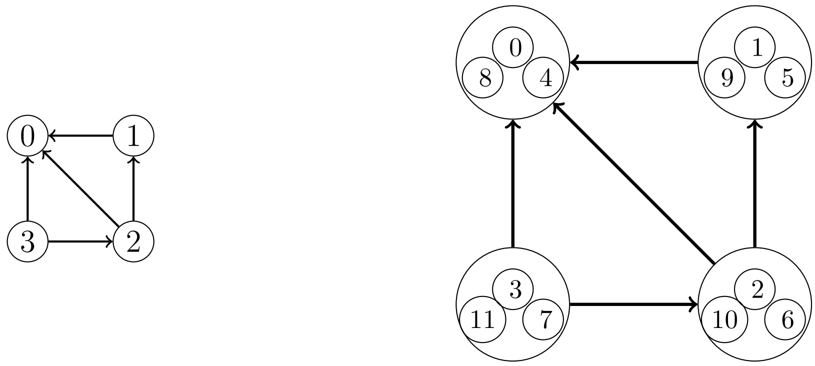
The next result serves as a cautionary tale for the lexicographic product. Let J_n denote the empty graph on n vertices.

Corollary 7. *Let $n = n_1 + n_2 \equiv 1, 3 \pmod 4$ and $k \equiv 2 \pmod 4$. Then K_{n_1,n_2} is orientable $\mathbb{Z}_n$-distance magic, but $K_{n_1,n_2} \circ J_k$ is not orientable $\mathbb{Z}_{nk}$-distance magic.*

Proof. The proof is clear since $K_{n_1,n_2} \circ J_k \cong K_{kn_1,kn_2}$ and $kn_1 + kn_2 \equiv 2 \pmod 4$. □

To conclude this section, we recall a theorem regarding distance magic labelings in [4] and prove an analagous theorem in the setting of directed graphs.

Theorem 8. [4] *If H is an r-regular graph, then $G = H \circ J_{2k}$ is distance magic for any k.*



(a) Directed $\mathbb{Z}_4$-distance magic labeling of a graph H

(b) Directed $\mathbb{Z}_{12}$-distance magic labeling of the graph $G = H \circ \overline{K_3}$

Figure 2: Illustration of Theorem 9

Figure 2 illustrates the following theorem. In Figure 2b, each bold arc represents the edges in a directed $K_{3,3}$.

Theorem 9. *If H is an orientable $\mathbb{Z}_n$-distance magic graph of order n, then the lexicographic product $G = H \circ J_k$ is orientable $\mathbb{Z}_{nk}$-distance magic except possibly when $k \equiv 2 \pmod 4$ and H contains a vertex x such that* $\text{indeg}(x) \not\equiv \text{outdeg}(x) \pmod 2$.

Proof. Let $\overrightarrow{H}$ have directed $\mathbb{Z}_n$-distance magic labeling $f : V(H) \to \mathbb{Z}_n$ with magic constant μ. Construct the graph $G = H \circ J_k$ with vertex set $V(G) = \{(i,j) : i \in V(H),\ j = 1, 2, \dots, k\}$ by replacing each vertex i of H with k isolated vertices such that two of the new vertices are adjacent whenever their counterparts in H are adjacent. We will orient the edges of G later. For all $i \in V(H)$, let B_i represent the set of k vertices which have replaced the vertex i. We will now label the vertices in each set B_i. For $i = 0, 1, \dots, n-1$, define cosets $A_i = \{i + \langle n \rangle\} \subseteq \mathbb{Z}_{nk}$, where $\langle n \rangle$ is the subgroup generated by n. For all $i = 0, 1, \dots, n-1$, let $\ell : A_i \to B_i$ be an arbitrary bijection.

Case 1. *k odd or* $\text{indeg}(x) \not\equiv \text{outdeg}(x) \pmod 2$ *for all $x \in V(H)$.*

Orient the edges of G so that for $(i,j) \in B_i$ and $(p,q) \in B_p$, each edge $(i,j)(p,q)$ has the same orientation as its counterpart ip in $\overrightarrow{H}$. Let

$$\begin{aligned} S_i &= \sum_{x \in B_i} \ell(x) \\ &= \sum_{a \in A_i} a \\ &= i + (n+i) + \cdots + ((k-1)n + i) \\ &= \tfrac{k[2i+(k-1)n]}{2}, \end{aligned}$$

with all arithmetic performed modulo nk. Let $x \in B_i$ and let $N_H^+(i) = \{a_1, \dots, a_p\}$ and $N_H^-(i) = \{b_1, \dots, b_q\}$ where $p = \text{indeg}(i)$ and $q = \text{outdeg}(i)$. If $\sum_{i=1}^{p} a_i = a$

and $\sum_{i=1}^{q} b_i = b$, then $a - b = \mu$, (with all arithmetic performed in $\mathbb{Z}_n$) since H is orientable $\mathbb{Z}_n$-distance magic. Recalling that k is odd or $p \equiv q \pmod 2$, we have

$$\begin{array}{rcl} w(x) & = & \sum_{i=1}^{p} S_{a_i} - \sum_{i=1}^{q} S_{b_i} \\ & = & \frac{k[2(a_1+\cdots+a_p)+p(k-1)n]}{2} - \frac{k[2(b_1+\cdots+b_q)+q(k-1)n]}{2} \\ & = & S_{a_1+\cdots+a_p} + \frac{k(k-1)(p-1)n}{2} - S_{b_1+\cdots+b_q} - \frac{k(k-1)(q-1)n}{2} \\ & = & S_a - S_b + \frac{(k-1)(p-q)}{2} nk \\ & \equiv & S_a - S_b \pmod{nk} \\ & \equiv & k(a-b) \pmod{nk} \\ & \equiv & k\mu \pmod{nk}, \end{array}$$

which shows ℓ is a directed $\mathbb{Z}_{nk}$-distance magic labeling of G.

Case 2. $k \equiv 0 \pmod 4$.

Notice that every vertex in B_i can be expressed uniquely as $i + tn$ for some $t \in \{0, 1, \ldots, k-1\}$. For every edge $ij \in E(H)$, orient the edges in G between B_i and B_j as follows. For all $a, b \in [k]$, orient the edges between $i + an \in B_i$ and $j + bn \in B_j$ such that, if $a \equiv 0, 3 \pmod 4$, then

$$\begin{array}{rcl} N_G^+(i+an) & = & \{j + bn : b \equiv 0, 3 \pmod 4\}, \\ N_G^-(i+an) & = & \{j + bn : b \equiv 1, 2 \pmod 4\}, \end{array}$$

and, if $a \equiv 1, 2 \pmod 4$, then

$$\begin{array}{rcl} N_G^+(i+an) & = & \{j + bn : b \equiv 1, 2 \pmod 4\}, \\ N_G^-(i+an) & = & \{j + bn : b \equiv 0, 3 \pmod 4\}. \end{array}$$

Let $i + an \in B_i$ for some $a \in [k]$ and let $ij \in E(H)$. Let $w_{ij}(i + an)$ be the weight of $i + an$ in $\overrightarrow{G}$ induced by the edge $ij \in E(H)$. If $a \equiv 0, 3 \pmod 4$, then

$$\begin{array}{rcl} w_{ij}(i+an) & = & \sum\limits_{b\equiv 0,3 \pmod 4} (j+bn) - \sum\limits_{b\equiv 1,2 \pmod 4} (j+bn) \\ & = & [(0n+3n) - (1n+2n)] + \cdots + [(4n+7n) - (5n+6n)] \\ & & + [((k-4)n + (k-1)n) - ((k-3)n + (k-2)n)] \\ & = & 0. \end{array}$$

If $a \equiv 1, 2 \pmod 4$, a similar calculation shows $w_{ij}(i+an) = 0$. Therefore, each edge ij in H induces 0 weight in $\overrightarrow{G}$, so the graph G is orientable $\mathbb{Z}_{nk}$-distance magic. □

3 Strong product of cycles

Throughout this section, let $m, n \geq 3$. Assume $V(C_m) = \{g_0, g_1, \ldots, g_{m-1}\}$ and $V(C_n) = \{h_0, h_1, \ldots, h_{n-1}\}$. The strong product of two cycles $C_m \boxtimes C_n$ is an 8-regular graph on $V = \{(i, j) : i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1, \ldots, n-1\}\}$ which is the union of the direct product $C_m \times C_n$ and the Cartesian product $C_m \square C_n$. Orient $C_m \boxtimes C_n$ so that $N^+(i, j) = \{(i+1, j), (i+1, j+1), (i, j+1), (i-1, j+1)\}$ and $N^-(i, j) = \{(i-1, j), (i-1, j-1), (i, j-1), (i+1, j-1)\}$ where the arithmetic is taken modulo m in the first coordinate and modulo n in the second coordinate.

Theorem 10. *If* $\gcd(m,n) = 1, 2,$ *or* 4*, then the strong product* $C_m \boxtimes C_n$ *is orientable* $\mathbb{Z}_{mn}$*-distance magic.*

Proof. Assume $m \leq n$ and let a *diagonal* of $C_m \boxtimes C_n$ be the sequence of $\operatorname{lcm}(m,n)$ vertices $(i,0),(i+1,1),\ldots,(m-1,m-1),(0,m),(1,m+1),\ldots,(i-1,n-1)$. The number of diagonals is $g = \gcd(m,n)$. For $i = 0,1,\ldots,g-1$ let $D_i = (d_{(0,i)}, d_{(1,i)}, \ldots, d_{(l-1,i)})$ be the i^{th} diagonal. Similarly, a *back diagonal* is the sequence of vertices $(i,0),(i-1,1),\ldots,(0,i),(m-1,i+1),\ldots,(i+1,n-1)$ and denote the i^{th} by $B_i = (b_{(0,i)}, b_{(1,i)}, \ldots, b_{(l-1,i)})$.

Let $H \cong \langle g \rangle$, the subgroup of $\mathbb{Z}_{mn}$ generated by g. Define $\ell\colon V(C_m \boxtimes C_n) \to \mathbb{Z}_{mn}$ by labeling the vertices of the diagonal D_i with the elements of the coset $H + i$ in increasing order for $i = 0,1,\ldots,g-1$. Write $n = km + r$ for $0 \leq r \leq m-1$.

If $g = 1$ or 2 it must be the case that $b_{(1,i)} = d_{(h,i)}$ for some h. Counting steps through the lattice, it is not difficult to see that when $g = 1$, then $h = c(km+r)+1$ where c is the unique number such that $c \in \{1,2,\ldots,m-1\}$ and $cr \equiv -2 \pmod{m}$ and when $g = 2$, $h = \frac{(m-2)n}{r}+1$. Therefore the two sequences, $(b_{(0,i)}, b_{(1,i)}, \ldots, b_{(l-1,i)})$ and $(d_{(0,i)}, d_{(h,i)}, d_{(2h,i)}, \ldots)$ are equal since $|B_i| = |D_i|$. Notice that for any vertex $(i,j) = d_{(a,t)}$ on D_t, we have $N^+(i,j) = \left\{d_{(a+1,t)}, b_{(c+2,t)}, d_{(p+1,t+1)}, d_{(q+1,t+1)}\right\}$ and $N^-(i,j) = \left\{d_{(a-1,t)}, b_{(c,t)}, d_{(p,t+1)}, d_{(q,t+1)}\right\}$ for some numbers c, p, q, while $t+1$ is performed modulo g. Therefore

$$\begin{aligned}
w(i,j) &= (d_{(a+1,t)} - d_{(a-1,t)}) + (d_{(p+1,t+1)} - d_{(p,t+1)}) \\
&\quad + (d_{(q+1,t+1)} - d_{(q,t+1)}) + (b_{(c+2,t)} - b_{(c,t)}) \\
&= 2g + g + g + 2gh \\
&= 2g(2g+h).
\end{aligned}$$

If $g = 4$, the graph contains exactly four diagonals, so $b_{(2,i)} = d_{(h,i)}$ for some h. Counting steps through the lattice, we obtain $h = \frac{(m-4)n}{r} + 2$. For any vertex $(i,j) = d_{(a,t)}$ on D_t, we have $N^+(i,j) = \left\{d_{(a+1,t)}, b_{(c+2,t')}, d_{(p+1,t+1)}, d_{(q+1,t-1)}\right\}$ and $N^-(i,j) = \left\{d_{(a-1,t)}, b_{(c,t')}, d_{(p,t+1)}, d_{(q,t-1)}\right\}$ for some numbers t', c, p, q, and $t+1$ and $t-1$ are performed modulo 4. Observing $b_{(c,i)} - b_{(c-2,i)} = gh$, we obtain

$$\begin{aligned}
w(i,j) &= (d_{(a+1,t)} - d_{(a-1,t)}) + (d_{(p+1,t+1)} - d_{(p,t+1)}) \\
&\quad + (d_{(q+1,t-1)} - d_{(q,t-1)}) + (b_{(c+2,t')} - b_{(c,t')}) \\
&= 2g + g + g + gh \\
&= g(4+h).
\end{aligned}$$

Since h is independent of i and j in all cases, we have proven the result. ☐

Theorem 11. *Let* $d = \gcd(m,n) = 3, 5,$ *or* 6*. If* $d^2 \nmid m$ *and* $d^2 \nmid n$*, then* $C_m \boxtimes C_n$ *is orientable* $\mathbb{Z}_{mn}$*-distance magic.*

Proof. Let $\gcd(m,n) = d \in \{3,5,6\}$, $d^2 \nmid m$, and $d^2 \nmid n$. Therefore $\gcd(\frac{m}{d}, d) = \gcd(\frac{n}{d}, d) = 1$ which implies $\gcd(\frac{m}{d}, \frac{n}{d}) = \gcd(\frac{m}{d}, d^2) = \gcd(\frac{n}{d}, d^2) = 1$ and $\mathbb{Z}_{mn} \cong \mathbb{Z}_{d^2} \times \mathbb{Z}_{\frac{m}{d}} \times \mathbb{Z}_{\frac{n}{d}}$. Let $r(i)$ and $r(j)$ represent the unique elements of $\mathbb{Z}_{d^2}$ congruent to i and j, respectively, modulo d.

Define $\ell : V\left(C_m \boxtimes C_n\right) \to \mathbb{Z}_{d^2} \times \mathbb{Z}_{\frac{m}{d}} \times \mathbb{Z}_{\frac{n}{d}}$ by

$$\ell(i,j) = (\alpha_d, \beta, \gamma),$$

where $0 \le \beta < \frac{m}{d}$, $\beta \equiv i \pmod{\frac{m}{d}}$, $0 \le \gamma < \frac{n}{d}$, $\gamma \equiv j \pmod{\frac{n}{d}}$, and where α_d is defined as follows. If $d = 3$, let $\alpha_3 = 3r(i) + r(j)$. If $d = 5$, let $\alpha_5 = 5r(j) + r(i-2j)$. If $d = 6$, let

$$\alpha_6 = \begin{cases} 6r(i) + 2r(j),\ i & \text{even} \\ 6r(i-1) + 2r(j) + 1,\ i\&\text{odd} \end{cases}$$

where the arithmetic is performed modulo d^2.

To show that ℓ is injective, we have $\ell(i,j) = (\alpha_d, \beta, \gamma) = (\alpha'_d, \beta', \gamma') = \ell(i', j')$ if and only if $\alpha_d \equiv \alpha'_d \pmod{d^2}$ and $\beta \equiv \beta' \pmod{\frac{m}{d}}$ and $\gamma \equiv \gamma' \pmod{\frac{n}{d}}$. Consequently, $i \equiv i' \pmod{\frac{m}{d}}$ and $j \equiv j' \pmod{\frac{n}{d}}$. If $d = 3$, then $\alpha_3 \equiv \alpha'_3 \pmod{3^2}$ implies $r(j) = r(j')$ so $j \equiv j' \pmod 3$. Since $\gcd(3, \frac{n}{3}) = 1$ by assumption, we have $j \equiv j' \pmod n$, and hence $j = j'$. It follows easily that $i = i'$. Therefore ℓ is injective, and hence bijective when $d = 3$. A similar argument can be made to show ℓ is bijective when $d = 5$ or 6.

Let $(i,j) \in V(C_m \boxtimes C_n)$. We calculate $w(i,j)$ component-wise. Let

$$w(i,j) \;=\; (w_1, w_2, w_3),$$

where $w_1 \in \mathbb{Z}_{d^2}$, $w_2 \in \mathbb{Z}_{\frac{m}{d}}$, and $w_3 \in \mathbb{Z}_{\frac{n}{d}}$. First we determine w_1 for each $d \in \{3, 5, 6\}$. Consequently, the arithmetic will be performed in $\mathbb{Z}_{d^2}$. The full computation is shown only in the case of $d = 3$, where

$$\begin{aligned} w_1 &= [3r(i+1) + r(j)] + [3(r(i+1)) + (r(j+1))] \\ &\quad + [3r(i) + (r(j+1))] + [3(r(i-1)) + (r(j+1))] \\ &\quad - [3(r(i-1)) + r(j)] - [3(r(i+1)) + (r(j-1))] \\ &\quad - [3r(i) + (r(j-1))] - [3(r(i+1)) + (r(j-1))] \\ &= 3. \end{aligned}$$

If $d = 5$, we obtain $w_1 = 5$, and for $d = 6$, we have $w_1 = 24$. Next we calculate $w_2 \in \mathbb{Z}_{\frac{m}{d}}$ and $w_3 \in \mathbb{Z}_{\frac{n}{d}}$. We have

$$\begin{aligned} w_2 &= 2(i-1) + i + (i+1) - 2(i+1) - i - i + 1 \\ &= -2, \end{aligned}$$

and

$$\begin{aligned} w_3 &= j + 3(j+1) - j - 3(j-1) \\ &= 6. \end{aligned}$$

Hence $w(i,j) = (w_1, -2, 6)$, proving the theorem. □

Theorem 12. *If $mn \equiv 2 \pmod 4$, then $C_m \boxtimes C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic.*

Proof. If $mn \equiv 2 \pmod 4$, then 2 divides exactly one of m or n. Without loss of generality, we may assume $2 \mid m$. By the Fundamental Theorem of Finite Abelian Groups, we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_2 \times \mathbb{Z}_{\frac{mn}{2}}$. Define $\ell\colon V(C_m \boxtimes C_n) \to \mathbb{Z}_2 \times \mathbb{Z}_{\frac{mn}{2}}$ by

$$\ell(i,j) = \begin{cases} \left(0, \frac{i}{2}n + j\right), i & \text{even} \\ \left(1, \frac{i-1}{2}n + j\right), i & \text{odd} \end{cases}.$$

Clearly ℓ is a bijection. If i is even we have,

$$\begin{aligned} w(i,j) &= \left(1, \tfrac{i}{2}n + j\right) + \left(1, \tfrac{i}{2}n + j + 1\right) + \left(1, \tfrac{i}{2}n + j\right) \\ &\quad + \left(1, \tfrac{i}{2}n + j + 1\right) - \left(1, \tfrac{i-2}{2}n + j\right) - \left(1, \tfrac{i-2}{2}n + j - 1\right) \\ &\quad - \left(0, \tfrac{i}{2}n + j - 1\right) - \left(1, \tfrac{i}{2}n + j - 1\right) \\ &= (0, n+6). \end{aligned}$$

If i is odd then,

$$\begin{aligned} w(i,j) &= \left(0, \tfrac{i+1}{2}n + j\right) + \left(0, \tfrac{i+1}{2}n + j + 1\right) + \left(1, \tfrac{i-1}{2}n + j + 1\right) \\ &\quad + \left(1, \tfrac{i-1}{2}n + j + 1\right) - \left(0, \tfrac{i-1}{2}n + j\right) - \left(0, \tfrac{i-1}{2}n + j - 1\right) \\ &\quad - \left(1, \tfrac{i-1}{2}n + j - 1\right) - \left(0, \tfrac{i+1}{2}n + j - 1\right) \\ &= (0, n+6). \end{aligned}$$

□

Our final theorem is a partial result on what can be said of $C_m \boxtimes C_n$ if $mn \equiv 0 \pmod 4$.

Theorem 13. *If $m \equiv n \equiv 2 \pmod 4$, then $C_m \boxtimes C_n$ is orientable $\mathbb{Z}_{mn}$-distance magic.*

Proof. Since $\frac{mn}{4}$ is odd we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_4 \times \mathbb{Z}_{\frac{mn}{4}}$. Define $\ell\colon V(C_m \boxtimes C_n) \to \mathbb{Z}_4 \times \mathbb{Z}_{\frac{mn}{4}}$ by

$$\ell(i,j) = \begin{cases} (0, \frac{in}{4} + j), & i \text{ even}, & j \text{ even} \\ (1, \frac{in}{4} + j), & i \text{ even}, & j \text{ odd} \\ (2, \frac{(i-1)n}{4} + j), & i \text{ odd}, & j \text{ even} \\ (3, \frac{(i-1)n}{4} + j), & i \text{ odd}, & j \text{ odd} \end{cases}.$$

Clearly, ℓ is a bijection and we proceed to determine the weights. For all $(i,j) \in V(C_m \boxtimes C_n)$, let $w(i,j) = (w_1, w_2)$ where $w_1 \in \mathbb{Z}_4$ and $w_2 \in Z_{\frac{mn}{4}}$. It is easy to check that $w_1 = 0$. If i is even,

$$\begin{aligned} w_2 &= \left(\tfrac{i}{2}\tfrac{n}{2} + j\right) + \left(\tfrac{i}{2}\tfrac{n}{2} + j + 1\right) + \left(\tfrac{i}{2}\tfrac{n}{2} + j + 1\right) + \left(\tfrac{i-2}{2}\tfrac{n}{2} + j + 1\right) \\ &\quad - \left[\left(\tfrac{i-2}{2}\tfrac{n}{2} + j\right) + \left(\tfrac{i-2}{2}\tfrac{n}{2} + j - 1\right) + \left(\tfrac{i}{2}\tfrac{n}{2} + j - 1\right) + \left(\tfrac{i}{2}\tfrac{n}{2} + j - 1\right)\right] \\ &= \tfrac{n}{2} + 6. \end{aligned}$$

If i is odd,

$$\begin{aligned} w_2 &= \left(\tfrac{i+1}{2}\tfrac{n}{2} + j\right) + \left(\tfrac{i+1}{2}\tfrac{n}{2} + j + 1\right) + \left(\tfrac{i-1}{2}\tfrac{n}{2} + j + 1\right) + \left(\tfrac{i-1}{2}\tfrac{n}{2} + j + 1\right) \\ &\quad - \left[\left(\tfrac{i-1}{2}\tfrac{n}{2} + j\right) + \left(\tfrac{i-1}{2}\tfrac{n}{2} + j - 1\right) + \left(\tfrac{i-1}{2}\tfrac{n}{2} + j - 1\right) + \left(\tfrac{i+1}{2}\tfrac{n}{2} + j - 1\right)\right] \\ &= \tfrac{n}{2} + 6. \end{aligned}$$

Hence $w(i,j) = \left(0, \frac{n}{2} + 6\right)$, proving the theorem. □

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