AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **70(3)** (2018), Pages 336–339

On star-packings having a large matching

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Abstract

Let G be a graph, and let $f : V(G) \to \{2, 3, \dots\}$ be a function. A family $\mathcal{P}$ of vertex-disjoint subgraphs of G is an f-star-packing if each element of $\mathcal{P}$ is a star of order at least 2 and for $x \in \bigcup_{P\in\mathcal{P}} V(P)$, the degree of x in the graph $\bigcup_{P\in\mathcal{P}} P$ is at most $f(x)$. In this paper we prove that G has a maximum f-star-packing $\mathcal{P}$ such that $|\mathcal{P}|$ is equal to the matching number of G. As an application of our result, we show a corollary concerning a bound on the number of components of order 2 in a path-factor.

1 Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For terms and symbols not defined here, we refer the reader to [4].

A family $\mathcal{P}$ of vertex-disjoint connected subgraphs of G is called a *packing*. We let $V(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} V(P)$ and $E(\mathcal{P}) = \bigcup_{P\in\mathcal{P}} E(P)$. For each $x \in V(\mathcal{P})$, let $d_{\mathcal{P}}(x)$ denote the degree of x in the graph $\bigcup_{P\in\mathcal{P}} P$. A packing $\mathcal{P}$ of G is *perfect* if $V(\mathcal{P}) = V(G)$. A packing $\mathcal{P}$ of G is called a *matching* if each element of $\mathcal{P}$ is a complete graph of order 2. For a function $f : V(G) \to \{2, 3, \dots\}$, a packing $\mathcal{P}$ is called an *f-star-packing* if each element of $\mathcal{P}$ is a star and $1 \le d_{\mathcal{P}}(x) \le f(x)$ for all $x \in V(\mathcal{P})$. A matching $\mathcal{M}$ (resp. an f-star-packing $\mathcal{P}$) of G is *maximum* if there is no matching $\mathcal{M}'$ (resp. no f-star-packing $\mathcal{P}'$) of G with $|V(\mathcal{M}')| > |V(\mathcal{M})|$ (resp. $|V(\mathcal{P}')| > |V(\mathcal{P})|$). The cardinality of a maximum matching of G, denoted by $\alpha'(G)$, is called the *matching number* of G.

Note that a matching of a graph G is an f-star-packing for any function $f : V(G) \to \{2, 3, \ldots\}$. Note also that for an f-star-packing $\mathcal{P}$, since edges from distinct elements of $\mathcal{P}$ form a matching, we have $|\mathcal{P}| \leq \alpha'(G)$. Thus it is natural to seek for a maximum f-star packing $\mathcal{P}$ with $|\mathcal{P}| = \alpha'(G)$. In other words, we are interested in the existence problem of a maximum f-star packing containing a maximum matching. Our main result is the following.

Theorem 1.1 *Let G be a graph, and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then G has a maximum f-star-packing $\mathcal{P}$ with $|\mathcal{P}| = \alpha'(G)$.*

Now we consider a special kind of f-star-packing. A perfect f-star-packing of a graph G is called a *path-factor* if $f(x) = 2$ for all $x \in V(G)$. Note that each element of a path-factor is a path of order 2 or 3. A min-max theorem concerning an f-star-packing is known (see Theorem 7.9 in [2]). In particular, a necessary and sufficient condition for the existence of a path-factor is given as follows (here $i(G)$ denotes the number of isolated vertices of a graph G):

Theorem A (Akiyama, Avis and Era [1]) *A graph G has a path-factor if and only if $i(G - S) \leq 2|S|$ for all $S \subseteq V(G)$.*

Berge [3] gave the following theorem concerning a maximum matching (here odd(G) denotes the number of components having odd order of a graph G).

Theorem B (Berge [3]) *Let G be a graph, and let α be a real number with $0 \leq \alpha \leq \frac{|V(G)|}{2}$. Then $\alpha'(G) \geq \alpha$ if and only if odd$(G - S) \leq |S| + |V(G)| - 2\alpha$ for all $S \subseteq V(G)$.*

By Theorems 1.1, A and B, we obtain the following corollary concerning the existence of a path-factor which contains at least as many components of order 2 as required.

Corollary 1.2 *Let G be a graph, and let t be a real number with $0 \leq t \leq \frac{|V(G)|}{2}$. Then G has a path-factor $\mathcal{P}$ such that the number of elements of order 2 is at least t if and only if $i(G - S) \leq 2|S|$ and odd$(G - S) \leq |S| + \frac{|V(G)| - 2t}{3}$ for all $S \subseteq V(G)$.*

2 Proof of Theorem 1.1

Let $\mathcal{M}$ be a maximum matching of G, and let $\mathcal{P}$ be a maximum f-star-packing of G. We choose $\mathcal{M}$ and $\mathcal{P}$ so that

(P1) $|E(\mathcal{P}) \cap E(\mathcal{M})|$ is as large as possible.

Set $\mathcal{P}_1 = \{P \in \mathcal{P} : |V(P)| \geq 3\}$ and $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$. Let $Z = \{x \in V(\mathcal{P}) : d_{\mathcal{P}}(x) \geq 2\}$. Note that $Z \subseteq V(\mathcal{P}_1)$. Let M_1 be the set of edges in $E(\mathcal{M})$ incident with a vertex in Z, and let $M_2 = E(\mathcal{M}) - M_1$. Let H be the subgraph of G induced by the set $(M_2 - E(\mathcal{P}_2)) \cup (E(\mathcal{P}_2) - M_2)$.

Claim 2.1 *For each component C of H, we have $|E(C) \cap M_2| \leq |E(C) \cap E(\mathcal{P}_2)|$.*

Proof. Since M_2 and $E(\mathcal{P}_2)$ are sets of independent edges of G, C is a path or a cycle. By way of contradiction, we suppose that $|E(C) \cap M_2| > |E(C) \cap E(\mathcal{P}_2)|$. It follows that C is a path of even order and, if we write $C = u_1u_2 \cdots u_{2m}$ ($m \geq 1$), then $u_{2i-1}u_{2i} \in M_2$ ($1 \leq i \leq m$) and $u_{2i}u_{2i+1} \in E(\mathcal{P}_2)$ ($1 \leq i \leq m-1$). Furthermore, $u_1, u_{2m} \in (V(\mathcal{P}_1) - Z) \cup (V(G) - V(\mathcal{P}))$. Let P^i be the path $u_{2i}u_{2i+1}$ for each i ($1 \leq i \leq m-1$), and let Q^i be the path $u_{2i-1}u_{2i}$ for each i ($1 \leq i \leq m$). Note that $P^i \in \mathcal{P}_2$ and $E(\mathcal{P}) \cap (\bigcup_{1 \leq i \leq m} E(Q^i)) = \emptyset$.

We first suppose that $\{u_1, u_{2m}\} \cap (V(G) - V(\mathcal{P})) \neq \emptyset$. If $\{u_1, u_{2m}\} \subseteq V(G) - V(\mathcal{P})$, then $\mathcal{Q}_1 = (\mathcal{P} - \{P^1, \ldots, P^{m-1}\}) \cup \{Q^1, \ldots, Q^m\}$ is an f-star-packing of G with $|V(\mathcal{Q}_1)| > |V(\mathcal{P})|$, which contradicts the maximality of $\mathcal{P}$. Thus, without loss of generality, we may assume that u_1 belongs to an element R of $\mathcal{P}_1$. Then $\mathcal{Q}_2 = (\mathcal{P} - \{R, P^1, \ldots, P^{m-1}\}) \cup \{R - u_1, Q^1, \ldots, Q^m\}$ is an f-star-packing of G with $|V(\mathcal{Q}_2)| > |V(\mathcal{P})|$, which contradicts the maximality of $\mathcal{P}$. Consequently, $\{u_1, u_{2m}\} \subseteq V(\mathcal{P}_1) - Z$.

For $i \in \{1, 2m\}$, let R^i be the element of $\mathcal{P}_1$ containing u_i. If $R^1 \neq R^{2m}$, let $\mathcal{Q}_3 = (\mathcal{P} - \{R^1, R^{2m}, P^1, \ldots, P^{m-1}\}) \cup \{R^1 - u_1, R^{2m} - u_{2m}, Q^1, \ldots, Q^m\}$; if $R^1 = R^{2m}$ and $|V(R^1)| \geq 4$, let $\mathcal{Q}_3 = (\mathcal{P} - \{R^1, P^1, \ldots, P^{m-1}\}) \cup \{R^1 - \{u_1, u_{2m}\}, Q^1, \ldots, Q^m\}$; if $R^1 = R^{2m}$ and $|V(R^1)| = 3$, let $\mathcal{Q}_3 = (\mathcal{P} - \{R^1, P^1, \ldots, P^{m-1}\}) \cup \{vu_1u_2, Q^2, \ldots, Q^m\}$ where v is the vertex in $Z \cap V(R^1)$. In each case, $\mathcal{Q}_3$ is an f-star-packing of G with $|V(\mathcal{Q}_3)| = |V(\mathcal{P})|$ and $|E(\mathcal{Q}_3) \cap E(\mathcal{M})| > |E(\mathcal{P}) \cap E(\mathcal{M})|$, which contradicts (P1) (note that this argument works even if $m = 1$). $\square$

It follows from Claim 2.1 that $|M_2| = \sum_C |E(C) \cap M_2| + |M_2 \cap E(\mathcal{P}_2)| \leq \sum_C |E(C) \cap E(\mathcal{P}_2)| + |M_2 \cap E(\mathcal{P}_2)| = |\mathcal{P}_2|$, where C runs over all components of H. Furthermore, we have $|M_1| \leq |Z| = |\mathcal{P}_1|$. Consequently,

$$|\mathcal{P}| = |\mathcal{P}_1| + |\mathcal{P}_2| \geq |M_1| + |M_2| = |\mathcal{M}| = \alpha'(G).$$

As we mentioned before the statement of Theorem 1.1, we have $|\mathcal{P}| \leq \alpha'(G)$. Therefore, $|\mathcal{P}| = \alpha'(G)$.

Acknowledgment

This work was supported by JSPS KAKENHI Grant number 26800086 (to M.F.).

References

[1] J. Akiyama, D. Avis and H. Era, On a $\{1,2\}$-factor of a graph, TRU Math. **16** (1980), 97–102.

[2] J. Akiyama and M. Kano, *Factors and factorizations of graphs*, Lecture Notes in Mathematics **2031**, Springer, 2010.

[3] C. Berge, Sur le couplage maximum d'un graphe, C. R. Acad. Sci. Paris **247** (1958), 258–259.

[4] R. Diestel, *Graph Theory (4th edition)*, Graduate Texts in Mathematics **173**, Springer, 2010.

(Received 17 May 2017)