AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **71(3)** (2018), Pages 403–426

Symmetric graphs with complete quotients*

A. Gardiner

13, Bincleaves Road
Weymouth DT4 8RL
United Kingdom

Cheryl E. Praeger

School of Mathematics and Statistics
University of Western Australia
Crawley, WA 6009
Australia

Dedicated with affection and admiration to the memory of our friend and colleague Anne Penfold Street

Abstract

Let Γ be a G-symmetric graph with vertex set V. We suppose that V admits a G-invariant partition $\mathcal{B} = \{B = B_0, B_1, \ldots, B_b\}$, with parts B_i of size v, and that the quotient graph $\Gamma_{\mathcal{B}}$ induced on $\mathcal{B}$ is a complete graph K_{b+1}. Then, for each pair of suffices i, j $(i \neq j)$, the graph $\langle B_i, B_j \rangle$ induced on $B_i \cup B_j$ is bipartite with each vertex of valency 0 or t (a constant). When $t = 1$, it was shown earlier how a flag-transitive 1-design $\mathcal{D}(B)$ induced on the part B can sometimes be used to classify possible triples $(\Gamma, G, \mathcal{B})$. Here we extend these ideas to $t \geq 1$ and prove that, if $G(B)^B$ is 2-transitive and the blocks of $\mathcal{D}(B)$ have size less than v, then either (i) $v < b$, or (ii) the triple $(\Gamma, G, \mathcal{B})$ is known explicitly.

1 Introduction

A graph Γ is G-*symmetric* if $G \leq \mathrm{Aut}\,\Gamma$ acts transitively on the vertices of Γ, and the stabiliser $G(x)$ of a vertex x acts transitively on the edges incident with x. For vertex-transitive graphs, these properties are equivalent to requiring that G act transitively on the *arcs* of Γ, that is, the ordered pairs of adjacent vertices in Γ, and for this reason G-symmetric graphs are also called G-arc-transitive graphs. Let Γ be a G-symmetric graph with vertex set V, and let $\mathcal{B} = \{B = B_0, B_1, B_2, \ldots\}$ be a

* The research for this project was partially supported by an Australian Research Council large grant.

G-invariant partition of V. Then we obtain a natural quotient graph $\Gamma_{\mathcal{B}}$ with vertex set $\mathcal{B}$ (where two parts $B_i, B_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ precisely when they are joined by some edge of Γ). In seeking to analyse Γ in terms of the G-invariant partition $\mathcal{B}$ and the quotient graph $\Gamma_{\mathcal{B}}$, it is natural to exclude the case where $\Gamma_{\mathcal{B}}$ is a null graph.

To investigate such triples $(\Gamma, G, \mathcal{B})$, a framework [6] was introduced by the authors in 1995 which made use also of two other associated combinatorial structures: namely a 1-design $\mathcal{D}(B)$ induced on a part $B \in \mathcal{B}$, and the (bipartite) restriction $\langle B_i, B_j \rangle$ of Γ to the union $B_i \cup B_j$ of adjacent parts B_i, B_j of $\Gamma_{\mathcal{B}}$. The subgraph $\langle B_i, B_j \rangle$ has edges involving k vertices of each of B_i, B_j, with k independent of i, j. The paper [6] studies various properties of these graphs, and then focuses on the situations where $\Gamma_{\mathcal{B}}$ is a complete graph or a cycle, and the cases where k is $1, 2$, $v - 1$ or v, where $v = |B_i|$. Many of these special cases, and also the case $k = v - 2$ (see [15, 18]), are explored further in the literature, especially by Sanming Zhou and his co-authors. In particular they classified all triples $(\Gamma, G, \mathcal{B})$ such that $\Gamma_{\mathcal{B}}$ is a complete graph and $k = v - 1$, a culmination of work in [2, 4, 10, 19]; and they studied more general quotients $\Gamma_{\mathcal{B}}$, especially the case where G acts 2-arc transitively on $\Gamma_{\mathcal{B}}$, see [12, 14, 16, 20, 21, 22].

This paper was motivated by studies initiated in [6], and developed further in [7, 9], of the case where $\Gamma_{\mathcal{B}}$ is a complete graph K_{b+1}, and the induced action $G(B)^B$ of the stabiliser of a part $B \in \mathcal{B}$ on B is 2-transitive. The results from these papers typically assert that either the triple $(\Gamma, G, \mathcal{B})$ is known explicitly, or the part-size $|B|$ is bounded above by a certain function $f(b)$. For example, [7, Theorem] proves this with $f(b) = b + 1$ if the bipartite graph $\langle B_i, B_j \rangle$ is a perfect matching. An earlier special case (of [7, Theorem]), namely [6, Theorem 4.3], identified G-distance-transitive, antipodal covers of K_{b+1} as examples (and all such covers are known, see [11, 17]).

Here we focus on the situation where $\langle B_i, B_j \rangle$ has isolated vertices and may, or may not, be a partial matching. Our main result Theorem 1.1 identifies a rich class of examples in this case, and finds all examples for which $|B| \geq b$. Two of the families of examples (first identified in an early version of this paper [8]), namely the cross ratio graphs and twisted cross ratio graphs, have been studied in detail in [9]; their full automorphism groups are determined there, as are the small number of exceptional isomorphisms between them. First we summarise briefly some fundamental theory of G-imprimitive symmetric graphs, and then we state Theorem 1.1.

1.1 Basic facts and parameters

Let $(\Gamma, G, \mathcal{B})$ be as above, that is, $\Gamma = (V, E)$ is a G-symmetric (simple undirected) graph with vertex set V and edge set E, $\mathcal{B}$ is a G-invariant vertex-partition, and the quotient $\Gamma_{\mathcal{B}}$ is not a null graph. Since G is transitive on V, Γ is regular of valency $s = |\Gamma(x)|$, where $\Gamma(x) = \{y | \{x, y\} \in E\}$; all parts $B_i \in \mathcal{B}$ have the same size $v := |B|$; and the setwise stabiliser $G(B)$ of $B \in \mathcal{B}$ acts transitively on B. Since G acts transitively on the edges of Γ and since $\Gamma_{\mathcal{B}}$ is not a null graph, the number of edges joining each pair of adjacent parts B_i, B_j is a non-zero constant m (say), and there are no edges joining two vertices in the same part. If K is the kernel of the

v	$=$	$\|B\|$	the number of points of $\mathcal{D}(B)$
b	$=$	$\|\Gamma_{\mathcal{B}}(B)\|$	the valency of Γ_B and the number of blocks of $\mathcal{D}(B)$
r	$=$	s/t	the number of blocks on a point of $\mathcal{D}(B)$
k	$=$	$\|\Gamma(B') \cap B\|$	if $y \in B'$ and $\Gamma(y) \cap B \neq \emptyset$; the block-size of $\mathcal{D}(B)$
s	$=$	$\|\Gamma(x)\|$	the valency of Γ
t	$=$	$\|\Gamma(x) \cap B'\|$	if $x \in B, \Gamma(x) \cap B' \neq \emptyset$

Table I: Parameters of $(\Gamma, G, \mathcal{B})$

action of G on $\mathcal{B}$, then the quotient graph $\Gamma_{\mathcal{B}}$ is G/K-symmetric, so $\Gamma_{\mathcal{B}}$ is regular of valency b, where $v \cdot s = b \cdot m$.

If B_i, B_j are adjacent parts, then since G acts transitively on the *arcs* of Γ (ordered pairs of adjacent vertices), each vertex of B_i is adjacent to either 0 or t vertices of B_j, for some constant $t \geq 1$. Thus t divides both s and m. Set

$$r := s/t \quad \text{and} \quad k := m/t. \tag{1}$$

The study in [6] looked solely at the case where $t = 1$, that is, where $\langle B_i, B_j \rangle$ $(i \neq j)$ is a partial matching. However, in general, the parameter t can be arbitrarily large (even if we demand not only that Γ is G-symmetric, but also that $G(B)^B$ is 2-transitive and that $\Gamma_{\mathcal{B}}$ is complete: see Example 2.10, and line 3 of Table II).

For $B \in \mathcal{B}$, set $\Gamma(B) := \{y \mid \Gamma(y) \cap B \neq \emptyset\}$ and $\Gamma_{\mathcal{B}}(B) := \{B' \in \mathcal{B} \mid \Gamma(B') \cap B \neq \emptyset\}$. The 1-design $\mathcal{D}(B)$ defined in [6] has point set B, block set $\Gamma_{\mathcal{B}}(B)$, and a 'block' B' is adjacent to a 'point' x if $\Gamma(x) \cap B' \neq \emptyset$. We define the *parameters* of $(\Gamma, G, \mathcal{B})$ to be (v, b, r, k, s, t) which we summarise in Table I.

1.2 Main result

Our goal, which we partially achieve in this paper, is as follows:

> *For arbitrary $t \geq 1$ and arbitrary $k \leq v$, we would like to classify all 'exceptional' triples $(\Gamma, G, \mathcal{B})$ with $G(B)^B$ 2-transitive and $\Gamma_{\mathcal{B}} = K_{b+1}$, and with $v \geq b$.*

In [7] we explained why such a result is of interest (for example, as an extension of Fisher's inequality $v \leq b$ for 2-designs, and as a generalisation of the inequality $r \leq k$ for an r-fold antipodal distance transitive cover of a k-valent graph). In this paper we accomplish this when $k < v$. The case $k = v$ remains open.

First we comment on the notation used in the statement of Theorem 1.1.

- We denote by $c \cdot \Delta$ the graph consisting of c disjoint copies of a given graph Δ.
- The complete graph on n vertices is denoted K_n while $K_{n[a]}$ denotes the complete multipartite graph with n parts of size a; if $n = 2$ we usually write $K_{2[a]}$ as $K_{a,a}$ (a complete bipartite graph).

- Several of the examples listed are "$*$-transforms" of more familiar graphs (see Definition 2.7).

- The cross-ratio graphs $\mathrm{CR}\,(q; d, s)$ and $\mathrm{TCR}\,(q; d, s)$ arising in Theorem 1.1 (c) are defined in Definition 2.1. For these graphs $d \in \mathrm{GF}\,(q) \setminus \{0, 1\}$, and s divides $s(d)$, where the subfield of $\mathrm{GF}\,(q)$ generated by d has order $p^{s(d)}$ (with p the prime dividing q). The relevant groups G for these graphs are subgroups of $\mathrm{P\Gamma L}\,(2, q)$ which are 3-transitive on $\mathrm{GF}\,(q) \cup \{\infty\}$; such subgroups are classified in [9, Theorem 2.1].

- The graphs associated with designs, which occur in the tables for Theorem 1.1 are introduced in Subsection 2.2. In particular, the collinear pairs graph $\mathrm{ColPairs}(\mathcal{D}')$, where $\mathcal{D}'$ is the design of points and planes in the affine space $\mathrm{AG}\,(d, 2)$, is isomorphic to $(2^d - 1) \cdot K_{2^{d-1}[2]}$. The design $S(22, 6, 1)$ on line 2 of Table IV is the Steiner system with automorphism group $\mathrm{Aut}\,(M_{22})$, and the design $\mathcal{D}(M_{11})$ on line 3 of Table IV is the unique $3-(12, 6, 2)$ design with automorphism group M_{11}.

Theorem 1.1 *Let $\Gamma = (V, E)$ be a G-symmetric graph with a G-invariant vertex-partition $\mathcal{B} = \{B_0, B_1, \ldots, B_b\}$ and parameters (v, b, r, k, s, t). Suppose that the quotient $\Gamma_{\mathcal{B}}$ is a complete graph K_{b+1}, and that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive, and $k < v$. Then one of the following holds.*

(a) *$v < b$;*

(b) *$v = b$, $r = k$, $V = \{ij \ : \ i, j \in X, \ i \neq j\}$ where $X = \{0, 1, \ldots, v\}$, $B_i = \{ij \ : \ j \in X, j \neq i\}$ (for $i \in X$), $G \leq \mathrm{Sym}\,(X)$ acts coordinate-wise, and Γ, G, t are as in one of the lines of Table II, where $k = 1$ in line 1, and $k = v - 1$ in the other lines.*

(c) *$v = b$, V is the set of flags $i\beta$ of a design $\mathcal{D}'$ with point set $X = \{0, 1, \ldots, v\}$, B_i is the set of flags $i\beta$ on i (for $i \in X$), $G \leq \mathrm{Aut}\,(\mathcal{D}') \leq \mathrm{Sym}\,(X)$ acts coordinate-wise, and $\Gamma, G, \mathcal{D}', t$ are as in one of the lines of Table III.*

In Section 2 we introduce the main families of triples $(\Gamma, G, \mathcal{B})$ that arise in our classification. In Section 3 we indicate how the framework which was introduced in [6], for the case where $t = 1$ and $G(x)^{\Gamma(x)}$ is primitive, can be extended to arbitrary symmetric graphs with $t \geq 1$. In particular we show that in any such graph we get a 1-design $\mathcal{D}(B)$ induced on the part B, with $G(B)$ acting flag-transitively on $\mathcal{D}(B)$. This framework is then used in Sections 4 and 5 to analyse triples $(\Gamma, G, \mathcal{B})$ in which $G(B)^B$ is 2-transitive and $\Gamma_{\mathcal{B}} = K_{b+1}$ is a complete graph. We treat separately the cases $t = 1$ (Section 4) and $t \geq 2$ (Section 5).

2 Examples and general constructions.

In this section we introduce the main families of graphs that arise in the situation analysed in Theorem 1.1. We begin by defining the cross-ratio graphs. Next we look

Line	Γ	Edges $\{ij, i'j'\}$	t	Conditions on G and parameters
1	$\binom{v+1}{2} \cdot K_2$	$(i', j') = (j, i)$	1	G 3-transitive on X
2	$(v+1) \cdot K_v$	$j = j'$	1	G 3-transitive on X
3	$*$-transf. of line 2	Def. 2.7	$v-2$	$G = A_{v+1}$ $(v \geq 5)$, S_{v+1} $(v \geq 4)$ or M_{v+1} $(v+1 = 11, 12, 23, 24)$
4	$\Gamma = \mathrm{CR}\,(v; d, s(d))$ or $\mathrm{TCR}\,(v; d, s(d))$	Def. 2.1	$s/s(d)$	$r = k = v-1$, v a prime power, $G \leq \mathrm{P\Gamma L}\,(2, v)$, 3-trans. on X, $G(\infty 01)$-orbit cont'g d has size t
5	ColPairs($\mathcal{D}'$)	Def 2.6	t_1	$\mathcal{D}', G, t_1$ as in Table IV, lines 1,2,3
6	NonCol($\mathcal{D}'$)	Def 2.6	t_2	$\mathcal{D}', G, t_2$ as in Table IV, lines 1,2,3

Table II: Examples for Theorem 1.1 (b)

Line	Γ	Edges $\{i\beta, i'\beta'\}$	t	Conditions on G and parameters
1	$b' \cdot K_{k'}$	$\beta = \beta'$	1	$\mathcal{D}', G, b', k'$ as in Table IV, lines 2–4
2	$(b'/2) \cdot K_{k',k'}$	$\beta \cap \beta' = \emptyset$	1	$\mathcal{D}', G, b', k'$ as in Table IV, lines 3, 4
3	$*$-transf. of line 1	Def. 2.7	$k'-2$	$\mathcal{D}', G, b', k'$ as in Table IV, lines 2–4
4	$*$-transf. of line 2	Def. 2.7	$k'-1$	$\mathcal{D}', G, b', k'$ as in Table IV, lines 3, 4
5	$\Gamma_1(M_{22})$	$i \notin \beta', i' \notin \beta$, $\beta \cap \beta' = \emptyset$	6	$\mathcal{D}', G$ as in Table IV, line 2
6	$\Gamma_2(M_{22})$	$i \notin \beta', i' \notin \beta$, $\lvert\beta \cap \beta'\rvert = 2$	10	$\mathcal{D}', G$ as in Table IV, line 2

Table III: Examples for Theorem 1.1 (c); in lines 3–6, $k' = k+1$

Line	$\mathcal{D}'$	$v+1$	k'	λ	G	t_1	t_2	b'
1	$\mathrm{AG}_2(d, 2)$	2^d	4	1	$\mathrm{AGL}\,(d, 2)$, or $d = 4, G = Z_2^4 \cdot A_7$	1	$2^d - 4$	
2	$S(22, 6, 1)$	22	6	1	M_{22} or $\mathrm{Aut}\,(M_{22})$	3	16	77
3	$\mathcal{D}'(M_{11})$	12	6	2	M_{11}	6	3	22
4	$\mathrm{AG}_{d-1}(d, 2)$	2^d	2^{d-1}	$2^{d-2} - 1$	Same as line 1	–	–	$2^{d+1}-2$

Table IV: Designs $\mathcal{D}'$ for Theorem 1.1 (b) and (c); $\mathcal{D}'$ has point set X and b' blocks and is a $3-(v+1, k', \lambda)$ design at some generic examples arising from the action of a 3-transitive permutation group (G, Ω) on the set of ordered pairs of distinct elements of Ω, and from the flags in a 3-design. We then introduce and illustrate the "$*$-transform" construction which links certain pairs of triples $(\Gamma, G, \mathcal{B})$ and $(\Gamma^*, G, \mathcal{B})$.

2.1 Cross ratio graphs

First we introduce the *untwisted cross ratio graphs* $\mathrm{CR}\,(q; d, s)$ and the *twisted cross ratio graphs* $\mathrm{TCR}\,(q; d, s)$ which feature in Sections 4 and 5. They may be defined as orbital graphs of a transitive permutation group with respect to nontrivial self-paired orbits of a point stabiliser. If G is a transitive permutation group on a set V, and $x \in V$, then the $G(x)$-orbit Y containing a point y is *nontrivial* provided $y \neq x$, and is *self-paired* if there exists an element in G which interchanges x and y. For a nontrivial self-paired $G(x)$-orbit Y, the corresponding *orbital graph* is defined as the graph with vertex set V and edges the pairs $\{x^g, y^g\}$, for $y \in Y$ and $g \in G$. It is straightforward to show that each orbital graph for G is G-symmetric.

Let $q = p^n$ where p is prime and n is a positive integer. The projective line $\mathrm{PG}\,(1, q)$ over the field $\mathrm{GF}\,(q)$ of order q can be identified with the set $\mathrm{GF}\,(q) \cup \{\infty\}$, where ∞ satisfies the usual arithmetic rules such as $1/\infty = 0, \infty + y = \infty$, etc. The two-dimensional projective group $\mathrm{PGL}\,(2, q)$ then consists of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az+b}{cz+d} \quad (\text{with } a, b, c, d \in \mathrm{GF}\,(q), \text{ and } ad - bc \neq 0)$$

of $\mathrm{PG}\,(1, q)$ (see, for example, [5, p. 242]). Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if the 4-tuple (a, b, c, d) is a non-zero multiple of (a', b', c', d'). The group $\mathrm{PGL}\,(2, q)$ is sharply 3-transitive in this action on $\mathrm{PG}\,(1, q)$, that is, it is 3-transitive and only the identity element $t_{1,0,0,1}$ fixes three elements of $\mathrm{PG}\,(1, q)$. The Frobenius automorphism $\sigma : x \mapsto x^p$ of the field $\mathrm{GF}\,(q)$ induces an automorphism of $\mathrm{PGL}\,(2, q)$ by $\sigma : t_{a,b,c,d} \mapsto t_{a^p,b^p,c^p,d^p}$, and the group generated by $\mathrm{PGL}\,(2, q)$ and σ is the semidirect product $\mathrm{PGL}\,(2, q) \cdot \langle\sigma\rangle$ and is denoted by $\mathrm{P\Gamma L}\,(2, q)$. The group $\mathrm{P\Gamma L}\,(2, q)$ is the automorphism group of $\mathrm{PGL}\,(2, q)$, and it too acts on $\mathrm{PG}\,(1, q)$ (with $\sigma : z \mapsto z^p$, where $\infty^p = \infty$). The 3-transitive subgroups of $\mathrm{P\Gamma L}\,(2, q)$ are (see [9, Theorem 2.1]) precisely the subgroups $\mathrm{PGL}\,(2, q) \cdot \langle\sigma^s\rangle$ for divisors s of n, and, in the case where p is odd and n is even, also the subgroups $\mathrm{M}(s, q) = \langle \mathrm{PSL}\,(2, q), \sigma^s t_{a,0,0,1}\rangle$, for divisors s of $n/2$, where a is a primitive element of $\mathrm{GF}\,(q)$.

Each 3-transitive subgroup of $\mathrm{P\Gamma L}\,(2, q)$ acts transitively on the set V of ordered pairs of distinct points of the projective line $\{\infty\} \cup \mathrm{GF}\,(q)$. The cross-ratio graphs are certain orbital graphs for the actions on V of 3-transitive subgroups of $\mathrm{P\Gamma L}\,(2, q)$. The stabiliser in $\mathrm{PGL}\,(2, q)$ of $\infty 0$ is cyclic of order $q - 1$ and consists of the transformations $z \mapsto az$ $(a \in \mathrm{GF}\,(q)^*)$; while if q is odd then the stabiliser in $\mathrm{PSL}\,(2, q)$ of $\infty 0$ consists of the transformations $z \mapsto az$ with a a square. Let G be a 3-transitive subgroup of $\mathrm{P\Gamma L}\,(2, q)$: namely either $G = \mathrm{PGL}\,(2, q) \cdot \langle\sigma^s\rangle$ for some divisor s of n, or p is odd, n is even, and $G = \mathrm{M}(s/2, q)$ for some even divisor s of n. Then the stabiliser $G(\infty 0)$ is transitive on $\mathrm{PG}\,(1, q) \setminus \{\infty, 0\}$, and $G(\infty 01)$ is $\langle\sigma^s\rangle$ (see [9, Corollary 2.2]).

Let $q \geq 3$. Then each element $d \in \mathrm{GF}(q) \setminus \{0, 1\}$ generates a subfield of $\mathrm{GF}(q)$ of order $p^{s(d)}$ for some divisor $s(d)$ of n. Suppose that d is such that s divides $s(d)$, so that in particular $s(d)$ is even if $G = \mathrm{M}(s/2, q)$. Then the $G(\infty01)$-orbit in V containing $1d$ is the set $\{1d^{\sigma^{si}} \mid 0 \leq i < s(d)/s\}$ of size $s(d)/s$. If $G = \mathrm{PGL}(2, q) \cdot \langle\sigma^s\rangle$, then the $G(\infty0)$-orbit $\Delta(\infty0)$ containing $1d$ consists of the $(q-1)s(d)/s$ pairs ab, where $a \in \mathrm{GF}(q) \setminus \{0\}$ and $b = ad^{\sigma^{si}}$ with $0 \leq i < s(d)/s$. The orbit $\Delta(\infty0)$ is self-paired because the element $t_{1,-d,1,-1}$ interchanges $\infty0$ and $1d$. If $G = \mathrm{M}(s/2, q)$, then the $G(\infty0)$-orbit $\Delta'(\infty0)$ containing $1d$ consists of the $(q-1)s(d)/s$ pairs ab, where $b = ad^{\sigma^{si}}$ with $0 \leq i < s(d)/s$ if a is a square in $\mathrm{GF}(q)$, and $b = ad^{\sigma^{si+s/2}}$ with $0 \leq i < s(d)/s$ if a is a non-square in $\mathrm{GF}(q)$. If $d-1$ is a square then the orbit $\Delta'(\infty0)$ is self-paired because the element $t_{1,-d,1,-1} \in \mathrm{PSL}(2, q) \subseteq G$. However if $d-1$ is a non-square then $\Delta'(\infty0)$ is not self-paired. (This may be proved by an argument similar to that used in the last paragraph of the proof of [9, Theorem 4.1].) We are now able to define the cross-ratio graphs.

Definition 2.1 Let $q = p^n$ for a prime p, where $n \geq 1$ and $q \geq 3$, and let V denote the set of ordered pairs of distinct points from the projective line $\mathrm{PG}(1, q) = \mathrm{GF}(q) \cup \{\infty\}$. Let $d \in \mathrm{GF}(q)$, $d \neq 0, 1$, and let s be a divisor of $s(d)$.

(a) The *untwisted cross ratio graph* $\mathrm{CR}(q; d, s)$ is defined as the orbital graph for the action on V of $G := \mathrm{PGL}(2, q) \cdot \langle\sigma^s\rangle$ corresponding to the self-paired $G(\infty0)$-orbit $\Delta(\infty0)$.

(b) Suppose now that p is odd, n is even (so $q \geq 9$), and also that $d-1$ is a square and both s and $s(d)$ are even. Then the *twisted cross ratio graph* $\mathrm{TCR}(q; d, s)$ is defined as the orbital graph for the action on V of the group $G := \mathrm{M}(s/2, q)$ corresponding to the self-paired $G(\infty0)$-orbit $\Delta'(\infty0)$.

Remark 2.2 (a) The graphs $\mathrm{CR}(q; d, s)$ are defined in [9, Definition 3.2] in terms of the cross-ratio. However their definition given here as orbital graphs is equivalent, see [9, Remark 3.3 (b)].

(b) The graphs $\mathrm{CR}(3; 2, 1)$ and $\mathrm{CR}(5; 4, 1)$ are disconnected: $\mathrm{CR}(3; 2, 1) \cong 3 \cdot C_4$ and $\mathrm{CR}(5; 4, 1) \cong 5 \cdot (C_3[\overline{K_2}])$ (a lexicographic product). All other twisted and untwisted cross-ratio graphs are connected, by [9, Proposition 5.2].

(c) All isomorphisms between cross-ratio graphs are specified in [9, Theorem 6.2]. In particular there are no isomorphisms between a twisted cross-ratio graph and an untwisted cross-ratio graph.

(d) The full automorphism groups of all twisted and untwisted cross-ratio graphs are determined in [9, Theorem 6.1]. For $\Gamma = \mathrm{CR}(q; d, s)$ we have $\mathrm{Aut}(\Gamma) \cap \mathrm{P\Gamma L}(2, q) = \mathrm{PGL}(2, q) \cdot \langle\sigma^s\rangle$ by [9, Theorem 3.4], and for $\Gamma = \mathrm{TCR}(q; d, s)$ we have $\mathrm{Aut}(\Gamma) \cap \mathrm{P\Gamma L}(2, q) = \mathrm{M}(s/2, q)$ by [9, Theorem 3.7]. In either case, $G := \mathrm{Aut}(\Gamma) \cap \mathrm{P\Gamma L}(2, q)$ acts symmetrically on Γ and preserves the G-invariant partition

$$\mathcal{B} := \{B(x) \mid x \in \mathrm{PG}(1, q)\}, \text{ where } B(x) = \{xy \mid y \in \mathrm{PG}(1, q), y \neq x\}.$$

The part size is $v = |B(x)| = q \geq 3$ and the quotient $\Gamma_{\mathcal{B}} \cong K_{q+1}$. Moreover, by [9, Theorems 3.4 and 3.7], the parameters $k = q-1$ and $t = s(d)/s$.

2.2 Generic examples from a 3-transitive group

We now give some examples which are in many ways typical of the triples $(\Gamma, G, \mathcal{B})$ which we shall meet in later sections. The examples given reflect the fact that we are primarily interested in cases where $G(B)^B$ is 2-transitive. Our first two examples are of triples $(\Gamma, G, \mathcal{B})$ in which $G^{\mathcal{B}}$ is 3-transitive, and in which the vertices of Γ are labelled by ordered pairs of distinct parts of $\mathcal{B}$. Such a situation is typical of the case where $k = v - 1$ (Proposition 4.1 and 5.1). The first example, which extends naturally to the case where the set $\mathcal{P}$ of labels is the set of points of the affine geometry $\mathrm{AG}\,(d, 2), d \geq 4$, illustrates the case $t = 1$ (as in Section 4); the second and third examples illustrate the case $t \geq 2$ (as in Section 5).

Example 2.3 Let $\mathcal{P} = \{0, 1, \ldots, 7\}$ be the set of points of the affine geometry $\mathrm{AG}\,(3, 2)$, and let $V = \{ij : 0 \leq i, j \leq 7, i \neq j\}$. Define a graph Γ on the vertex set V, by joining vertex ij to vertex kl precisely when i, j, k, l are the four points of some 2-dimensional affine subspace. Then $G = \mathrm{AGL}\,(3, 2)$ acts 3-transitively on $\mathcal{P}$, Γ is G-symmetric, and we have a natural G-invariant partition $\mathcal{B} = \{B = B_0, B_1, \ldots, B_7\}$ on V, where $B_i = \{ij : 0 \leq j \leq 7, j \neq i\}$. Moreover $G(B)^B$ is 2-transitive. The pair $(\Gamma, \mathcal{B})$ has parameters $v = b = 7, r = k = 6, t = 1$. The graph Γ is the disjoint union of 7 copies of the complete multipartite graph $K_{4;2}$ with 4 parts of size 2, and the quotient $\Gamma_{\mathcal{B}} = K_8$.

Example 2.4 The group $G = S_5$ acts naturally on the set $\{0, 1, 2, 3, 4\}$. Let the graph Γ have vertex set $V = \{ij : 0 \leq i, j \leq 4, i \neq j\}$, with vertex ij adjacent to vertex kl if and only if $|\{i, j, k, l\}| = 4$. The set V admits the G-partition $\mathcal{B} = \{B = B_0, B_1, B_2, B_3, B_4\}$, where $B_i = \{ij : 0 \leq j \leq 4, j \neq i\}$ with $G(B)^B = S_4$. This defines a connected G-symmetric graph of valency 6, and a triple $(\Gamma, G, \mathcal{B})$ with parameters $v = b = 4, r = k = 3, t = 2$, and with quotient graph $\Gamma_{\mathcal{B}} = K_5$.

In fact $\Gamma = \bar{K}_2 \mathrm{wr}\, O_3$ is obtained from the Petersen graph O_3 by replacing each vertex v of O_3 by two vertices v_1, v_2, and each edge vw of O_3 by the four edges $v_i w_j$ $(1 \leq i, j \leq 2)$. The full automorphism group $\mathrm{Aut}\,\Gamma = S_2 \mathrm{wr}\, S_5 = S_2^5 \cdot S_5$ admits only the invariant partition $\{\{v_1, v_2\} : v \in VO_3\}$. However the subgroup $G := \langle A_5, d(12)\rangle \cong S_5$ (where d is the generator of the diagonal subgroup of S_2^5 and (12) lies in the top group S_5) acts transitively on 1-arcs, and the subgroup chain $G > G(B) > G(v_1)$ (with both containments proper) gives rise to the G-invariant partition $\mathcal{B}$ (compare [6, Example 2.3]).

Example 2.5 Let V be the set of flags, that is, the incident point-hyperplane pairs $i\beta$, in the affine geometry $\mathrm{AG}\,(3, 2)$. Define a graph Γ on V by joining $i\beta$ to $i'\beta'$ if and only if $i \neq i', \beta \neq \beta', i \in \beta', i' \in \beta$. Then Γ is a connected graph of valency 6. The group $G = \mathrm{AGL}\,(3, 2)$ acts symmetrically on Γ, and preserves the partition $\mathcal{B}$ with 8 parts B_i where, for any given point i, $B_i = \{i\beta : \beta$ a hyperplane containing $i\}$ has size $v = 7$, $k = 3$, and $t = 2$. The quotient graph $\Gamma_{\mathcal{B}} = K_8$.

The examples arising in Theorem 1.1 are often associated with designs. For a positive integer s, an s-(v', k', λ) *design* $\mathcal{D}' = (X, \mathcal{L})$, where $s \leq k' \leq v'$, consists of a set X of cardinality v', whose elements are called points, and a set $\mathcal{L}$ of k'-element subsets of X, called blocks, such that each s-element subset of X is contained in exactly λ blocks. The relevant designs $\mathcal{D}'$ for the statement of Theorem 1.1 are 3-designs and are listed in Table IV; in these cases the cardinality of $\mathcal{L}$ is $b' = \lambda\frac{v'(v'-1)(v'-2)}{k'(k'-1)(k'-2)}$. We will meet the following graphs associated with designs.

Definition 2.6 Let $\mathcal{D}' = (X, \mathcal{L})$ be a 3-(v', k', λ) design with b' blocks. Let $V = \{ij \ : \ i, j \in X, \ i \neq j\}$, and let $F = \{i\beta \ : \ i \in X, \ \beta \in \mathcal{L}, \ i \in \beta\}$ (the set of flags of $\mathcal{D}'$).

(a) The *collinear pairs graph* ColPairs$(\mathcal{D}')$ has vertex set V, and vertices ij and $i'j'$ are adjacent if and only if i, j, i', j' are pairwise distinct and are all contained in some block in $\mathcal{L}$.

(b) The *non-collinear pairs graph* NonCol$(\mathcal{D}')$ has vertex set V, and vertices ij and $i'j'$ are adjacent if and only if i, j, i', j' are pairwise distinct but no block of $\mathcal{L}$ contains all of them.

(c) Let $\mathcal{D}'$ be the 3-$(22, 6, 1)$ Steiner system in line 2 of Table IV, so $\mathrm{Aut}\,(\mathcal{D}') = \mathrm{Aut}\,(M_{22})$. The graphs $\Gamma_1(M_{22})$ and $\Gamma_2(M_{22})$ in lines 5 and 6 of Table III both have vertex set F, and two flags $i\beta$ and $i'\beta'$ are adjacent if and only if $i \notin \beta', i' \notin \beta$, and either $\beta \cap \beta' = \emptyset$ (in $\Gamma_1(M_{22})$), or $|\beta \cap \beta'| = 2$ (in $\Gamma_2(M_{22})$).

2.3 The star-transform

Next we introduce the "$*$-transform" of a triple $(\Gamma, G, \mathcal{B})$ which relates certain triples with $t \geq 2$ in Theorem 1.1 to triples with $t = 1$. Given any triple $(\Gamma, G, \mathcal{B})$, if $B_i, B_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$, then $|\Gamma(B_i) \cap B_j| = k$, with k as in equation (1), see Table I.

Definition 2.7 Let Γ be a finite graph and let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a partition of the vertex set V. For $i \neq j$, let $X_{ij} = \Gamma(B_i) \cap B_j$, and suppose that, whenever $X_{ij} \neq \emptyset$ the cardinality $|X_{ij}| \geq 2$. Then the $*$-*transform* Γ^* of Γ relative to $\mathcal{B}$ has vertex set V, and edges all pairs $\{x, y\}$ such that, for some $i \neq j$, $x \in X_{ij}, y \in X_{ji}$, and $\{x, y\}$ is not an edge of Γ. Note that $(\Gamma^*)^* = \Gamma$.

Thus, if $A(\Gamma)$ denotes the set of arcs of Γ, then $A(\Gamma^*) \cap (B_i \times B_j) = (X_{ji} \times X_{ij}) \setminus A(\Gamma)$.

Lemma 2.8 *Let Γ be a finite G-symmetric graph admitting a G-invariant partition $\mathcal{B}$ of the vertex set V, where $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$, such that $(\Gamma, G, \mathcal{B})$ has parameters (v, b, k, r, s, t), with $k \geq 2$. Let Γ^*, X_{ij} be as in Definition 2.7, relative to $\mathcal{B}$.*

(a) Then $G \leq \mathrm{Aut}\,(\Gamma^*)$ and, whenever B_i, B_j are adjacent in $\Gamma_{\mathcal{B}}$, $G(B_i, B_j)$ is transitive on $A(\Gamma) \cap (B_i \times B_j) = A(\Gamma) \cap (X_{ji} \times X_{ij})$.

(b) Γ^* is G-symmetric if and only if, for some i, j such that $X_{ij} \neq \emptyset$, the stabiliser $G(B_i, B_j)$ is transitive on $(X_{ji} \times X_{ij}) \setminus A(\Gamma)$.

(c) *If $t = k$ then Γ^* is a null graph, while if $t < k$ and Γ^* is G-symmetric, then Γ^* has parameters $(v^*, b^*, r^*, k^*, s^*, t^*) = (v, b, r, k, rk - s, k - t)$.*

Proof: It is straightforward to show that $G \leq \mathrm{Aut}(\Gamma^*)$. From the definition of the X_{ij} it follows that $A(\Gamma) \cap (B_i \times B_j) = A(\Gamma) \cap (X_{ji} \times X_{ij})$, the set of arcs (x, y) with $x \in B_i$ and $y \in B_j$. If $(x, y), (x', y')$ are two arcs in this set, then since G is transitive on $A(\Gamma)$, there is an element $g \in G$ such that $(x', y') = (x, y)^g = (x^g, y^g)$. Since $\mathcal{B}$ is G-invariant, and since $x, x' \in B_i$ we have $B_i^g = B_i$, and similarly $B_j^g = B_j$. Thus $g \in G(B_i, B_j)$, proving part (a).

If Γ^* is a null graph, then it is G-symmetric since G is transitive on vertices, and also the local transitivity property holds vacuously, so the equivalence holds in this case. Suppose now that Γ^* is not null, and suppose first that Γ^* is G-symmetric. Then part (a) may be applied to Γ^* yielding that $G(B_i, B_j)$ is transitive on $(X_{ji} \times X_{ij}) \setminus A(\Gamma)$ whenever $X_{ij} \neq \emptyset$. Conversely supose that $G(B_i, B_j)$ is transitive on $(X_{ji} \times X_{ij}) \setminus A(\Gamma)$ for some i, j such that $X_{ij} \neq \emptyset$. Since Γ is G-symmetric, and since Γ^* is not null, it follows that $A(\Gamma^*) \cap (B_i \times B_j) = (X_{ji} \times X_{ij}) \setminus A(\Gamma)$ is non-empty. Thus there exists an arc (x, y) of Γ^* with $x \in B_i$ and $y \in B_j$. Let (x', y') be arbitrary arc of Γ^*, say with $x' \in B_{i'}$ and $y' \in B_{j'}$, so also $X_{i'j'} \neq \emptyset$. By the definition of Γ^* there is an edge of Γ between B_i and B_j, and similarly there is an edge of Γ between the parts $B_{i'}$ and $B_{j'}$. Since G is transitive on $A(\Gamma)$, some element $g \in G$ maps $(B_{i'}, B_{j'})$ to (B_i, B_j), and replacing x', y' by their images under g we may assume that $x' \in B_i$ and $y' \in B_j$. Thus both (x, y) and (x', y') lie in $(X_{ji} \times X_{ij}) \setminus A(\Gamma)$, and transitivity of $G(B_i, B_j)$ on this set shows that (x, y) and (x', y') lie in the same G-orbit. Thus Γ^* is G-symmetric, proving (b).

Now $t \leq k$ (see Table I), and if $t = k$ then by its definition, Γ^* is a null graph. Suppose that $t < k$ and that Γ^* is G-symmetric. To verify the assertions about the parameters (see Table I) we observe that $\Gamma_{\mathcal{B}} = \Gamma^*_{\mathcal{B}}$, and hence $b^* = b$. Also $v^* = |B_i| = v$ and $k^* = |X_{ij}| = k$ (for $X_{ij} \neq \emptyset$). From Definition 2.7, we see that $t^* = k - t$. Since $t < k$, for $x \in B_i$, say, the set of parts of $\mathcal{B}$ containg vertices of $\Gamma(x)$ is the same as that containing vertices of $\Gamma^*(x)$, so $r^* = r$, and hence, by (1), $s^* = r^* t^* = r(k - t) = rk - s$, completing the proof. ∎

The only instance of the above construction that we shall need arises when the pair $(\Gamma, \mathcal{B})$ has $t = 1$ and $G(B_i, B_j)$ is 2-transitive on X_{ij}. In such a setting we obtain a natural correspondence between certain triples $(\Gamma, G, \mathcal{B})$ with $t = 1, k \geq 3$ and their $*$-transforms $(\Gamma^*, G, \mathcal{B})$ with $t^* = k - t = k - 1 \geq 2$. We note that when $t = 1$ and $k = |X_{ij}| = 2$, then $G(B_i, B_j)$ is automatically 2-transitive on X_{ij}, so Γ^* is always G-symmetric, and is precisely the graph called Γ^{opp} in [6, Section 6].

Example 2.9 The $*$-transform of the triple $(\Gamma, G, \mathcal{B})$ in Example 2.4 of valency 6 (with $k = 3, t = 2$) is a triple $(\Gamma^*, G, \mathcal{B})$ of valency 3 with $t^* = 1$ and $\Gamma^*_{\mathcal{B}} = \Gamma_{\mathcal{B}} = K_5$ (in fact $\Gamma^* = 5 \cdot K_4$). The $*$-transform of the triple $(\Gamma, G, \mathcal{B})$ of valency 6 in Example 2.5 (with $k = 3, t = 2$) is a triple $(\Gamma^*, G, \mathcal{B})$ of valency 3 with $k^* = 3, t^* = 1$, and $\Gamma^*_{\mathcal{B}} = \Gamma_{\mathcal{B}} = K_8$; vertices (i, β) and (i', β') are adjacent in Γ^* if and only if $\beta = \beta'$, so $\Gamma^* = 14 \cdot K_4$.

Example 2.10 Example 2.5 generalises naturally to flags in AG(d, q), or in PG(d, q), giving G-symmetric triples $(\Gamma, G, \mathcal{B})$ with $\Gamma_{\mathcal{B}}$ a complete graph and with $G(B)^B$ 2-transitive, for $B \in \mathcal{B}$. The resulting triples are $*$-transforms of simpler triples with $t^* = 1$ and with the same quotient. (These examples show that the parameter t can be arbitrarily large, as does the natural partition $\mathcal{B}$ in the complete multipartite graphs $K_{(b+1)[v]}$ with $b+1$ parts each of size v.)

3 Imprimitive symmetric graphs with $t \geq 1$

Let Γ be an arbitrary G-symmetric, imprimitive graph with vertex set V. If $x \in V$, then $\Gamma_i(x)$ denotes the set of vertices at distance i from x, for $i \geq 1$, and we usually write $\Gamma(x) = \Gamma_1(x)$. Let $\mathcal{B} = \{B = B_0, B_1, B_2, \ldots\}$ be a G-invariant partition of V. If B_i, B_j are two parts joined by an edge of Γ, then (since G acts transitively on directed edges of Γ) each vertex $x \in B_i$ is joined to either 0 or t vertices of B_j, where $t \geq 1$ is independent of the choice of the parts B_i, B_j.

In [6] we considered only the case $t = 1$. In fact we assumed that $G(x)$ acts primitively on $\Gamma(x)$, whence either the valency b of the quotient graph $\Gamma_{\mathcal{B}}$ is 1 and Γ is bipartite, or $t = 1$. However the approach developed in [6] can often be used in situations where $t \geq 2$. In this section we run through the basic ideas following [6, Sections 3 and 4]. The proofs are straightforward and are omitted.

By our notational convention, if $B \in \mathcal{B}$, then $\Gamma_{\mathcal{B}}(B)$ denotes the set of "vertices" in the graph $\Gamma_{\mathcal{B}}$ which are adjacent to the "vertex" B in the graph $\Gamma_{\mathcal{B}}$. That is to say, $\Gamma_{\mathcal{B}}(B)$ denotes the set of parts in $\mathcal{B}$ which are joined to B by an edge of Γ. Recall the definition of $\mathcal{D}(B)$ at the end of Subsection 1.1. The next result justifies the descriptions of the parameters in Table I.

Proposition 3.1 *Let Γ be a G-symmetric graph whose vertex set V admits a G-invariant partition $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$.*

(a) *There exists a constant $t \geq 1$ such that, for each $x \in B$ and each $B_i \in \Gamma_{\mathcal{B}}(B)$, the cardinality $|\Gamma(x) \cap B_i| = 0$ or t.*

(b) *Γ is regular of valency s (say), and t divides s. Set $r := s/t$; then each point of $\mathcal{D}(B)$ lies in r blocks of $\mathcal{D}(B)$.*

(c) *Each pair of adjacent parts is joined by a constant number m (say) of edges of Γ, and t divides m. Set $k := m/t$; then if $B_i \in \Gamma_{\mathcal{B}}(B)$, $k = |\{x \in B : \Gamma(x) \cap B_i \neq \emptyset\}|$ is the number of points incident with each block of $\mathcal{D}(B)$.*

(d) *Each part $B \in \mathcal{B}$ has constant size v (say), and the quotient graph $\Gamma_{\mathcal{B}}$ is regular of valency b (say), where $vs = bm$ (so $vr = bk$).*

(e) *$\mathcal{D}(B)$ is a 1-design with parameters (v, b, r, k).*

(f) *$G(B)$ induces a group of automorphisms of $\mathcal{D}(B)$, which is transitive on "points" (that is, on B), on "blocks" (that is, on $\Gamma_{\mathcal{B}}(B)$), and on "flags" (that is, on $\{(x, B_i) : x \in B, B_i \in \Gamma_{\mathcal{B}}(B)$ and $\Gamma(x) \cap B_i \neq \emptyset\}$).*

Corollary 3.1.1 *Let Γ be a G-symmetric graph and let $\mathcal{B} = \{B = B_0, B_1, B_2, \ldots\}$ be a G-invariant partition of the vertex set. Suppose in addition that $G(B)^B$ is 2-transitive.*

(a) *Then each pair of "points" of B lies in a constant number λ (say) of "blocks" of $\mathcal{D}(B)$. Hence either $\lambda = 0$ or $\mathcal{D}(B)$ is a 2-design – possibly with repeated "blocks".*

(b) *$\lambda = 0$ if and only if $k = 1$ (whence $t = k = m = 1$, $b = vr$).*

(c) *Suppose that $\lambda \geq 1$ (whence $k \geq 2$) and that $\mathcal{D}(B)$ has repeated "blocks". Then each "block" is repeated the same number of times, say ρ, so $\rho \geq 2$ and ρ divides λ, r, and b; if we ignore repetitions then $\mathcal{D}(B)$ becomes a $2-(v, k, \lambda/\rho)$ design with b/ρ "blocks", so either*

(i) *$v = k$, $\rho = r = b = \lambda$; or*

(ii) *$v > k$, whence $v \leq b/\rho < b$ (by Fisher's inequality).*

(d) *Suppose that $\lambda \geq 1$ and that $\mathcal{D}(B)$ has no repeated "blocks" (so $\rho = 1$ and $k \geq 2$). Then either $b = 1$ or we must have $v > k$, so $v \leq b$ (by Fisher's inequality). Hence*

(i) *$b = 1, v = k$, (so, if $\Gamma_{\mathcal{B}}$ is connected, then $\Gamma = v \cdot K_2$ or $\Gamma = K_{v,v}$); or*

(ii) *$k < v < b$; or*

(iii) *$k < v = b$, $\mathcal{D}(B)$ is a non-degenerate symmetric 2-design, and $G(B)$ acts 2-transitively on both "points" and "blocks" of $\mathcal{D}(B)$.*

Corollary 3.1.2 *Let Γ be a G-symmetric graph and let $\mathcal{B} = \{B = B_0, B_1, B_2, \ldots\}$ be a G-invariant partition of the vertex set. Suppose that $G(B)^B$ is 2-transitive, and that, in addition, the quotient graph $\Gamma_{\mathcal{B}} = K_{b+1}$ is a complete graph. Then one of the following holds.*

(a) *$v < b$;*

(b) *$v = b$, $\lambda = 0$, $t = k = m = 1 = r$, so $\Gamma = \binom{v+1}{2} \cdot K_2$, and (as in* [6, Theorem 4.2]*) $V = \{ij : 0 \leq i, j \leq v, i \neq j\}$, $B_i = \{ij : 0 \leq j \leq v, j \neq i\}$ $(0 \leq i \leq v)$, vertex 'ij' is joined only to 'ji', and G may be any 3-transitive group on $\{0, 1, \ldots, v\}$;*

(c) *$v = k$ and $\mathcal{D}(B)$ has a single "block" of size v repeated $b = r$ times;*

(d) *$v = b$, $2 \leq k < v$, and $\mathcal{D}(B)$ is a symmetric 2-design with no repeated blocks, whence $G(B)$ is 2-transitive on "blocks" of $\mathcal{D}(B)$ (that is, on $\Gamma_{\mathcal{B}}(B) = \mathcal{B} \setminus \{B\}$) as well as on "points", so G is 3-transitive on $\mathcal{B}$.*

If we wish to classify all such triples $(\Gamma, G, \mathcal{B})$ with $v \geq b$, it remains to analyse the triples occurring in parts (c) and (d) of Corollary 3.1.2. In [7] we proved that when $t = 1$ in case (c), then either $v \leq b + 1$ or $(\Gamma, G, \mathcal{B})$ is uniquely determined;

we also made inroads into classifying the triples with $v = b + 1$ or $v = b$. Thus the following problem remains open. Its solution would complete the classification of all exceptional triples $(\Gamma, G, \mathcal{B})$ with $v \geq b$.

Problem 3.1.3 *Classify all triples* $(\Gamma, G, \mathcal{B})$ *satisfying the conditions of Corollary* 3.1.2 *(c) with* $v \geq b$ *and* $t \geq 2$.

The rest of this paper is devoted to classifying the triples $(\Gamma, G, \mathcal{B})$ in part (d) of Corollary 3.1.2: Section 4 classifies the triples with $t = 1$, while Section 5 classifies those with $t \geq 2$.

Note that when $k < v$ and $G(B)^B$ is 2-transitive, the group G is faithful on $\mathcal{B}$ (see the first paragraph of the proof of Lemma 3.2 below). Let G_B denote the pointwise stabiliser of the part B. If $\mathcal{D}(B)$ has no repeated blocks, then G_B fixes each part in $\Gamma_{\mathcal{B}}(B)$, so if $\Gamma_{\mathcal{B}}$ is a complete graph then $G_B = 1$; hence $G(B)$ is faithful on B. Thus when classifying triples which arise in part (d) of Corollary 3.1.2 we do not need to worry about unfaithful actions. (This is very different from case (c) of Corollary 3.1.2; see [7].)

In part (d) of Corollary 3.1.2, $G^{\mathcal{B}}$ is 3-transitive and the subgroup $G(B)$ is 2-transitive both on $\mathcal{B} \setminus \{B\}$ and on B of the same degree $v = b$. The case $k = v - 1$ is dealt with in Proposition 4.1 (for $t = 1$) and Proposition 5.1 (for $t \geq 2$). If $2 \leq k \leq v-2$, or equivalently $2 \leq r \leq b-2$, then a vertex $x \in B$ will be adjacent to vertices from r parts of $\mathcal{B} \setminus \{B\}$, and we note that $|\mathcal{B} \setminus \{B\}| = b \geq r+2$. Consequently the stabiliser $G(x)$ of a point $x \in B$ is not transitive on $\mathcal{B} \setminus \{B\}$ (since $r < b$) and, although $G(x)$ has the same order as the stabiliser in G_B of a part of $\mathcal{B} \setminus \{B\}$, $G(x)$ is not equal to the stabiliser of such a part $B' \in \mathcal{B} \setminus \{B\}$ (for if it were then $G(x)$ would be transitive on $\mathcal{B} \setminus \{B, B'\}$ and this is not the case since $r \leq b - 2$). In particular the actions of $G(B)$ on $\mathcal{B} \setminus \{B\}$ and on B are not equivalent. The next lemma identifies the possibilities for G explicitly.

Lemma 3.2 *Let* Γ *be a* G*-symmetric graph and let* $\mathcal{B} = \{B = B_0, B_1, B_2, \ldots\}$ *be a* G*-invariant partition of the vertex set. Suppose that,* $G(B)^B$ *is* 2*-transitive, that the quotient graph* $\Gamma_{\mathcal{B}} = K_{b+1}$ *is a complete graph, and that part (d) of Corollary* 3.1.2 *holds with* $k \leq v-2$. *Then* G *is a group of automorphisms of a* 3*-design* $\mathcal{D}$ *with point set* $\mathcal{B}$ *such that, for* β *a block of* $\mathcal{D}$ *containing* B, $G(B, \beta)$ *is transitive on* $\beta \setminus \{B\}$ *and on* $\mathcal{B} \setminus \beta$. *Moreover,* $(\mathcal{D}, G)$ *are as in one of the lines* 2–4 *of Table IV.*

Proof: First we show that G acts faithfully on $\mathcal{B}$. Let K be the kernel of this action. If $K \neq 1$, then K^B is a nontrivial normal subgroup of the 2-transitive group $G(B)^B$, so K^B is transitive. Thus, for $x \in B$, $G(B) = KG(x)$. Hence $G(x)^{\mathcal{B}} = G(B)^{\mathcal{B}}$ is transitive on $\mathcal{B} \setminus \{B\}$, so $r = b$ whence $k = v$ (since $vr = bk$), contrary to Corollary 3.1.2(d). Therefore $K = 1$.

It follows from the remarks immediately before Lemma 3.2 that $(G, \mathcal{B})$ is a 3-transitive group of degree $b + 1$ such that, for $B \in \mathcal{B}$, $G(B)$ has two 2-transitive permutation representations of degree b which are not equivalent and which are such that a point stabiliser in one of the representations is intransitive in the other representation. Hence from the classification of finite 2-transitive groups (see [1] and [13, Appendix 1]) the socle of $G(B)$ is one of the following: $\mathrm{PSL}(d, q)$ with $d \geq 3$ and $b = (q^d - 1)/(q - 1)$; or A_7 with $b = 15$; or HS with $b = 176$; or $\mathrm{PSL}(2, 11)$ with $b = 11$. A 3-transitive extension of a group $G(B)$ in the first family exists if and only if either $q = 2$ and $G = \mathrm{AGL}(d, 2)$ as in Table IV line 4, or $(d, q) = (3, 4)$ and $G = M_{22}$ or $\mathrm{Aut}(M_{22})$ as in Table IV line 2. Similarly A_7 has a unique transitive extension $G = 2^4 \cdot A_7$ as in Table IV line 4, and $\mathrm{PSL}(2, 11)$ has a unique transitive extension $G = M_{11}$ as in Table IV line 3. The group HS has no transitive extension. In each case G acts on a 3-design $\mathcal{D}$ with point set $\mathcal{B}$ as claimed, and $G(B, \beta)$ is transitive on both $\beta \setminus \{B\}$ and $\mathcal{B} \setminus \beta$, where β is a block of $\mathcal{D}$ containing B. ∎

In each of the cases of Lemma 3.2 we may therefore identify $\mathcal{B}$ with the point set of the 3-design $\mathcal{D}$. We finish this section with a lemma which shows that the vertices of Γ may be identified with the flags of $\mathcal{D}$ and that the symmetric 2-design $\mathcal{D}(B)$ is related to $\mathcal{D}$ in one of two ways.

Definition 3.3 Let $\mathcal{D}$ be an s-design, where $s \geq 1$, and let $\mathcal{P}$ denote the point set of $\mathcal{D}$.

(a) For a point P of $\mathcal{D}$ the *derived design* $\mathcal{D}_P$ is the 2-design with point set $\mathcal{P} \setminus \{P\}$ and with blocks the sets $\beta \setminus \{P\}$, where β is a block of $\mathcal{D}$ containing P.

(b) The *dual design* of $\mathcal{D}$ is the design whose points are the blocks of $\mathcal{D}$, and whose blocks are the points of $\mathcal{D}$, with incidence unchanged. The dual design of $\mathcal{D}$ is, in general, a 1-design, and is a 2-design if and only if $\mathcal{D}$ is a symmetric 2-design.

(c) The *complementary design* $\mathcal{D}^c$ of $\mathcal{D}$ has the same point set $\mathcal{P}$ as $\mathcal{D}$, with blocks of $\mathcal{D}^c$ being the complements (in $\mathcal{P}$) of those of $\mathcal{D}$.

Lemma 3.4 *Let Γ be a G-symmetric graph and let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a G-invariant partition of the vertex set. Suppose that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive, that the quotient graph $\Gamma_{\mathcal{B}} = K_{b+1}$ is a complete graph, and that case (d) of Corollary 3.1.2 holds with $k \leq v - 2$. Then the following also hold.*

(a) *$(G, \mathcal{B})$ is a 3-transitive permutation group which is equivalent to the action of G on the point set $\mathcal{P}$ of a 3-design $\mathcal{D}$ as in Table IV, lines 2–4.*

(b) *The vertex set V may be identified with the set of flags of $\mathcal{D}$ in such a way that G acts coordinate-wise; each part B_P of $\mathcal{B}$ is the set of flags with a fixed first coordinate P.*

(c) *For $B = B_P \in \mathcal{B}$ corresponding to a point $P \in \mathcal{P}$, there are just two possibilities for the non-degenerate 2-design $\mathcal{D}(B_P)$: either*

(i) *$\mathcal{D}(B_P)$ is the dual design of the derived design $\mathcal{D}_P$ (that is, the point set B_P of $\mathcal{D}(B_P)$ corresponds to the set of blocks of $\mathcal{D}_P$, and the block set $\Gamma_{\mathcal{B}}(B_P) = \mathcal{B} \setminus \{B_P\}$ of $\mathcal{D}(B_P)$ corresponds to the set of points of $\mathcal{D}_P$; moreover if β is a block of $\mathcal{D}$ incident with P, then the point (P, β) in B_P is incident with a block $B_{P'}$ of $\mathcal{D}(B_P)$ if and only if $P' \in \beta$); or*

(ii) *$\mathcal{D}(B_P)$ is the complement of the dual design of the derived design $\mathcal{D}_P$: that is, a point (P,β) of $\mathcal{D}(B_P)$ (which corresponds to a block β of $\mathcal{D}$ incident with P) is adjacent to a block $B_{P'}$ of $\mathcal{D}(B_P)$ (which is a point P' of $\mathcal{D}_P$) if and only if $P' \notin \beta$.*

(d) *Let $x = (P,\beta) \in B_P \in \mathcal{B}$, and let $B_{P'} \in \mathcal{B} \setminus \{B_P\}$ be such that $\Gamma(x) \cap B_{P'} \neq \emptyset$.*

(i) *In case (c)(i) (that is, if $P' \in \beta$), the $G(x, B_{P'})$-orbits in $B_{P'}$ are $\{(P',\beta)\}$, $\{(P',\beta') : P, P' \in \beta'$ and $\beta' \neq \beta\}$, and $\{(P',\beta') : P' \in \beta', P \notin \beta'\}$. The orbit lengths are as follows: $1, 4, 16$ in line 2 of Table IV; and $1, 4, 6$ in line 3 of Table IV; and $1, 2^{d-1} - 2, 2^{d-1}$ in line 4 of Table IV.*

(ii) *In case (c)(ii) (that is, if $P' \notin \beta$), the $G(x, B_{P'})$-orbits in $B_{P'}$, in line 2 of Table IV, are $\{(P',\beta') : P, P' \in \beta'\}$ of length 5, $\{(P',\beta') : P' \in \beta', \beta' \cap \beta = \emptyset\}$ of length 6, and $\{(P',\beta') : P \notin \beta', P' \in \beta', |\beta' \cap \beta| = 2\}$ of length 10. In lines 3 and 4 of Table IV, the $G(x, B_{P'})$-orbits in $B_{P'}$ are as follows: $\{(P',\bar{\beta})\}$ (where $\bar{\beta} = \mathcal{P} \setminus \beta$), $\{(P',\beta') : P \notin \beta', P' \in \beta', \beta' \neq \bar{\beta}\}$, and $\{(P',\beta') : P, P' \in \beta'\}$, of lengths $1, 5, 5$ in line 3, and of lengths $1, 2^{d-1} - 1, 2^{d-1} - 1$ in line 4.*

Proof: By our comments before Lemma 3.2, and by Lemma 3.2 itself, we may identify $\mathcal{B}$ with the point set $\mathcal{P}$ of one of the 3-designs $\mathcal{D}$ of Table *IV*, lines 2, 3, 4. Let $B = B_P \in \mathcal{B}$ correspond to the point $P \in \mathcal{P}$. As we noted before the statement of Lemma 3.2, the 2-transitive actions of $G(B)$ on $\mathcal{B} \setminus \{B\}$ and on B are not equivalent and it follows that the action of $G(B)$ on B is isomorphic to its action on the blocks of $\mathcal{D}_P$. It follows that we may label each point $x \in B$ as (P,β), where $\beta \setminus \{P\}$ is the block of $\mathcal{D}_P$ to which it corresponds, in such a way that $G(B)$ acts naturally on the second coordinates. Thus we may label elements of the set V by flags (P,β) of $\mathcal{D}$ in such a way that G acts coordinate-wise and each part of $\mathcal{B}$ consists of the set of flags with a fixed first coordinate. Thus parts (a) and (b) are proved.

Let $x = (P,\beta) \in B$. By Lemma 3.2, $G(x) = G(P,\beta)$ is transitive on both $\beta \setminus \{P\}$ and $\mathcal{P} \setminus \beta$, and since the G-actions on $\mathcal{B}$ and $\mathcal{P}$ are equivalent, $G(x)$ has two orbits in $\mathcal{B} \setminus \{B\}$, namely $\mathcal{B}_1 := \{B_{P'} : P' \in \beta, P' \neq P\}$ and $\mathcal{B}_2 := \{B_{P'};\ P' \in \mathcal{P} \setminus \beta\}$. It follows, since $G(x)$ is transitive on $\Gamma(x)$, that $\Gamma(x)$ meets either each block in $\mathcal{B}_1$ or each block in $\mathcal{B}_2$ (but not both of these since $r \leq b-2$). Now points of B_P correspond to blocks of $\mathcal{D}_P$; and the block set of $\mathcal{D}(B_P)$, namely $\mathcal{B} \setminus \{B_P\}$, is the point set of $\mathcal{D}_P$. If $\Gamma(x)$ meets the parts of $\mathcal{B}_1$, then a point (P,β) and a block $B_{P'}$ of $\mathcal{D}(B_P)$ are incident in $\mathcal{D}(B_P)$ if and only if $P' \in \beta \setminus \{P\}$, and therefore $\mathcal{D}(B_P)$ is the dual design of $\mathcal{D}_P$. On the other hand if $\Gamma(x)$ meets the parts of $\mathcal{B}_2$ then $\mathcal{D}(B_P)$ is the dual design of $\mathcal{D}_P$ with incidence reversed - that is, $\mathcal{D}(B_P)$ is the complement of the dual of $\mathcal{D}_P$.

To prove part (d) we make a careful examination of the cases. Suppose first that $\mathcal{D}(B_P)$ is the dual design of $\mathcal{D}_P$ as in case (c) (i). Let $x = (P,\beta) \in B_P$ and $P' \in \beta \setminus \{P\}$. Then in all the cases of Lemma 3.2, $G(x, B_{P'}) = G(P, P', \beta)$ has three orbits in $B_{P'}$, namely the fixed point (P',β), the set of all (P',β') where β' contains both P and P', and the set of all (P',β') where β' contains P' but not

P. The orbit lengths are as stated in (d) (i). Now suppose that $\mathcal{D}(B_P)$ is the dual design of $\mathcal{D}_P$ with incidence reversed, as in case (c) (ii). This time consider $x = (P, \beta) \in B_P$ and $P' \notin \beta$. In lines 3 and 4 of Table IV, $\bar{\beta} = \mathcal{P} \setminus \beta$ is also a block of $\mathcal{D}$ and $P' \in \bar{\beta}$. Moreover $G(x, B_{P'}) = G(P, P', \beta)$ also fixes $(P', \bar{\beta}) \in B_{P'}$ and so $G(x, B_{P'}) = G(B_P, (P', \bar{\beta}))$; it can be checked that the $G(P, P', \beta)$-orbits in $B_{P'}$ are $\{(P', \bar{\beta})\}$, $\{(P', \beta');\ P, P' \in \beta'\}$, and $\{(P', \beta');\ P \notin \beta', P' \in \beta', \beta' \neq \bar{\beta}\}$. The orbit lengths are as stated in (d) (ii).

In line 2 of Table IV, $G(B_P, B_{P'}) = G(P, P')$ has orbits of lengths 5, 16 in B_P and in $B_{P'}$, and $x = (P, \beta)$ lies in a $G(P, P')$-orbit of length 16 (since $P' \notin \beta$). Thus (since 5 and 16 are coprime) $G(x, B_{P'}) = G(P, P', \beta) = A_5$ is still transitive on the $G(P, P')$-orbit of length 5 in $B_{P'}$. We claim that this is the set of vertices (P', β') such that β' contains P, P' and one point of $\beta \setminus \{P\}$. By [3, p.39] there are 60 blocks of $\mathcal{D}$ which meet β in two points; each such block contains 4 points of $\mathcal{P} \setminus \beta$ and consequently (since $G(\beta)$ is transitive on $\mathcal{P} \setminus \beta$) the point P' of $\mathcal{P} \setminus \beta$ lies in 15 of these blocks. Since each triple of points of $\mathcal{D}$ lies in a unique block of $\mathcal{D}$, there are exactly 5 blocks of $\mathcal{D}$ containing P, P' and one point of $\beta \setminus \{P\}$. Thus P' lies in exactly 10 blocks of $\mathcal{D}$ which do not contain P and which meet β in 2 points. Moreover, from [3, p.39], we see that there are 16 blocks of $\mathcal{D}$ which are disjoint from β and, as $G(\beta)$ is transitive on $\mathcal{P} \setminus \beta$, it follows that P' lies in exactly 6 blocks disjoint from β. It follows that the three subsets listed in (d) (ii) in this case are all invariant under $G(P, P', \beta) \cong A_5$. From our discussion above, we can now conclude that the set of size 5 is an orbit, and, since $G(P, P', \beta)$ is transitive on the unordered pairs from $\beta \setminus \{P\}$, the set of size 10 is also an orbit. From the character table for M_{22} in [3, p. 40], we see that elements of $G(P, P', \beta)$ of order 3 fix exactly 5 blocks of $\mathcal{D}$. Since they must fix two blocks of the 5 containing P and P', and they must fix one block of the 10 containing P' but not P and meeting β in two points, they can fix at most two blocks of the 6 containing P' and disjoint from β. It follows that they fix none of the latter blocks and that the set of size 6 is also a $G(P, P', \beta)$-orbit. ■

4 $\mathcal{D}(B)$ a symmetric 2-design with $t = 1$

We assume throughout this section that $(\Gamma, G, \mathcal{B})$ is a triple arising in case (d) of Corollary 3.1.2 with $t = 1$. We begin by looking briefly at the case $k = 2 < v$. This is not strictly necessary but gives some insight into the general case where $k = v - 1$. We then classify triples $(\Gamma, G, \mathcal{B})$ with $k = v - 1$, before analysing the non-degenerate case $3 \leq k \leq v - 2$.

Case $k = 2 < v$: Here each of the $\binom{v}{2}$ pairs of "points" in B lies in λ "blocks" of $\mathcal{D}(B)$, so $\binom{v}{2} \cdot \lambda = b = v$. Hence $v = 3, \lambda = 1, r = k = 2$, so $\mathcal{B} = \{B_0, B_1, B_2, B_3\}$ and each vertex $x \in B_i$ is joined to one vertex in each of two of the other three parts, but to no vertex of the third part, say $B_{i'}$. Thus each vertex $x \in V$ receives a natural label ii' $(0 \leq i \neq i' \leq 3)$, where $x \in B_i$ and $\Gamma(x) \cap B_{i'} = \emptyset$. In particular no vertex whose label has second coordinate i' is joined to any vertex whose label has i' as its first coordinate. The G-partition $\mathcal{B}$ is determined by this labelling; $G(B) = S_3$, so $G = S_4$ acting coordinate-wise on labels, as in Theorem 1.1 (b). There are just two possibilities for Γ: either

(i) any two adjacent vertices have the same second coordinate (and different first coordinates), so $\Gamma = 4 \cdot K_3$; or

(ii) any two adjacent vertices have labels involving four different coordinates, so $\Gamma = 3 \cdot C_4$.

The case $k = 2$ is instructive in that $k = 2$ implies $k = v - 1$ (since $v = 3$ in (i) and (ii)), and both the approach used when $k = 2$, and the outcomes (i) and (ii) constitute a useful 'first approximation' to the general case $k = v - 1$. Possibility (i) ($\Gamma = 4 \cdot K_3$) occurs in line 2 of Table II. We interpret the graph $\Gamma = 3 \cdot C_4$ in (ii) above in two different ways. The first interpretation is to observe that, since $b + 1 = 4$, the parts B_i could have been indexed with the points of the projective line $\mathrm{PG}(1, 3)$; the vertices of Γ would then have been labelled by ordered pairs of points from $\mathrm{PG}(1, 3)$, and $\infty 0$ would then be joined to both 12 and 21, whence Γ is visibly the cross ratio graph $\mathrm{CR}(3; 1)$ (as defined in Subsection 2.1); this exhibits $3 \cdot C_4$ as the first member of the family of graphs in line 4 of Table II. The second interpretation is to view the coordinates $\{0, 1, 2, 3\}$ in the vertex labels as the four points of an affine geometry $\mathrm{AG}(2, 2)$ of dimension $d = 2$, with ij joined to $i'j'$ in Γ if and only if i, j, i', j' are pairwise distinct and lie in a single 2-dimensional subspace! This exhibits $3 \cdot C_4 = (2^2 - 1) \cdot K_{2^{2-1}[2]}$ as the first member of the family of graphs in Table II line 5 with $\mathcal{D}'$ as in Table IV line 1.

Proposition 4.1 *Let Γ be a G-symmetric graph with vertex set V. Let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a G-partition of V with $|\mathcal{B}| = b + 1$ and each $|B_i| = v$, and with quotient $\Gamma_{\mathcal{B}} = K_{b+1}$. Suppose that $v = b$ and that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive. Suppose further that $t = 1$ and that either $k = 2 < v$ or $k = v - 1 \geq 2$. Then the vertex set V may be taken to be $\{ij : \ 0 \leq i, j \leq v, i \neq j\}$, where $x = ij$ if and only if $x \in B_i$ and $\Gamma(x) \cap B_j = \emptyset$. The G-invariant partition $\mathcal{B}$ is determined by this labelling and $\Gamma, G, \mathcal{B}$ are as in Theorem* 1.1 *(b) satisfying one of Table II line* 2, 4, *or* 5 *(with $\mathcal{D}'$ as in Table IV line* 1*).*

Proof: We have already seen that this result is true when $k = 2$, so we may assume that $k = v - 1 \geq 3$. Since $v = b$, we have $r = k = v - 1$ by Proposition 3.1 (d), so each vertex $x \in B_i \in \mathcal{B}$ is joined to one vertex in each of $r = v - 1$ parts $B_{i'} \neq B_i$. Moreover, if $B_{i'} \neq B_i$ and $\Gamma(x) \cap B_{i'} = \emptyset$, then x is the only vertex of B_i not joined to $B_{i'}$ (since $k = v - 1$). Thus each vertex $x \in V$ receives a natural label ii' as claimed with G acting coordinate-wise. In particular, the G-partition is determined by the labelling, and no vertex whose label has second coordinate j can be joined to a vertex whose label has first coordinate j. This leaves just two possible cases; we consider each in turn.

Suppose two adjacent vertices have labels with the same second coordinate; that is, suppose that some vertex $il \in B_i$ is joined to $jl \in B_j$, where $i \neq j$. Let $B := B_l = \{ll' : \ 0 \leq l' \leq v, l' \neq l\}$. Then $G(B)^B$ is 2-transitive, and the actions of $G(B)$ on $X = \{il : \ 0 \leq i \leq v, i \neq l\}$ and on B are equivalent. Hence the induced subgraph

$\langle X\rangle = K_v$, so Γ is the disjoint union of $v+1$ copies of K_v, with $G^{\mathcal{B}}$ 3-transitive, as in Table II line 2. Conversely, once the vertices have been labelled, the graph Γ in which vertices are joined if they have the same second coordinate has $\mathrm{Aut}\,(\Gamma,\mathcal{B}) = S_{v+1}$, so G may be any 3-transitive subgroup of S_{v+1}.

There remains the possibility that each pair of adjacent vertices x, y have labels involving four different symbols, say $x = 01,\ y = cd$. Let $B = B_0 = \{0i\ :\ 1 \le i \le v\}$. Each coordinate symbol "i" corresponds to a part $B_i \in \mathcal{B}$. As in the previous paragraph, $G(B)$ is 2-transitive on the set $X = \{i0 :\ 1 \le i \le v\}$ and hence G is 3-transitive on $\mathcal{B}$, that is, on the coordinate symbols $\{0, 1, \ldots, v\}$. Also $G(x) = G(01)$ is transitive on the remaining $v-1$ vertices in B, namely $\{0i :\ 2 \le i \le v\}$. Let $w = 0c$, for some $c > 1$. Then $G(x, w) = G(01c)$ fixes $c0$, so leaves B_c invariant; hence $G(x, w)$ fixes $\Gamma(x) \cap B_c = \{y = cd\}$, say, so fixes d. Thus G must be a 3-transitive group on $\{0, 1, \ldots, v\}$ in which the stabiliser $G(01c)$ of three points fixes a fourth point d. Checking the list of 3-transitive groups (for example in [1, 13]) we see that there are just two possible cases: either

(i) $\mathrm{PSL}\,(2, q) \le G \le \mathrm{P\Gamma L}\,(2, q)$ for some prime power $q \ge 3$, or

(ii) $G = \mathrm{AGL}\,(d, 2)$ for some $d \ge 2$, or $G = Z_2^4 \cdot A_7 < \mathrm{AGL}\,(4, 2)$ with $d = 4$.

The graphs arising in case (i) are classified in [9, Theorem 4.1]. Since $k \ge 3$ and since adjacent vertices have labels involving four different symbols, the only examples are cross-ratio graphs $\mathrm{CR}\,(q; d, s)$ or $\mathrm{TCR}\,(q; d, s)$ as defined in Definition 2.1. Since $t = 1$ it follows (see Remark 2.2 (d)) that $s = s(d)$. Thus in this case Γ, G satisfy line 4 of Table *II*.

Finally suppose that $G = \mathrm{AGL}\,(d, 2)$ or $Z_2^4 \cdot A_7 < \mathrm{AGL}\,(4, 2)$. The vertices of Γ are labelled by ordered pairs of distinct points of the affine geometry $\mathrm{AG}\,(d, 2)$. Moreover we showed above that in this case, if 01 is joined to cd, then $G(01c)$ fixes d; thus 01 is joined to cd if and only if $0, 1, c, d$ are the four points of a 2-dimensional subspace. Hence line 5 of Table II holds with $\mathcal{D}'$ as in line 1 of Table IV. Note that, even when $G = Z_2^4 \cdot A_7$, $G(01c)$ has only four fixed points in $\mathrm{AG}\,(d, 2)$, so the possibilities for G are also as listed. (Here $\Gamma = \mathrm{ColPairs}(\mathrm{AG}_2(d, 2)) \cong (2^d - 1) \cdot K_{2^{d-1}[2]}$, and each 2-dimensional subspace $\{0, 1, c, d\}$ in $\mathrm{AG}\,(d, 2)$ gives rise to three cycles of length 4 in Γ, for example, $\langle 01, cd, 10, dc\rangle = C_4$.) ■

We end this section by considering the remaining case $3 \le k \le v-2$, $t = 1$.

Proposition 4.2 *Let Γ be a G-symmetric graph with vertex set V. Let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a G-invariant partition of V with $|\mathcal{B}| = b+1$ and each $|B_i| = v$, and with quotient $\Gamma_{\mathcal{B}} = K_{b+1}$. Suppose that $v = b$ and that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive. Suppose further that $3 \le k \le v-2$, and $t = 1$. Then we may take V to be the set of flags $P\beta$ in a $3-(v+1, k+1, \lambda)$ design $\mathcal{D}$, where $\Gamma, G, \mathcal{B}, \mathcal{D}$ are as in Theorem* 1.1 *(c), and moreover satisfy one of Table III line* 1 *(with $\mathcal{D}$ as in Table IV line* 2, 3 *or* 4*), or line* 2 *(with $\mathcal{D}$ as in Table IV line* 3 *or* 4*).*

Proof: Now $\mathcal{D}(B)$ is a symmetric design with $3 \le k \le v-2$, and part (d) of Corollary 3.1.2 holds. It follows from Lemma 3.4 that V may be identified with the flags of a 3-design $\mathcal{D}$ such that each part $B \in \mathcal{B}$ is the set of flags on a certain point of $\mathcal{D}$; moreover $\mathcal{D}$ is as in one of lines 2–4 of Table IV. Thus each vertex x is identified with a flag $P\beta$, where P is a point of $\mathcal{D}$ and β is a block of $\mathcal{D}$ incident with P, and hence the first assertions of Theorem 1.1 (c) hold. Since $t = 1$, it follows that, if $x = P\beta$ is adjacent to $P'\beta'$, then $P \neq P'$ and $G(x, B_{P'}) = G(P, P', \beta)$ must fix $P'\beta'$ and hence must fix β'. By Lemma 3.4 (d), it follows that either

(i) $P\beta$ is adjacent to $P'\beta'$ if and only if $\beta = \beta'$ and β contains both P and P'; or

(ii) $P\beta$ is adjacent to $P'\beta'$ if and only if β and β' are disjoint.

In case (i) the graph Γ is the disjoint union of b' complete graphs $K_{k'}$, where b' is the number of blocks of $\mathcal{D}$ and k' is the number of points of $\mathcal{D}$ incident with a given block. It follows from Lemmas 3.2 and 3.4 that $\Gamma, G, \mathcal{B}, \mathcal{D}$ satisfy line 1 of Table II, with $\mathcal{D}$ as in one of lines 2–4 of Table IV. On the other hand in case (ii) above, the graph Γ is the disjoint union of $b'/2$ complete bipartite graphs $K_{k',k'}$, and by Lemmas 3.2 and 3.4, line 2 of Table II holds, with $\mathcal{D}$ as in one of lines 3 or 4 of Table IV. ■

5 $\mathcal{D}(B)$ a symmetric 2-design with $t \geq 2$

We assume throughout this section that $(\Gamma, G, \mathcal{B})$ is a triple arising in case (d) of Corollary 3.1.2 with $t \geq 2$. We begin by looking briefly at the cases $k = 2(= r)$ and $k = 3(= r)$. Both the analysis and the examples that arise emphasise the importance of the vertex labelling; we therefore make this labelling explicit at the outset. When $t \geq 2$, the 'degenerate case' $k = v - 1$ behaves differently for different groups G; hence our classification of the triples $(\Gamma, G, \mathcal{B})$ with $k = v - 1$ (Proposition 5.1) proceeds by considering each family of groups separately. Finally we deal with the non-degenerate case $3 \leq k \leq v - 2$ (Proposition 5.2) where we have available to us the results of Lemmas 3.2 and 3.4.

Case $k = 2$: Here each pair of points of B lies in a unique block of $\mathcal{D}(B)$ (and, since Corollary 3.1.2 (d) holds, there are no repeated blocks), so $\binom{v}{2} = b = v$. Hence $v = 3$, $k = v-1$, $\mathcal{B} = \{B_0, B_1, B_2, B_3\}$, $G(B) = S_3$, and $G = S_4$. As at the beginning of Section 4, each vertex x of Γ receives a unique label $x = ii'$, where $x \in B_i$, $i \neq i'$, and $\Gamma(x) \cap B_{i'} = \emptyset$. Let $x = 01$. Then $G(x) = G(01) = Z_2$ must act transitively on $\Gamma(x)$, so $|\Gamma(x) \cap B_2| = t \leq 1$. Hence $k = 2$ does not occur when $t \geq 2$.

Case $k = 3$: Here each pair of points in B lies in $\lambda \geq 1$ blocks of $\mathcal{D}(B)$, so $\binom{v}{2}\lambda = b\binom{k}{2} = v \cdot 3$. Hence (since $k = 3 < v$) either $v = 4, \lambda = 2$, or $v = 7, \lambda = 1$. Suppose first that $v = 4, \lambda = 2$; then $k = v - 1$, $\mathcal{D}(B)$ is the complete $2-(4,3,2)$ design, $G(B) = A_4$ or S_4, $G = A_5$ or S_5. Since $k = v-1$, each vertex x of Γ receives a unique label $x = ii'$ $(0 \leq i, i' \leq 4,\ i \neq i')$, where $x \in B_i$, and $\Gamma(x) \cap B_{i'} = \emptyset$. Let $x = 01$. Then $G(x, B_2)$ fixes 21 and 20, so $t \geq 2$ implies that $\Gamma(x) \cap B_2 = \{23, 24\}$ and that $G = S_5$. So $t = 2$, and $\Gamma = \Delta^*$ is obtained as the "$*$-transform" of a pair $(\Delta, \mathcal{B})$ with $t = 1$ from Proposition 4.1. By uniqueness of Γ, the graph $\Delta = (4+1)\cdot K_4$ (recall the definition of Δ^* from Definition 2.7), so Γ, G satisfy Theorem 1.1 (b), namely line 3 of Table II. Note that Γ also occurs as the cross-ratio graph CR $(4; d, 1)$, where $d \in \mathrm{GF}(4) \setminus \{0, 1\}$ in line 4 of Table II.

Suppose next that $v = 7, \lambda = 1$. Since $k = 3$, $\mathcal{D}(B)$ is PG $(2, 2)$ and $G(B) = \mathrm{PSL}(3, 2)$. Since G is a transitive extension of the 2-transitive group "$G(B)$ acting on the lines of PG $(2, 2)$", we have $G = \mathrm{AGL}(3, 2)$. The eight parts of $\mathcal{B}$ are the points of $\mathcal{D} = \mathrm{AG}(3, 2)$. Each line of $\mathcal{D}(B)$ corresponds to a "point" $B_i \in \mathcal{B} \setminus \{B\}$; and each point x of $\mathcal{D}(B)$ is determined by the three lines of $\mathcal{D}(B)$ which contain it, that is, the three points $B_i \in \mathcal{B} \setminus \{B\}$ such that $\Gamma(x) \cap B_i \neq \emptyset$. Since $k = 3 = r$, these three points B_i in $\mathcal{B} \setminus \{B\}$, together with B, form a hyperplane of $\mathcal{D} = \mathrm{AG}_2(3, 2)$. Thus each vertex x is naturally labelled by a flag $B\beta$ of $\mathcal{D} = \mathrm{AG}_2(3, 2)$, where $x \in B$, and β is the hyperplane of $\mathcal{D}$ which contains B and which gives rise to the line $\beta \setminus \{B\}$ which corresponds to x in the derived design $\mathcal{D}_B \cong \mathrm{PG}(2, 2)$. Thus $x = B\beta$ is joined to $x' = B'\beta'$ only if $B \neq B'$, $B \in \beta'$ and $B' \in \beta$. If the labels of adjacent vertices x, x' could have the same second coordinate $\beta = \beta'$, then $G(x, B')$ would fix x'; but $G(x, B')$ must act transitively on $\Gamma(x) \cap B'$, so $t = |\Gamma(x) \cap B'| = 1$. Hence when $t \geq 2$, $x = (B, \beta)$ is joined to $x' = (B', \beta')$ only if $\beta \neq \beta'$; so since $G(x, B')$ is transitive on the two hyperplanes which contain B and B' and which are not equal to β, we have $t = 2$ and adjacency in Γ is determined. This time $\Gamma = \Delta^*$ is the "$*$-transform" of a pair $(\Delta, \mathcal{B})$ with $t = 1$ from Proposition 4.2. By uniqueness of Γ, the graph $\Delta = (2^4 - 2) \cdot K_{2^2}$, so Γ, G satisfy Theorem 1.1 (c), namely line 1 of Table III with $\mathcal{D}$ in line 4 of Table IV.

These two examples illustrate what we expect to find in general: $k = 3, v = 4, \lambda = 2$ is an instance of the 'degenerate' case $k = v - 1$ (Proposition 5.1); $k = 3, v = 7, \lambda = 1$ is an instance of the non-degenerate case $3 \leq k \leq v - 2$ (Proposition 5.2). Many of the triples $(\Gamma, G, \mathcal{B})$ arising in Propositions 5.1 and 5.2 are the $*$-transforms of triples with $t = 1$ from Section 4. In addition we shall obtain cross-ratio graphs and a few sporadic examples.

We begin by considering the 'degenerate' case $k = v - 1$. Here the design $\mathcal{D}(B)$ is the complete $2 - (v, v - 1, v - 2)$ design.

Proposition 5.1 *Let Γ be a G-symmetric graph with vertex set V. Let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a G-partition of V with $|\mathcal{B}| = b + 1$ and each $|B_i| = v$, and with quotient $\Gamma_{\mathcal{B}} = K_{b+1}$. Suppose that $v = b$ and that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive. Suppose further that $t \geq 2$ and that either $k = 2 < v$, or $k = v - 1 \geq 2$. Then $k > 2$ and the vertex set V may be taken to be $\{ij : 0 \leq i, j \leq v, i \neq j\}$, where $x = ij$ if and only if $x \in B_i$ and $\Gamma(x) \cap B_j = \emptyset$. Further $\Gamma, G, \mathcal{B}$ are as in Theorem* 1.1 *(b), and satisfy one of Table II line* 3 *or* 4*, or line* 5 *or* 6 *(with $\mathcal{D}'$ as in Table IV line* 1, 2 *or* 3*).*

Proof: We have already seen that $k \neq 2$ and that, when $k = 3$, the conclusion holds (with $v = 4, G = S_5$). Thus we may assume that $v - 1 = k \geq 4$. Since $k = v - 1$, $\mathcal{D}(B)$ is the complete $2 - (v, v - 1, v - 2)$ design. In particular, for any $B_j \in \mathcal{B} \setminus \{B_i\}$, there is a unique $x \in B_i$ such that $\Gamma(x) \cap B_j = \emptyset$. Conversely, for each $x \in B_i$, there is a unique $B_j \in \mathcal{B} \setminus \{B_i\}$ such that $\Gamma(x) \cap B_j = \emptyset$. Thus each x receives a natural label ij as claimed with G acting coordinate-wise. In particular the G-invariant partition $\mathcal{B}$ is determined by the labelling, and no vertex whose label has second coordinate j can be joined to a vertex whose label has first coordinate j.

For $x = ij \in B_i$, the stabiliser $G(x)$ fixes B_i and B_j setwise: since G acts coordinate-wise on labels, $G(B_i, B_j) = G(ij) = G(x)$. Thus the actions of $G(B_i)$ on $\mathcal{B} \setminus \{B_i\}$ and on B_i are equivalent. In particular (since $G(B_i)^{B_i}$ is 2-transitive) G is 3-transitive on $\mathcal{B}$, and (since G acts coordinate-wise on labels) G acts faithfully on $\mathcal{B}$. Now $G(x)$ is transitive on $\mathcal{B} \setminus \{B_i, B_j\}$ and on $\Gamma(x)$, so $\Gamma(x)$ consists of t vertices from each part $C \in \mathcal{B} \setminus \{B_i, B_j\}$. Moreover, $\Gamma(x) \cap C$ is an orbit of $G(x, C)$ of length $t \geq 2$. Now $G(x, C)$ is the stabiliser in G of the three parts B_i, B_j and C. Consequently $G(x, C)^C$ is the stabiliser in the 2-transitive group $G(C)^C$ of two vertices, say u, u', and $\Gamma(x) \cap C$ is a $G(x, C)$-orbit in $C \setminus \{u, u'\}$ of length $t \geq 2$. In particular, $G(B_i, B_j, C)^C = G(x, C)^C \neq 1$. If $C = B_l$ then we may assume that u, u' are the vertices li, lj, respectively.

Suppose first that $G(x, C)$ is transitive on $C \setminus \{u, u'\}$, that is, that G is 4-transitive on $\{0, 1, \ldots v\}$ (so G is one of the groups listed in line 3 of Table II). Then $\Gamma(x) \cap C = C \setminus \{u, u'\}$, that is, ij is adjacent to ll' if and only if l' is distinct from i, j, l. So Γ is as in line 3 of Table II, that is, Γ is the $*$-transform of the graph in in line 2 of Table II.

Thus we may suppose that G is 3-transitive, but not 4-transitive, on $\mathcal{B}$, and also that the stabiliser of 3 points in this action is nontrivial (since $G(B_i, B_j, C)^{\mathcal{B}} \cong G(B_i, B_j, C) \neq 1$). Then G is one of the following groups.

(i) $\mathrm{PSL}\,(2, q) \leq G \leq \mathrm{P\Gamma L}\,(2, q)$ on the $v + 1 = q + 1$ points of the projective line $\mathrm{PG}\,(1, q)$, such that G is 3-transitive, and $G(\infty 01) \neq 1$ (so $q \geq 8$ since we are assuming that $v = q > 4$); or

(ii) $G = \mathrm{AGL}\,(d, 2)$ with $v + 1 = 2^d$ for some $d \geq 3$, or $v + 1 = 2^4$ and $G = Z_2^4 \cdot A_7 < \mathrm{AGL}\,(4, 2)$; or

(iii) $G = M_{22}$ or $\mathrm{Aut}\,(M_{22})$ with $v + 1 = 22$; or

(iv) $G = M_{11}$ with $v + 1 = 12$.

Suppose first that $\mathrm{PSL}\,(2, q) \leq G \leq \mathrm{P\Gamma L}\,(2, q)$, where $q = p^n$ for some prime p and $n \geq 2$. Then the vertices of Γ are labelled by ordered pairs of distinct points from the projective line $\mathrm{PG}\,(1, q)$. If the vertex x is labelled $\infty 0$, then $\Gamma(x) \cap B_1$ must be a single $G(\infty 01)$-orbit of length $t \geq 2$. Thus if $y \in \Gamma(x) \cap B_1$, then $y = 1d$, where $d \neq \infty, 0, 1$. It follows from [9, Theorem 4.1] that Γ is a cross-ratio graph $\mathrm{CR}\,(q; d, s)$ or $\mathrm{TCR}\,(q; d, s)$ for some divisor s of $s(d)$, where the subfield of $\mathrm{GF}\,(q)$ generated by d has order $p^{s(d)}$. Moreover, by [9, Theorems 3.4 and 3.7], $t = s(d)/s \geq 2$, and hence Γ, G satisfy line 4 of Table II.

Next suppose that $G = \mathrm{AGL}\,(d, 2)$ $(d \geq 3)$ or $Z_2^4 \cdot A_7 < \mathrm{AGL}\,(4, 2)$. The vertices of Γ are labelled by ordered pairs of distinct points of the affine geometry $\mathrm{AG}\,(d, 2)$. If 01 is joined to cd, then $G(01c)$ does not fix d (since $t \geq 2$); hence $d \notin \{0, 1\}$ and d is not equal to the fourth point c' of the affine plane spanned by $0, 1, c$. Now $G(01c)$ is transitive on $\mathrm{AG}\,(d, 2) \setminus \{0, 1, c, c'\}$. This is easily checked if $G = \mathrm{AGL}\,(d, 2)$, while if $G = Z_2^4 \cdot A_7$, then $G(01c) = A_4$ and each nonidentity element of this subgroup has exactly three fixed points in $\mathrm{AG}\,(4,2) \setminus \{0\}$ (see [3, p. 10]) so it must act regularly on the 12 points of $\mathrm{AG}\,(4,2) \setminus \{0,1,c,c'\}$. Hence 01 is joined to cd if and only if $0,1,c,d$ are distinct points and span an affine space of dimension 3. Thus Γ, G are as in line 6 of Table II (with $\mathcal{D}'$ as in line 1 of Table IV).

Now we treat the cases where $G = M_{22}$ or $\mathrm{Aut}\,(M_{22})$ $(v+1 = 22)$, or M_{11} $(v+1 = 12)$. Here the vertices of Γ are labelled by ordered pairs of distinct points of the design $\mathcal{D}$, where $\mathcal{D}$ is the 3-(22,6,1) Steiner system or the unique 3-(12,6,2) design respectively. Suppose that $x = ij \in B_i$ is joined to $y = i'j' \in B_{i'}$. Then $G(iji')$ has two fixed points in $B_{i'}$, namely $i'i$ and $i'j$, and two nontrivial orbits, namely the set C_1 of vertices $i'l$ where i,j,i',l lie in some block of $\mathcal{D}$, and the set C_2 of vertices $i'l$ where i,j,i',l are contained in no block of $\mathcal{D}$. Thus in each case we have exactly two graphs, corresponding to $\Gamma(x) \cap B_{i'}$ equal to C_1 or C_2 respectively, and line 5 or 6 of Table II holds (with $\mathcal{D}'$ as in line 2 or 3 of Table IV). ■

Finally we treat the "non-degenerate" case where $3 \le k \le v-2$.

Proposition 5.2 *Let Γ be a G-symmetric graph with vertex set V. Let $\mathcal{B} = \{B_0, B_1, B_2, \ldots\}$ be a G-partition of V with $|\mathcal{B}| = b+1$ and each $|B_i| = v$, and with quotient $\Gamma_{\mathcal{B}} = K_{b+1}$. Suppose that $v = b$ and that, for $B \in \mathcal{B}$, $G(B)^B$ is 2-transitive. Suppose further that $3 \le k \le v-2$, and $t \ge 2$. Then we may take V to be the set of flags in a $3-(v+1, k+1, \lambda)$ design $\mathcal{D}$, where $\Gamma, G, \mathcal{B}, \mathcal{D}$ are as in Theorem 1.1 (c), and moreover satisfy one of Table III line 3 (with $\mathcal{D}$ as in Table IV line 2, 3 or 4), or line 4 (with $\mathcal{D}$ as in Table IV line 3 or 4), or line 5 or 6 (with $\mathcal{D}$ as in Table IV line 2).*

Proof: The 2-design $\mathcal{D}(B)$ is symmetric with $3 \le k \le v-2$, and part (d) of Corollary 3.1.2 holds. It follows from Lemma 3.4 that V may be identified with the flags of a 3-design $\mathcal{D}$ as claimed, where $\mathcal{D}$ is as in one of lines 2-4 of Table IV. Moreover, since $t \ge 2$, each vertex $x = P\beta$ is adjacent to all the points of some nontrivial orbit (of length t) of $G(x, B_{P'}) = G(P, P', \beta)$ in $B_{P'}$. By Lemma 3.4 (d), it follows that either

(i) $\mathcal{D}(B_P)$ is the dual design of the derived design $\mathcal{D}_P$, so that, for $P' \neq P$, we have $\Gamma(B_{P'}) \cap B_P = \{P\beta' : \ P' \in \beta'\}$; or

(ii) $\mathcal{D}(B_P)$ is the dual design of the derived design $\mathcal{D}_P$ with incidence reversed so that, for $P' \neq P$, $\Gamma(B_{P'}) \cap B_P = \{P\beta' : \ P' \notin \beta'\}$.

In case (i), $x = P\beta$ is adjacent to a vertex of $B_{P'}$ if and only if $P' \in \beta$. Thus if $x = P\beta$ is adjacent to $P'\beta'$, then we must have $P \in \beta'$. Moreover,by Lemma 3.4 (d) (i), $G(x, B_{P'})$ is transitive on the set of all points $P'\beta'$ such that β' contains P, and hence in this case, $P\beta$ is adjacent to $P'\beta'$ if and only if $P \neq P'$, $\beta \neq \beta'$, and β, β' contain both P and P'. Thus Γ is the $*$-transform of the corresponding graph in Proposition 4.2, namely of a graph from line 1 of Table III, and hence line 3 of Table III holds for Γ, and the values of t can be read off from the orbit lengths in Lemma 3.4 (d) (i).

In case (ii), $x = P\beta$ is adjacent to a vertex of $B_{P'}$ if and only if $P' \notin \beta$. Thus if $x = P\beta$ is adjacent to $P'\beta'$, then we must have $P \notin \beta'$. Suppose first that $\mathcal{D}$ is in line 3 or 4 of Table IV, that is $\mathcal{D}$ is either the unique 3-(12,6,2) design or the affine geometry of points and hyperplanes of AG $(d, 2)$, respectively. Then by Lemma 3.4 (d) (ii), $G(x, B_{P'})$ is transitive on the set of all vertices $P'\beta'$ such that $P \notin \beta'$ and β' is not the complement $\bar{\beta}$ of β, and $G(x, B_{P'})$ fixes $P'\bar{\beta}$. Since $t > 1$ it follows that $P\beta$ is adjacent to $P'\beta'$ if and only if $P \neq P'$, $P \notin \beta'$, $P' \notin \beta$, and β' is not the complement $\bar{\beta}$ of β. This graph Γ is the $*$-transform of the corresponding graph in Proposition 4.2, namely of a graph from line 2 of Table III, and hence line 4 of Table III holds for Γ, and the values of t can be read off from the orbit lengths in Lemma 3.4 (d) (ii).

Finally suppose that $\mathcal{D}$ is the 3-(22,6,1) Steiner system as in line 2 of Table IV. Then by Lemma 3.4 (d) (ii), $G(x, B_{P'})$ has two orbits on the set of all vertices $P'\beta'$ such that $P \notin \beta'$, namely the set of those for which $\beta \cap \beta' = \emptyset$ (of length $t = 6$) and the set of those for which $|\beta \cap \beta'| = 2$ (of length $t = 10$). Thus there are two possible rules for incidence, giving two graphs Γ in this case. For the first graph, $P\beta$ is adjacent to $P'\beta'$ if and only if $P \neq P'$, $P \notin \beta'$, $P' \notin \beta$, and $\beta \cap \beta' = \emptyset$, while for the second graph, $P\beta$ is adjacent to $P'\beta'$ if and only if $P \neq P'$, $P \notin \beta'$, $P' \notin \beta$, and $|\beta \cap \beta'| = 2$. Thus we have line 5 or 6 of Table III respectively. ■

The results of this section complete the proofs of Theorem 1.1, noting that the examples in Table II line 1 arise from Corollary 3.1.2 (b), and that the assertions of Theorem 1.1 (b) hold for these examples.

References

[1] P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.

[2] Y. Q. Chen, T. Fang and S. Zhou, Affine flag graphs and classification of a family of symmetric graphs with complete quotients, (submitted).

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, England, 1985.

[4] T. Fang, X. G. Fang, B. Xia and S. Zhou, *A family of symmetric graphs with complete quotients*, *Electronic J. Combin.* **23** (2) (2016), P2.27, 40 pages.

[5] John D. Dixon and Brian Mortimer, *Permutation groups*, Springer-Verlag, New York, 1996.

[6] A. Gardiner and C. E. Praeger, A geometrical approach to imprimitive symmetric graphs, *Proc. London Math. Soc.* **71** (3) (1995), 524–546.

[7] A. Gardiner and C. E. Praeger, Topological covers of complete graphs, *Math. Proc. Cambridge Philos. Soc.* **123** (1998), 549–559.

[8] A. Gardiner and C. E. Praeger, Symmetric graphs with complete quotients, unpublished. Available on arXiv: 1403.07308.

[9] A. Gardiner, C. E. Praeger and S. Zhou, Cross-ratio graphs, *J. London Math. Soc.* **64** (2001), 257–272.

[10] M. Giulietti, S. Marcugini, F. Pambianco and S. Zhou, Unitary graphs and classification of a family of symmetric graphs with complete quotients, *J. Alg. Combin.* **38** (2013), 745–765.

[11] C. D. Godsil, R. A. Liebler and C. E. Praeger, Antipodal distance transitive covers of complete graphs, *European J. Combin.* **19** (1998), 455–478.

[12] M. A. Iranmanesh, C. E. Praeger and S. Zhou, Finite symmetric graphs with two-arc transitive quotients, *J. Combin. Theory Ser. B* **94** (2005), 79–99.

[13] M. W. Liebeck, The affine permutation groups of rank 3, *Proc. London Math. Soc.* **54** (3) (1987), 477-516.

[14] C. H. Li, C. E. Praeger and S. Zhou, A class of finite symmetric graphs with 2-arc transitive quotients, *Math. Proc. Cambridge Philos. Soc.* **129** (1) (2000), 19–34.

[15] C. H. Li, C. E. Praeger and S. Zhou, Imprimitive symmetric graphs with cyclic blocks, *European J. Combin.* **31** (2010), 362–367.

[16] Z. Lu, and S. Zhou, Finite symmetric graphs with two-arc transitive quotients. II. *J. Graph Theory* **56** (2007), 167–193.

[17] D. E. Taylor, Two-Graphs and doubly transitive groups, *J. Combin. Theory Ser. A* **61**, (1992), 113–122.

[18] G. Xu and S. Zhou, Solution to a question on a family of imprimitive symmetric graphs. *Bull. Aust. Math. Soc.* **82** (2010), 79–83.

[19] S. Zhou, Constructing a Class of Symmetric Graphs, *Europ. J. Combin.* **23** (2002), 741–760.

[20] S. Zhou, Almost covers of 2-arc transitive graphs, *Combinatorica* **24** (2004), 731–745. [Erratum: *Combinatorica* **27** (2007), 745–746.]

[21] S. Zhou, On a class of finite symmetric graphs. *Europ. J. Combin.* **29** (2008), 630–640.

[22] S. Zhou, Classification of a family of symmetric graphs with complete 2-arc-transitive quotients, *Discrete Math.* **309** (2009), 5404–5410.

(Received 1 June 2017)