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Pancyclic type properties of claw-free P_6-free graphs

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Abstract

Given integers k and m, a graph G on n vertices is said to be (k, m)-*pancyclic* if every set of k vertices in G is contained in a cycle of length r for each integer $r \in \{m, m + 1, \cdots, n\}$. This property, which generalizes the notion of a vertex pancyclic graph, provides one way to measure the prevalence of cycles in a graph. The property was introduced by Faudree, Gould, Jacobson, and Lesniak (2004). We show that any 2-connected claw-free P_6-free graph is $(k, 3k + 4)$-pancyclic for every integer $k \geq 1$. We provide an infinite family of graphs that shows this result is best possible.

1 Introduction

Let $G = (V, E)$ denote a simple graph of order $n \geq 3$. We say G is *pancyclic* if G contains a cycle of each possible length, from 3 up to n. The notion of vertex pancyclicity was defined by Bondy [2]. The graph G is *vertex pancyclic* if every vertex of G is contained in a cycle of each length, from 3 to n. We consider the property (k, m)-pancyclicity, defined in 2004 by Faudree et al. [11], which is a generalization of vertex pancyclicity.

Definition 1.1 (Faudree, Gould, Jacobson, and Lesniak [11])**.** Given integers k and m with $0 \leq k \leq m \leq n$, a graph G of order n is said to be (k, m)-*pancyclic* if for any k-set $S \subseteq V$ and any integer r with $m \leq r \leq n$, there exists a cycle of length r in G that contains S.

Whenever $m > n$ or $k > n$, we define (k, m)-pancyclicity to be the same as hamiltonicity. Note that (k, n)-pancyclicity represents hamiltonicity, $(0, 3)$-pancyclicity represents pancyclicity, and $(1, 3)$-pancyclicity represents vertex pancyclicity. Note also that whenever a graph is (k, m)-pancyclic for some $k \geq 1$, then it must also be $(k - 1, m)$-pancyclic and $(k, m + 1)$-pancyclic.

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Implications of Ore-type degree conditions for this type of generalized pancyclicity have recently been explored [11, 7]. Relationships between hamiltonian type properties and bounds on the quantity $\sigma_2(G) = \min\{d(x) + d(y) : xy \notin E(G)\}$ have been studied extensively. Ore [13] proved that if $\sigma_2(G) \geq n$, then G is hamiltonian. In 1971, Bondy [3] showed that the condition $\sigma_2(G) \geq n+1$ guarantees G is pancyclic. Faudree et al. (2004) showed that this bound ensures much more than pancyclicity. Their result, which uses the notion of (k, m)-pancyclicity, provides insight into the prevalence of cycles in such a graph.

Theorem A (Faudree, Gould, Jacobson, and Lesniak [11])**.** *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n+1$, then G is $(k, 2k)$-pancyclic for each integer $k \geq 2$.*

Another technique that has been employed to ensure hamiltonian-type properties is the forbidding of a subgraph or subgraphs. Given a graph H, we say G is *H-free* if G does not contain H as an induced subgraph. In this context, H is called a *forbidden subgraph*. If $\mathcal{F}$ is a family of graphs, we say G is *$\mathcal{F}$-free* if G is F-free for each $F \in \mathcal{F}$. Many results in hamiltonian theory that make use of forbidden subgraph conditions involve the star $K_{1,3}$, also known as the claw (see [9] for a survey that includes results in this area).

In 2015, it was shown that if only claw-free graphs are considered, we may lower the $\sigma_2(G)$ bound to n in Theorem A and simultaneously guarantee $(k, k+3)$-pancyclicity as opposed to $(k, 2k)$-pancyclicity.

Theorem B ([7])**.** *Let G be a claw-free graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is $(k, k+3)$-pancyclic for each integer $k \geq 1$.*

1.1 Pairs of Forbidden Subgraphs

For an integer $i \geq 1$ let P_i denote a path on i vertices, and for an integer $j \geq 3$ let C_j denote a cycle on j vertices. A number of hamiltonian-type results have been obtained involving forbidden families of subgraphs, such as the following result due to Broersma and Veldman [4] in 1990.

Theorem C (Broersma and Veldman [4])**.** *If G is a 2-connected graph that is $\{K_{1,3}, P_6\}$-free, then G is hamiltonian.*

In fact, it was shown by Faudree et al. in [12] that if such a graph has order $n \geq 10$, then it must be pancyclic.

Theorem D (Faudree, Ryjáček, and Schiermeyer [12])**.** *If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph of order $n \geq 10$, then G is pancyclic.*

Given integers $i, j, k \geq 0$, let $N(i, j, k)$ denote the generalized net, or the graph obtained by taking a triangle and three disjoint paths P_i, P_j, and P_k, and for each path, joining by an edge an end vertex from the path and a distinct vertex of the triangle. The net, denoted N, is the graph $N(1, 1, 1)$. The bull, denoted B, represents the graph $N(1,1,0)$. The wounded, denoted W, is the graph $N(2,1,0)$. Also, Z_i denotes the graph $N(i,0,0)$.

A characterization of all pairs of subgraphs that, when forbidden, imply hamiltonicity in 2-connected graphs of order $n \geq 10$ was given in Faudree and Gould [10]. Their result extended an earlier characterization by Bedrossian [1] that used graphs of small order to eliminate the pair $\{K_{1,3}, Z_3\}$.

Theorem E (Faudree and Gould [10]). *Let R and S be connected graphs ($R, S \neq P_3$) and let G be a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is one of the graphs $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$, or W.*

Since (k,m)-pancyclicity implies hamiltonicity, the ten pairs of forbidden subgraphs in Theorem E are the only pairs that could ensure (k,m)-pancyclicity for integers $k \leq m$ in 2-connected graphs. For each $k \geq 1$ and each of the nine pairs $\{K_{1,3}, S\}$ where $S \in \{C_3, P_4, P_5, Z_1, Z_2, Z_3, B, N, W\}$, the smallest integer m such that any 2-connected $\{K_{1,3}, S\}$-free graph is guaranteed to be (k,m)-pancyclic was given in [6, 8].

Theorem F ([6]). *Let G be a 2-connected $K_{1,3}$-free graph of order $n \geq 10$.*

(i) If $G \neq C_n$ is Z_1-free, then G is $(1,3)$-pancyclic, $(2,4)$-pancyclic, $(3,4)$-pancyclic, and (k,k)-pancyclic for $k \geq 4$.

(ii) If G is P_4-free, then G is $(1,4)$-pancyclic and $(k,k+2)$-pancyclic for $k \geq 2$.

(iii) If $G \neq C_n$ is Z_2-free, then G is $(1,4)$-pancyclic and $(k,3k)$-pancyclic for $k \geq 2$.

(iv) If G is S-free for some $S \in \{C_3, Z_3, B, N, W\}$ and $k \geq 0$, then G is (k,n)-pancyclic.

These results are best possible under the given conditions.

Theorem G ([8]). *Let G be a 2-connected $\{K_{1,3}, P_5\}$-free graph on $n \geq 5$ vertices. Then G is $(1,5)$-pancyclic and $(k,3k)$-pancyclic for $k \geq 2$. These results are best possible under the given conditions.*

In this paper, we complete the investigation of (k,m)-pancyclicity implied by forbidden pairs in 2-connected graphs by extending Theorem D in a natural way. We explore the prevalence of cycles in 2-connected $\{K_{1,3}, P_6\}$-free graphs. In particular, we show that such graphs are guaranteed to be not only pancyclic but, in fact, $(k, 3k+4)$-pancyclic for each integer $k \geq 1$. We also provide an example that shows this result is best possible.

1.2 Notation

For terms and notation not defined here, we refer the reader to [5]. For a vertex $v \in V$, we denote by $d(v)$ the degree of v, and by $N(v)$ the neighborhood of v. Given a subgraph H of G and a vertex $v \in V$, we let $N_H(v) = N(v) \cap V(H)$, and $d_H(v) = |N_H(v)|$. For $S \subseteq V$, let $N(S) = \{v \in V - S : vh \in E(G) \textit{ for some } h \in S\}$. Given a vertex u and a subgraph H in G such that $u \notin V(H)$, a (u, H)-path is any path in G from u to a vertex $v \in V(H)$.

Given a path P, we denote by (P) the set of all internal vertices of P, that is $V(P)$ minus the end vertices of P. If the end vertices of P are u and v, we denote by $[P]$ the set $V(P) = (P) \cup \{u, v\}$. Given a cycle C and a vertex $v \in V(C)$, we impose an orientation on C and let v^- (v^+) denote the vertex that appears directly before (after) v on C. We let xCy denote the path from x to y along C in the direction of the imposed orientation, while xC^-y will denote the path from x to y in the opposite direction along C.

2 Properties of Claw-free, P_6-free Graphs

The goal of this paper is to prove the following.

Theorem 2.1. *If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 7$ vertices, then G is $(k, 3k+4)$-pancyclic for all $k \geq 1$. This result is best possible under the given conditions.*

We begin with several definitions and lemmas that will establish useful properties of $\{K_{1,3}, P_6\}$-free graphs.

Definition 2.1. Given a cycle C and a vertex $x \in V(G)$, we say C *absorbs* x if there exists a cycle C' in G with vertex set $V(C) \cup \{x\}$. Given $X \subseteq V(G)$, we say C *absorbs* X if there exists a cycle C' in G with vertex set $V(C) \cup X$. In this context, we say C absorbs x (or X) *via* C'.

Given cycles C and C' with $V(C) = V(C')$, note that C absorbs x (or X) if and only if C' absorbs x (or X).

Definition 2.2. Given a cycle C in G, a set of $m \geq 2$ vertices $\{z_1, \cdots, z_m\} \subseteq V(G) - V(C)$ is called a *tab* of C if $z_1 z_2 \cdots z_m$ is a path in G and there exist distinct vertices $u, v \in V(C)$ such that $N_C(z_1) = \{u\}$ and $N_C(z_m) = \{v\}$. An *m-tab* of C is a tab of C with cardinality m.

The following lemma guarantees that any cycle C in G absorbs each of its tabs, as well as each vertex $z \in V - V(C)$ with $d_C(z) \geq 2$.

Lemma 2.1. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices, and let C be a cycle in G. If $\{z_1, z_2, \cdots, z_m\} \subseteq V - V(C)$ and there are distinct vertices $u, v \in V(C)$ such that $u z_1 z_2 \cdots z_m v$ is a path in G, then C absorbs $\{z_1, z_2, \cdots, z_m\}$. In particular, we have the following.*

(a) If $z \notin V(C)$ and $d_C(z) \geq 2$, then C absorbs z.

(b) If T is a tab of C, then C absorbs T.

Proof. Let C be a cycle of length j. We first observe that it is sufficient to prove parts (a) and (b). Now if $\{z_1, z_2, \cdots, z_m\}$ is a tab of C, then the result follows from part (b); otherwise $d_C(z_1) \geq 2$ without loss of generality, and C absorbs z_1 by part (a). Repeating this argument as needed, we see that C absorbs $\{z_1, z_2, \cdots, z_m\}$.

We now prove parts (a) and (b). We will assume $j \geq 8$, because both parts are easy to verify when $3 \leq j \leq 7$, using the fact that G is claw-free.

Proof of part (a). Suppose the conclusion of part (a) is false. Since for every cycle $\hat{C}$ with $V(\hat{C}) = V(C)$, C absorbs z if and only if $\hat{C}$ absorbs z, we may pick distinct vertices $u, v \in V(C) \cap N(z)$ and assume without loss of generality that

$$|(uCv)| = \min\{|(a\hat{C}b)| : \hat{C} \text{ is a cycle in } G \text{ with } V(\hat{C}) = V(C), \text{ and } a, b \in V(C) \cap N(z) \text{ are distinct}\}.$$

Now u and v do not occur consecutively on C, for otherwise part (a) clearly holds. The path uCv also satisfies $N(z) \cap (uCv) = \emptyset$ by our choice of u and v.

Since G is claw-free, we must have $u^-u^+, v^-v^+ \in E$. Now $uv^- \notin E$, for otherwise C absorbs z via $uzvCu^-u^+Cv^-u$. This implies $v^- \neq u^+$. By symmetry, we have $uv^+ \notin E$ and $v^+ \neq u^-$. Also $uv^{--} \notin E$, or else C absorbs z via $uzvv^-v^+Cu^-u^+Cv^{--}u$. By symmetry, $uv^{++} \notin E$. This implies $v^{--}, v^{++} \notin \{u^+, u^-\}$, and therefore $|(uCv)| \geq 3$. Due to the minimality of $|(uCv)|$, we must have $v^+, v^{++} \notin N(z)$.

If u has consecutive neighbors w_1 and w_2 occurring on uCv in that order, then the cycle $\hat{C} = vCu^-u^+Cw_1uw_2Cv$ whose vertex set is $V(C)$ contains a path $u\hat{C}v$ which is shorter than uCv, contradicting the minimality of $|(uCv)|$. Therefore u does not have consecutive neighbors on uCv.

Let $x \in (uCv)$ be the unique vertex such that $ux \in E$ and $uw \notin E$ for all $w \in (xCv)$. Since no two neighbors of u are consecutive on uCv, we have $ux^- \notin E$. In particular, this implies $x \neq u^{++}$. Since $uv^-, uv^{--} \notin E$, we also have $x \notin \{v^-, v^{--}\}$. We consider two cases.
Case 1. Suppose $x \neq u^+$.

Since G is claw-free, we have $x^-x^+ \in E$. Now $xv^- \notin E$, for otherwise C absorbs z via the cycle $uzvCu^-u^+Cx^-x^+Cv^-xu$. Similarly $xv^+ \notin E$ since otherwise C absorbs z via $uzvC^-x^+x^-C^-u^+u^-C^-v^+xu$, and $xv^{--} \notin E$ or else C absorbs z via $uzvv^-v^+Cu^-u^+Cx^-x^+Cv^{--}xu$. Since $xv^{--} \notin E$, we have $x \neq v^{---}$, and thus $uv^{---} \notin E$.

Let $y \in (xCv)$ be such that $xy \in E$ and $xw \notin E$ for all $w \in (yCv)$. Such a vertex y must exist since $xx^+ \in E$. Note that $y \in (xCv^{--})$. If there is a vertex $\alpha \in (yCv)$ such that $y\alpha^- \in E$ and $y\alpha \notin E$, then $\{z, u, x, y, \alpha^-, \alpha\}$ induces a P_6. This implies $y\alpha \in E$ for all $\alpha \in (yCv)$. Now $yv^+ \in E$, for otherwise $\{z, u, x, y, v^-, v^+\}$ induces a

P_6. Then $y^+v^+ \in E$, or else $\{y, x, y^+, v^+\}$ induces a claw. But now C absorbs z via $uzvC^-y^+v^+Cu^-u^+Cx^-x^+Cyxu$.

Case 2. Suppose $x = u^+$. Thus $N(u) \cap (u^+Cv) = \emptyset$.

We have $u^+v^+ \notin E$ or else C absorbs z via $uzvC^-u^+v^+Cu$, and $u^+v^{++} \notin E$ since otherwise C absorbs z via $uzvv^+v^-C^-u^+v^{++}Cu$. Let $y \in N(u^+) \cap (u^+Cv)$ be such that $u^+w \notin E$ for all $w \in (yCv)$.

Suppose $(yCv) \not\subset N(y)$. This implies $y \notin \{v^{--}, v^-\}$. Choose $w \in (yCv) - N(y)$ so that the path yCw is shortest possible. Hence $(yCw) \subset N(y)$. But now $\{z, u, u^+, y, w^-, w\}$ induces a P_6. Therefore it must be the case that $(yCv) \subset N(y)$.

Now $yv^+ \in E$, or else $y \neq v^-$ and $\{z, u, u^+, y, v^-, v^+\}$ induces a P_6. Similarly $yv^{++} \in E$ or else $\{z, u, u^+, y, v^+, v^{++}\}$ induces a P_6. Since $u^+w \notin E$ for all $w \in (yCv) \cup \{v^+, v^{++}\}$, it must be the case that $[yCv^-] \cup \{v^+, v^{++}\}$ induces a clique, in order to avoid a claw centered at y. Note that $v^{+++} \neq u^-$, for otherwise C absorbs z via $vzuu^-u^+Cyv^{++}v^+y^+Cv$.

Suppose $[v^{+++}Cu^{--}] \not\subset N(y)$. Choose $w \in [v^{+++}Cu^{--}] - N(y)$ so that the path yCw is shortest possible. Hence $[v^+Cw^-] \subset N(y)$. Now if $u^+\gamma \in E$ for some $\gamma \in (v^{++}Cw)$, then C absorbs z via $vzuC^-\gamma u^+Cy\gamma^-C^-v^+y^+Cv$. Therefore $u^+\gamma \notin E$ for all $\gamma \in (v^{++}Cw)$. Then in order to avoid a claw centered at y, it must be the case that $[yCv^-] \cup [v^+Cw^-]$ induces a clique.

Now $u^+w \notin E$, for otherwise C absorbs z via $uzvCw^-v^-C^-u^+wCu$. Also $uw \notin E$, or else C absorbs z via $vzuwCu^-u^+Cv^-w^-C^-v$. Similarly $uw^- \notin E$. Furthermore $zw \notin E$, since otherwise C absorbs z via $vzwCv^-w^-C^-v$. Similarly $zw^- \notin E$. But now $\{z, u, u^+, y, w^-, w\}$ induces a P_6.

Therefore we must have $[v^{+++}Cu^{--}] \subset N(y)$. However, now C absorbs z via the cycle $vzuu^-u^+Cyu^{--}C^-v^+y^+Cv$.

Proof of part (b). Let $T = \{z_1, z_2, \cdots, z_m\}$ be a tab of C such that $P = z_1z_2\cdots z_m$ is a path in G, $N_C(z_1) = \{u\}$, and $N_C(z_m) = \{v\}$. We may clearly follow the same argument from the proof of part (a), making the following changes. Replace each occurrence of the phrase "absorbs z" with "absorbs T"; replace each occurrence of $N(z)$ with $N(\{z_1, z_m\})$; replace each occurrence of the path uzv (or vzu) with the path uPv (or vP^-u); for each reference to a set of vertices that induces a P_6, replace z with z_1; and near the end of the proof, rather than argue that $zw, zw^- \notin E$, simply note that $z_1w, z_1w^- \notin E$ because T is a tab of C. □

The context for the next few results is as follows. Given a 2-connected $\{K_{1,3}, P_6\}$-free graph G, suppose C is a cycle in G and let $z \in V - V(C)$. Since G is 2-connected, we may pick a pair of (z, C)-paths P and Q that are vertex-disjoint except for z, such that $|(P)| + |(Q)|$ is minimal among all such sums. Let u and v denote the end vertices of P and Q, respectively, on C. Note that $|(P)|, |(Q)| \leq 3$ since G is P_6-free. We now prove Lemmas 2.2, 2.3, 2.4, and 2.5 which will allow us to handle the different possible values of $|(P)|$ and $|(Q)|$ in such a situation.

Lemma 2.2. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices, and let*

C, z, P, Q, u, and v be as described in the preceding paragraph. If $P = uu_1u_2u_3z$, then $(V(C) - \{v\}) \cup \{u_1\}$ induces a clique in G, and C absorbs u_1.

Proof. Note that $V(P)$ induces a P_5 in G. By the minimality of $|(P)| + |(Q)|$, we have $xz, xu_3, xu_2 \notin E$ for all $x \in V(C) - \{v\}$.

Suppose $[v^+Cu] \not\subset N(u_1)$, and choose $w \in [v^+Cu]$ so that $wu_1 \notin E$ and the path wCu is as short as possible. But now $u_1w^+ \in E$ by the minimality of $|(wCu)|$, and $\{z, u_3, u_2, u_1, w^+, w\}$ induces a P_6. Thus $[v^+Cu] \subset N(u_1)$. By symmetry we also have $[uCv^-] \subset N(u_1)$, and so $V(C) - \{v\} \subset N(u_1)$. Avoiding a claw centered at u_1, it must be the case that $V(C) - \{v\}$ induces a clique in G. Also $u_1x \in E$ for some $x \in \{u^-, u^+\}$, so C absorbs u_1. □

Lemma 2.3. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices, and let C, z, P, Q, u, and v be as previously described. If $P = uu_1u_2z$, then either $V(C) - \{v\}$ induces a clique in G, or C absorbs u_1.*

Proof. Suppose C does not absorb u_1. Now by the minimality of $|(P)| + |(Q)|$, we have $xz, xu_2 \notin E$ for all $x \in V(C) - \{v\}$. Also $xu_1 \notin E$ for all $x \in V(C) - \{u\}$, for otherwise C absorbs u_1 by Lemma 2.1.

Clearly $ux \in E$ for all $x \in (vCu)$, for otherwise there exists a vertex $w \in (vCu)$ satisfying $uw \notin E$ and $uw^+ \in E$, which implies that $w \neq u^-$ and $\{z, u_2, u_1, u, w^+, w\}$ induces a P_6. By symmetry $ux \in E$ for all $x \in (uCv)$, and so $ux \in E$ for all $x \in V(C) - \{u, v\}$. Avoiding a claw centered at u, it must be the case that $V(C) - \{v\}$ induces a clique in G. □

Lemma 2.4. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices, and let C, z, P, Q, u, and v be as previously described. If $|(P)| = 2$, then there is no induced P_5 that occurs consecutively on C.*

Proof. Let X denote a path $x_1x_2x_3x_4x_5 = x_1Cx_5$ on C that is an induced P_5. Let $P = uu_1u_2z$, and let $N_Q(v) = v'$. We begin with the following claim.

Claim 2.1. *If there exists a vertex $y \in V(C)$ such that $yu_2 \in E$ or $yz \in E$, then $y = v$ and $N_C(v') = \{v\}$.*

Proof. By the minimality of $|(P)| + |(Q)|$, we have $az, au_2 \notin E$ for all $a \in V(C) - \{v\}$, so such a vertex y must equal v. Let Q' denote the induced path from z to v' on Q. Now suppose there exists a vertex $w \in N_C(v') - \{v\}$. If $vu_2 \in E$, then the pair of paths $\{zu_2v, Q'w\}$ contradicts the minimality of $|(P)| + |(Q)|$. If $vz \in E$, then the pair of paths $\{zv, Q'w\}$ contradicts the minimality of $|(P)| + |(Q)|$. So in either case, we must have $N_C(v') - \{v\} = \emptyset$, and thus $N_C(v') = \{v\}$. This completes the proof of the claim. □

We now consider two cases corresponding to whether or not u_1 has a neighbor on X.

Case 1. Suppose $u_1x \in E$ for some $x \in V(X)$.

If there exists a vertex $y \in V(X)$ such that $yu_2 \in E$ or $yz \in E$, then $y = v$ and $N_C(v') = \{v\}$ by Claim 2.1. If $v \in \{x_2, x_3, x_4\}$, then the set $\{v, v', v^-, v^+\}$ induces a claw centered at v. Hence $v = x_5$ without loss of generality. But this is a contradiction, since now the path $x_1x_2x_3x_4x_5v'$ is an induced P_6. Therefore we must have $yu_2, yz \notin E$ for all $y \in V(X)$. We now consider three possibilities.

Suppose $u_1x_1 \in E$. Then for each $w \in \{x_3, x_4, x_5\}$ we have $wu_1 \notin E$, since otherwise $\{u_1, u_2, x_1, w\}$ induces a claw centered at u_1. Thus $u_1x_2 \in E$, or else the set $\{x_5, x_4, x_3, x_2, x_1, u_1\}$ induces a P_6. But now $\{x_5, x_4, x_3, x_2, u_1, u_2\}$ induces a P_6. Therefore $u_1x_1, u_1x_5 \notin E$ without loss of generality.

Suppose $u_1x_2 \in E$. Then for each $w \in \{x_4, x_5\}$ we have $wu_1 \notin E$, or else $\{u_1, u_2, x_2, w\}$ induces a claw centered at u_1. Hence $u_1x_3 \in E$, since otherwise $\{x_5, x_4, x_3, x_2, u_1, u_2\}$ induces a P_6. But now $\{x_5, x_4, x_3, u_1, u_2, z\}$ induces a P_6. Therefore $u_1x_2, u_1x_4 \notin E$ without loss of generality.

Thus we must have $u_1x_3 \in E$. But now the path $x_1x_2x_3u_1u_2z$ is an induced P_6.

Case 2. Suppose $u_1x \notin E$ for all $x \in V(X)$.

We have $u \notin V(X)$. Without loss of generality, we may assume $v \notin (x_3Cu)$. Now for each $y \in (x_3Cu)$, since $y \neq v$, we must have $yz, yu_2 \notin E$ by Claim 2.1. We must also have $x_3z, x_3u_2 \notin E$, or else Claim 2.1 implies $N_C(v') = \{x_3\}$, and thus the set $\{x_3, v', x_2, x_4\}$ induces a claw centered at x_3.

Let $w \in [x_5^+Cu]$ be the unique vertex such that $wu_1 \in E$ and $au_1 \notin E$ for all $a \in (x_5Cw)$. Now if $w\alpha \notin E$ for some $\alpha \in [x_3Cw^-]$, then there must exist a vertex $\beta \in [x_4Cw^-]$ such that $w\beta \in E$ and $w\beta^- \notin E$. But this implies that the path $zu_2u_1w\beta\beta^-$ is an induced P_6, since $u_1a \notin E$ for all $a \in V(X) \cup (x_5Cw)$.

Hence we must have $w\alpha \in E$ for all $\alpha \in [x_3Cw^-]$. But now $\{w, u_1, x_5, x_3\}$ induces a claw centered at w. □

We will need the following definition for Lemma 2.5.

Definition 2.3. Given a cycle C and a set $X \subset V(C)$, we say X is *skippable* with respect to C if for all $Y \subseteq X$, there exists a cycle $\hat{C}$ in G with $V(\hat{C}) = V(C) - Y$.

Lemma 2.5. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices. Let C, z, P, Q, u, and v be as previously described, and let $S \subset V(C)$ be such that $|S| \geq 1$ and $|V(C)| \geq |S| + 2$. If $V(C) - \{v\}$ induces a clique in G, then there exists a cycle C' and a set $X \subset V(C) - S$ such that C absorbs $V(P) \cup V(Q)$ via C', X is skippable with respect to C', and:*

(i) If $|(P)| = 3$ or $|(Q)| = 3$, then $|X| \geq |V(C)| - |S| - 2$.

(ii) If $|(P)|, |(Q)| \leq 2$, then $|X| \geq |V(C)| - |S| - 3$.

Proof. Let $S \subset V(C)$ satisfy $|V(C)| \geq |S| + 2 \geq 3$, and suppose $V(C) - \{v\}$ induces a clique in G.

Proof of part (i). Assume $P = uu_1u_2u_3z$ without loss of generality. By Lemma 2.2, $(V(C) - \{v\}) \cup \{u_1\}$ induces a clique in G.

Suppose $\{v^-, v, v^+\} \cap S = \emptyset$. Then there exists a vertex $s \in S - \{v^-, v, v^+\}$. Note that $u_1 s, v^- s \in E$. Let $C' = vCs^-s^+Cv^-su_1PzQv$. Then $X = V(C') - (S \cup (P) \cup [Q] \cup \{v, v^+, s\})$ is skippable with respect to C' and $|X| \geq |V(C)| - |S| - 2$.

So let $\{v^-, v, v^+\} \cap S \neq \emptyset$. If u and v occur consecutively on C, say $u = v^-$, then let $C' = vCv^-PzQv$. Now $X = V(C') - (S \cup V(P) \cup V(Q) \cup \{v, v^+, v^-\})$ is skippable with respect to C' and $|X| \geq |V(C)| - |S| - 2$.

Thus we may assume u and v do not occur consecutively on C. If $v \in S$ or $v^+ \in S$, consider $C' = vCu^-u^+Cv^-uPzQv$. Then $X = V(C') - (S \cup V(P) \cup V(Q) \cup \{v, v^+, u\})$ is skippable with respect to C' and $|X| \geq |V(C)| - |S| - 2$. If $v^- \in S$, let $C' = vQzPuv^+Cu^-u^+Cv$. Then $X = V(C') - (S \cup V(P) \cup V(Q) \cup \{u, v^-, v\})$ is skippable with respect to C' and $|X| \geq |V(C)| - |S| - 2$.

Proof of part (ii). Suppose $|(P)|, |(Q)| \leq 2$. If u and v occur consecutively on C, say $u = v^-$, then let $C' = vCuPzQv$. Otherwise, let $C' = vCu^-u^+Cv^-uPzQv$. Then $X = V(C') - (S \cup V(P) \cup V(Q) \cup \{v, v^+, u\})$ is skippable with respect to C' and $|X| \geq |V(C)| - |S| - 3$. □

Now if C is a cycle of length $l \geq 3k + 3$ that contains a k-set $S \subset V$ and has a 2-tab T, Lemmas 2.6 and 2.7 will allow us to effectively hop over a vertex in $V(C) - S$ to obtain a new cycle of length $l - 1$ which then absorbs the tab T. The end result is a cycle of length $l + 1$ that contains S.

Lemma 2.6. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices. Let C be a cycle in G with $|V(C)| \geq 6$, and let $x_1x_2x_3x_4x_5 = x_1Cx_5$ be a path contained on C. Let $\{z_1, z_2\}$ be a 2-tab of C with $N_C(z_1) = \{u\}$ and $N_C(z_2) = \{v\}$, and suppose one of the following two conditions holds:*

(i) $u, v \notin \{x_1, x_2, x_3, x_4, x_5\}$;

(ii) $u \in \{x_1, x_5\}$ and $v \notin \{x_2, x_3, x_4\}$.

Then there exists a cycle $\hat{C}$ in G and a vertex $x \in \{x_2, x_3, x_4\}$ such that $V(\hat{C}) = (V(C) - \{x\}) \cup \{z_1, z_2\}$.

Proof. Suppose the conclusion of the lemma is false. Note that each of the conditions (i) and (ii) implies $u, v \notin \{x_2, x_3, x_4\}$. Thus $x_1x_3, x_2x_4, x_3x_5 \notin E$, for otherwise we may hop over a vertex $x \in \{x_2, x_3, x_4\}$ to obtain the cycle $C' = x^-x^+Cx^-$ which has $\{z_1, z_2\}$ as a tab, and then apply Lemma 2.1 to obtain a cycle $\hat{C}$ with $V(\hat{C}) = V(C') \cup \{z_1, z_2\}$.

First suppose condition (i) holds. Without loss of generality we may assume $v \notin (x_5Cu)$. Let $y \in [x_1Cu^-]$ be the unique vertex such that $yu \in E$ and $au \notin E$ for all $a \in (x_1^-Cy)$.

Suppose $y = x_1$. Then $ux_3 \notin E$, or else $\{u, z_1, x_1, x_3\}$ induces a claw centered at u. Therefore $ux_2 \in E$, since otherwise $\{z_2, z_1, u, x_1, x_2, x_3\}$ induces a P_6. But now $ux_4 \in E$, or $\{z_2, z_1, u, x_2, x_3, x_4\}$ induces a P_6. This is a contradiction since now $\{u, z_1, x_2, x_4\}$ induces a claw.

Suppose $y = x_2$. Then $ux_4 \notin E$, or $\{u, z_1, x_2, x_4\}$ induces a claw. Thus $ux_3 \in E$, or else $\{z_2, z_1, u, x_2, x_3, x_4\}$ induces a P_6. However, now $ux_5 \in E$ since $\{z_2, z_1, u, x_3, x_4, x_5\}$ cannot induce a P_6. This is a contradiction as $\{u, z_1, x_3, x_5\}$ now induces a claw.

Therefore, it must be the case that $y \in [x_3Cu^-]$. Now if there is a vertex $\beta \in [x_2Cy^-]$ such that $y\beta \in E$ and $y\beta^- \notin E$, then $\{z_2, z_1, u, y, \beta, \beta^-\}$ induces a P_6. Thus there is no such vertex, which implies that $[x_1Cy^-] \subset N(y)$. But since $ua \notin E$ for all $a \in [x_1Cy^-]$, then avoiding a claw centered at y, it must be the case that $[x_1Cy]$ induces a clique in G. This is a contradiction, since clearly we may now hop over some vertex $x \in \{x_2, x_3, x_4\}$, and then apply Lemma 2.1 to obtain the desired cycle $\hat{C}$.

Now suppose condition (ii) holds. Assume $u = x_1$ without loss of generality. Then we must have $x_1x_4 \in E$, since otherwise $\{z_2, z_1, x_1, x_2, x_3, x_4\}$ induces a P_6. But this yields a contradiction, as $\{x_1, z_1, x_2, x_4\}$ now induces a claw centered at x_1. □

We now use Lemma 2.6 to prove the following.

Lemma 2.7. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices. Let $S \subset V$ satisfy $|S| = k \geq 1$, and suppose there exists a cycle C in G with $|V(C)| \geq 3k + 3$ and $S \subset V(C)$. If $\{z_1, z_2\}$ is a 2-tab of C, then there exists a cycle $\hat{C}$ in G and a vertex $x \in V(C) - S$ such that $V(\hat{C}) = (V(C) - \{x\}) \cup \{z_1, z_2\}$.*

Proof. Let $N_C(z_1) = u$ and $N_C(z_2) = v$. Since $S \subset V(C)$ and $|V(C)| \geq 3|S| + 3$, then by the Pigeonhole Principle at least one of the following two statements holds:

(A) There exist distinct ordered pairs $(w_1, w_1'), (w_2, w_2'), (w_3, w_3') \in S \times S$ such that $(w_jCw_j') \subset V(C) - S$ and $|(w_jCw_j')| = 3$ for each $j \in \{1, 2, 3\}$.

(B) There exists a path $y_1y_2y_3y_4y_5 = y_1Cy_5$ on C such that $y_1 \in S$ and $y_2, y_3, y_4, y_5 \notin S$.

First suppose (A) holds, and thus $k \geq 3$. Let $P_1 = w_1Cw_1', P_2 = w_2Cw_2'$, and $P_3 = w_3Cw_3'$. Note that $(P_i) \cap (P_j) = \emptyset$ for all $i \neq j$ since the ordered pairs from (A) are distinct. Now if $u, v \notin V(P_j)$ for some $j \in \{1, 2, 3\}$, then condition (i) of Lemma 2.6 applies since $P_j = w_jCw_j'$ is a path on 5 vertices, and we are done by Lemma 2.6.

Thus suppose for each $j \in \{1, 2, 3\}$, we have $\{u, v\} \cap V(P_j) \neq \emptyset$. This is only possible if u or v is an end vertex of P_j for some $j \in \{1, 2, 3\}$. Without loss of generality, assume $u = w_1$. Now $v \in (w_1Cw_1')$, since otherwise condition (ii) of Lemma 2.6 applies using the path w_1Cw_1', and we are done by Lemma 2.6. But this is a contradiction, since now $u, v \notin V(P_j)$ for some $j \in \{2, 3\}$.

Now suppose (B) holds. If $u, v \notin [y_1Cy_5]$, then condition (i) of Lemma 2.6 applies and we are done by Lemma 2.6. Thus we may assume $u \in [y_1Cy_5]$ without loss of generality. Let $y_6 = y_5^+$ and $y_7 = y_5^{++}$. We consider four cases.

Case 1. Suppose that $u \in \{y_1, y_2\}$. Then $u^+, u^{++}, u^{+++} \notin S$. We must have $v \in \{u^+, u^{++}, u^{+++}\}$ or else condition (ii) of Lemma 2.6 applies using the path uCu^{++++}, and we are done. If $v = u^{++}$, then uz_1z_2vCu is the desired cycle. If $v = u^{+++}$, then $v^-v^+ \in E$ in order to avoid a claw centered at v. But now $uz_1z_2vv^-v^+Cu$ is the desired cycle. Lastly, suppose $v = u^+$. Now assume $vv^{++}, v^+v^{+++} \notin E$, for otherwise the result clearly holds. Then we must have $vv^{+++} \in E$, or else $\{z_1, z_2, v, v^+, v^{++}, v^{+++}\}$ induces a P_6. This is a contradiction since now $\{v, z_2, v^+, v^{+++}\}$ induces a claw centered at v. Thus we may assume $u, v \notin \{y_1, y_2\}$.

Case 2. Let $u = y_3$. We have $v \neq y_5$ or else $y_3z_1z_2y_5Cy_3$ is the desired cycle. Now $v \neq y_6$, for otherwise $y_5y_7 \in E$ in order to avoid a claw centered at y_6, and then $y_3z_1z_2y_6y_5y_7Cy_3$ is the desired cycle. So we have $v \notin \{y_1, y_2, y_3, y_5, y_6\}$.

Suppose $v \neq y_4$. Then $y_3y_5, y_4y_6 \notin E$, or else there clearly exists a cycle C' with $V(C') = V(C) - \{y\}$ for some $y \in \{y_4, y_5\}$, and we are done by Lemma 2.1 since $\{z_1, z_2\}$ is a tab of C'. But now $y_3y_6 \in E$ since $\{z_2, z_1, y_3, y_4, y_5, y_6\}$ cannot induce a P_6, and so $\{y_3, z_1, y_4, y_6\}$ induces a claw centered at y_3.

Therefore we must have $v = y_4$. Now $y_4y_6 \notin E$ or else the result clearly holds. Suppose $y_6 \notin S$. Assume $y_5y_7 \notin E$ since otherwise the result certainly holds. But now $y_4y_7 \in E$, or else $\{z_1, z_2, y_4, y_5, y_6, y_7\}$ induces a P_6. This is a contradiction since $\{y_4, z_2, y_5, y_7\}$ induces a claw.

Now suppose $y_6 \in S$, and thus $k \geq 2$. Since $|V(C)| \geq 3|S| + 3$, by the Pigeonhole Principle there must exist a path $x_1x_2x_3x_4x_5 = x_1Cx_5$ such that $x_1 \in S - \{y_1\}$ and $x_2, x_3, x_4 \in V(C) - S$. But then $[y_2Cy_5] \cap [x_1Cx_5] = \emptyset$, and so $u, v \notin [x_1Cx_5]$. This fulfills condition (i) of Lemma 2.6, and we are done by Lemma 2.6. Hence we may assume $u, v \neq y_3$.

Case 3. Assume $u = y_4$. We know $v \notin \{y_1, y_2, y_3, y_4\}$. Also $y_1y_3, y_2y_4 \notin E$, since otherwise we may hop over y_2 or y_3 first, and then apply Lemma 2.1 to obtain the desired cycle. But now $y_1y_4 \in E$ since the set $\{z_2, z_1, y_4, y_3, y_2, y_1\}$ cannot induce a P_6. This is a contradiction because now $\{y_4, z_1, y_3, y_1\}$ induces a claw centered at y_4. Therefore we assume $u, v \neq y_4$.

Case 4. Suppose $u = y_5$. From the previous cases, we have $v \notin \{y_2, y_3, y_4\}$. Now condition (ii) of Lemma 2.6 applies using the path y_1Cy_5, and we are done by Lemma 2.6. □

Whenever a k-set S is contained in a cycle of length m with $3k+3 \leq m \leq n-1$, the next lemma guarantees that S is contained in a cycle of length $m+1$.

Lemma 2.8. *Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 8$ vertices. If a set $S \subset V$ of $k \geq 1$ vertices is contained in an m-cycle C where $3k+3 \leq m \leq n-1$, then S is also contained in an $(m+1)$-cycle.*

Proof. Pick a vertex $\gamma \in V - V(C)$ such that $\gamma u \in E$ for some $u \in V(C)$. As G is 2-connected, we may pick a path Q from γ to $V(C) - \{u\}$ that is shortest possible. Let v denote the end vertex of Q on C.

We may assume there does not exist a vertex in $V - V(C)$ that the cycle C absorbs, since otherwise the result holds. Therefore by Lemmas 2.1 and 2.2, $|(Q)| \notin \{0, 3\}$.

Suppose $Q = vv_1v_2\gamma$. Since C does not absorb v_1, Lemma 2.3 implies $V(C) - \{u\}$ induces a clique. By Lemma 2.5, there exists a cycle C' and a set $X \subset V(C) - S$ such that C absorbs $V(Q)$ via C', X is skippable with respect to C', and $|X| \geq |V(C)| - |S| - 3 \geq (3k + 3) - k - 3 \geq 2$. Since C' contains S and has length $m + 3$, the result holds.

Now suppose $Q = vv_1\gamma$. Since C does not absorb v_1, Lemma 2.1 implies that $d_C(v_1) = 1$, and thus $\{\gamma, v_1\}$ is a 2-tab of C. But now we may apply Lemma 2.7 to obtain a cycle of length $m + 1$ that contains S. □

3 Proof of Theorem 2.1

We nearly have all the tools in place to prove the main result. Let G be a 2-connected $\{K_{1,3}, P_6\}$-free graph. If $n = 7$, the result clearly holds since G is hamiltonian and $3k + 4 \geq 7$. Therefore we assume $n \geq 8$. First we will use an inductive proof to show that if $k \geq 1$ and $n \geq 3k + 4$, then any k-set is contained in a $(3k + 4)$-cycle. The following claim provides the base case for the proof.

Claim 3.1. *Every vertex $u \in V$ is contained in a cycle of length 7.*

Proof. Suppose $u \in V$ is not contained in a 7-cycle. A cycle that contains u and has shortest possible length, l, must satisfy $l \in \{3, 4, 5, 6\}$ since G is P_6-free. Therefore we may pick a cycle C of length $m \in \{3, 4, 5, 6\}$ that contains u.

Since G is connected and $n \geq 8$, we may choose $w \in V - V(C)$ such that $wx \in E$ for some $x \in V(C)$. As G is 2-connected, we may pick a path Q from w to $V(C) - \{x\}$ that is shortest possible. Let y denote the end vertex of Q on C. We may assume without loss of generality that $y \neq x^-$. We will consider four cases corresponding to the possible values of m.

Case 1. Suppose $m = 6$. Then by Lemma 2.8, u is contained in a 7-cycle.

Case 2. Suppose $m = 5$. By Case 1, we may assume u is not contained in a 6-cycle. Thus $d_C(v) \leq 1$ for all $v \notin V(C)$ by Lemma 2.1, and $|(Q)| \neq 3$ by Lemma 2.2. If $|(Q)| = 1$ then $\{w\} \cup (Q)$ is a 2-tab of C, and C absorbs $\{w\} \cup (Q)$ by Lemma 2.1. This is a contradiction, since now u is contained in a 7-cycle.

Suppose $Q = yy_1y_2w$. Without loss of generality we may assume $y \neq x^{--}$. Since $d_C(y_1) = 1$, Lemma 2.3 implies $V(C) - \{x\}$ induces a clique. If $u \in \{x, y, x^-, x^{--}\}$, then $yy_1y_2wxx^-x^{--}y$ is a 7-cycle which contains u. If $u \notin \{x, y, x^-, x^{--}\}$, then $yy_1y_2wxx^-uy$ is a 7-cycle.

Case 3. Suppose $m = 4$. By Cases 1 and 2, u is not contained in a 6-cycle or a 5-cycle. Hence $d_C(v) \leq 1$ for all $v \notin V(C)$ by Lemma 2.1, and $|(Q)| \neq 3$ by Lemma 2.2. If $|(Q)| = 1$, then clearly u is contained in a 6-cycle if $y = x^+$, and a 5-cycle if $y = x^{++}$. If $|(Q)| = 2$, then clearly u is contained in a 7-cycle if $y = x^+$, and a 6-cycle if $y = x^{++}$.

Case 4. Suppose $m = 3$. By the previous three cases, u is not contained in a 6-cycle, a 5-cycle, or a 4-cycle. Therefore $d_C(v) \leq 1$ for all $v \notin V(C)$ by Lemma 2.1, and $|(Q)| \neq 3$ by Lemma 2.2. If $|(Q)| = 1$, then u is contained in a 5-cycle. If $|(Q)| = 2$, then u is contained in a 6-cycle. This completes the proof of Claim 3.1. □

Now let $S \subset V$ be such that $|S| = k - 1 \geq 1$, and suppose there is a cycle C in G of length $3(k-1) + 4 = 3k + 1$ such that $S \subset V(C)$. Let $z \in V - S$. Assuming $n \geq 3k + 4$, we will show there exists a cycle C' of length $3k + 4$ in G such that $S \cup \{z\} \subset V(C')$. We begin by proving two claims.

Claim 3.2. *There exists a cycle $\hat{C}$ of length L in G such that $L \in \{3k+1, 3k+2, 3k+3, 3k+4\}$ and $S \cup \{z\} \subset V(\hat{C})$.*

Proof. If $z \in V(C)$, Claim 3.2 clearly holds. Suppose $z \notin V(C)$. Since G is 2-connected, we may pick a pair of (z, C)-paths P and Q that are vertex-disjoint except for z, such that $|(P)| + |(Q)|$ is minimal among all such sums. Let u and v denote the end vertices of P and Q, respectively, on C. Note that $|(P)|, |(Q)| \leq 3$ since G is P_6-free. Assume $|(P)| \leq |(Q)|$ without loss of generality.

If $P = uu_1u_2u_3z$, then $V(C) - \{v\}$ induces a clique in G by Lemma 2.2. By Lemma 2.5, there exists a cycle C' and a set $X \subset V(C) - S$ such that C absorbs $V(P) \cup V(Q)$ via C', X is skippable with respect to C', and $|X| \geq |V(C)| - |S| - 2 = (3k+1) - (k-1) - 2 = 2k \geq 4$. Hence Claim 3.2 clearly holds with $L = 3k + 4$. Thus we may assume $|(P)| \leq |(Q)| \leq 2$ without loss of generality.

Now C absorbs $V(P) \cup V(Q)$ by Lemma 2.1. Therefore if the ordered pair $(|(P)|, |(Q)|) \in \{(0,0), (0,1), (0,2), (1,1)\}$, then Claim 3.2 holds with $L \in \{3k + 2, 3k + 3, 3k + 4\}$.

Therefore we assume $|(Q)| = 2$ and $|(P)| \in \{1, 2\}$. Since $|V(C)| = 3|S| + 4$, then by the Pigeonhole Principle there exists a path $x_1x_2x_3x_4x_5 = x_1Cx_5$ on C such that $x_1 \in S$ and $x_2, x_3, x_4 \notin S$. By Lemma 2.4, x_1Cx_5 cannot be an induced P_5. This implies that there exists a cycle C_x such that $V(C_x) = V(C) - X$ for some nonempty set $X \subseteq \{x_2, x_3, x_4\}$. Note that $S \subset V(C_x)$ and $|V(C_x)| \in \{3k-2, 3k-1, 3k\}$.

Pick a pair of (z, C_x)-paths $\{P_x, Q_x\}$ that are vertex-disjoint except for z, such that $|(P_x)| + |(Q_x)|$ is minimal among all such sums. Let u_x and v_x denote the end vertices of P_x and Q_x, respectively, on C_x. Now $|(P_x) \cup (Q_x) \cup \{z\}| \geq |(P) \cup (Q) \cup \{z\}| \geq 4$ by the minimality of $|(P)| + |(Q)|$.

Suppose $V(C_x) - \{v_x\}$ induces a clique. By Lemma 2.5, there exists a cycle C'_x and a set $Z \subset V(C_x) - S$ such that C_x absorbs $V(P_x) \cup V(Q_x)$ via C'_x and Z is skippable with respect to C'_x. Suppose $|(P_x)| = 3$ or $|(Q_x)| = 3$. Then by Lemma 2.5, $|Z| \geq |V(C_x)| - |S| - 2$. If $|V(C_x)| = 3k$, then $|Z| \geq 3$ and Claim 3.2 holds with $L = 3k + 4$. If $|V(C_x)| = 3k - 1$, then $|Z| \geq 2$ and Claim 3.2 holds with $L \in \{3k + 3, 3k + 4\}$. If $|V(C_x)| = 3k - 2$, then $|Z| \geq 1$ and Claim 3.2 holds with $L \in \{3k + 2, 3k + 3, 3k + 4\}$.

If $|(P_x)|, |(Q_x)| \leq 2$, then by Lemma 2.5, $|Z| \geq |V(C_x)| - |S| - 3$. If $|V(C_x)| = 3k$, then $|Z| \geq 2$ and Claim 3.2 holds with $L = 3k+4$. If $|V(C_x)| = 3k-1$, then Claim 3.2

holds with $L \in \{3k+3, 3k+4\}$. If $|V(C_x)| = 3k-2$, then Claim 3.2 holds with $L \in \{3k+2, 3k+3\}$.

Therefore we assume neither $V(C_x) - \{v_x\}$ nor $V(C_x) - \{u_x\}$ induces a clique. By Lemma 2.2, we have $|(P_x)|, |(Q_x)| \leq 2$. Without loss of generality, this implies $|(Q_x)| = 2$ and $|(P_x)| \in \{1, 2\}$.

By Lemma 2.1, the cycle C_x absorbs $V(P_x) \cup V(Q_x)$. Thus if $|(P_x)| = 1$ and $|(Q_x)| = 2$, then Claim 3.2 holds with $L \in \{3k+2, 3k+3, 3k+4\}$.

So suppose $|(P_x)| = 2 = |(Q_x)|$. If $|V(C_x)| \in \{3k-2, 3k-1\}$, then by Lemma 2.1, Claim 3.2 holds with $L \in \{3k+3, 3k+4\}$.

Thus we assume $|V(C_x)| = 3k = 3|S|+3$. By the Pigeonhole Principle there exists a path $y_1y_2y_3y_4y_5 = y_1C_xy_5$ on C_x such that $y_1 \in S$ and $y_2, y_3, y_4 \notin S$. Lemma 2.4 implies $y_1C_xy_5$ cannot be an induced P_5. Therefore there exists a cycle C_y such that $V(C_y) = V(C_x) - Y$ for some set $Y \subseteq \{y_2, y_3, y_4\}$ with $Y \neq \emptyset$. Note that $S \subset V(C_y)$ and $|V(C_y)| \in \{3k-3, 3k-2, 3k-1\}$.

Pick a pair of (z, C_y)-paths $\{P_y, Q_y\}$ that are vertex-disjoint except for z, such that $|(P_y)| + |(Q_y)|$ is minimal among all such sums. Let u_y and v_y denote the end vertices of P_y and Q_y, respectively, on C_y. Now $|(P_y) \cup (Q_y) \cup \{z\}| \geq |(P_x) \cup (Q_x) \cup \{z\}| = 5$ by the minimality of $|(P_x)| + |(Q_x)|$.

Suppose $|(P_y)| = 3$ or $|(Q_y)| = 3$. Then without loss of generality, $V(C_y) - \{v_y\}$ induces a clique in G by Lemma 2.2. By Lemma 2.5, there exists a cycle C'_y and a set $Z \subset V(C_y) - S$ such that C_y absorbs $V(P_y) \cup V(Q_y)$ via C'_y, Z is skippable with respect to C'_y, and $|Z| \geq |V(C_y)| - |S| - 2$. If $|V(C_y)| = 3k-1$, then $|Z| \geq 2$ and Claim 3.2 holds with $L = 3k+4$. If $|V(C_y)| = 3k-2$, then $|Z| \geq 1$ and Claim 3.2 holds with $L \in \{3k+3, 3k+4\}$. If $|V(C_y)| = 3k-3$, then Claim 3.2 holds with $L \in \{3k+2, 3k+3, 3k+4\}$.

So assume $|(P_y)| = 2 = |(Q_y)|$. Since C_y absorbs $V(P_y) \cup V(Q_y)$ by Lemma 2.1, Claim 3.2 then holds with $L \in \{3k+2, 3k+3, 3k+4\}$. □

Claim 3.3. *If $L \in \{3k+1, 3k+2\}$ and $n \geq 3k+3$, then $S \cup \{z\}$ is contained in a $(3k+3)$-cycle or a $(3k+4)$-cycle.*

Proof. Pick a vertex $\gamma \in V - V(\hat{C})$ such that $\gamma u \in E$ for some $u \in V(\hat{C})$. Now pick a path Q from γ to $V(\hat{C}) - \{u\}$ that is shortest possible.

Case 1. Suppose $L = 3k+2$. If $|(Q)| \in \{0, 1\}$, then the result holds since $\hat{C}$ absorbs $V(Q)$ by Lemma 2.1. If $Q = vv_1v_2v_3\gamma$, then $\hat{C}$ absorbs v_1 by Lemma 2.2 and the result holds.

Suppose $Q = vv_1v_2\gamma$. Assume $\hat{C}$ does not absorb v_1, since the result holds otherwise. By Lemma 2.3, $V(\hat{C}) - \{u\}$ induces a clique. Hence Lemma 2.5 implies there exists a cycle $\hat{C}_1$ and a set $X \subset V(\hat{C}) - (S \cup \{z\})$ such that $\hat{C}$ absorbs $V(Q)$ via $\hat{C}_1$, X is skippable with respect to $\hat{C}_1$, and $|X| \geq |V(\hat{C})| - |S \cup \{z\}| - 3 = (3k+2) - k - 3 \geq 3$. Since $\hat{C}_1$ contains $S \cup \{z\}$ and has length $3k+5$, the result clearly holds in this case.

Case 2. Suppose $L = 3k+1$. If $|(Q)| \in \{1, 2\}$, then the result holds since $\hat{C}$ absorbs $V(Q)$ by Lemma 2.1.

If $Q = v\gamma$ or $Q = vv_1v_2v_3\gamma$, then $\hat{C}$ absorbs γ or v_1 by Lemma 2.1 or Lemma 2.2, respectively, yielding a cycle of length $3k + 2$ which contains $S \cup \{z\}$. We may now repeat the argument from Case 1, and Claim 3.3 holds. □

Claim 3.2 and Claim 3.3 together imply that the k-set $S \cup \{z\}$ is contained in a $(3k + 3)$-cycle or a $(3k + 4)$-cycle, assuming $n \geq 3k + 3$. If $S \cup \{z\}$ is contained in a non-hamiltonian $(3k + 3)$-cycle, then Lemma 2.8 allows us to obtain a cycle of length $3k + 4$ that contains $S \cup \{z\}$.

Hence by induction, we have shown that any set S of $k \geq 1$ vertices is contained in a $(3k + 4)$-cycle whenever $n \geq 3k + 4$. Furthermore, Lemma 2.8 guarantees that S is contained in a cycle of length m whenever $3k + 4 \leq m \leq n$. Therefore G is $(k, 3k + 4)$-pancyclic.

To see that this result is best possible, consider the graph H given in Figure 1, which is a 2-connected, $\{K_{1,3}, P_6\}$-free graph. It is easy to observe that H is not $(k, 3k + 3)$-pancyclic, since the set $\{y_1, y_2, \cdots, y_k\}$ is not contained in a cycle of length $3k + 3$. □

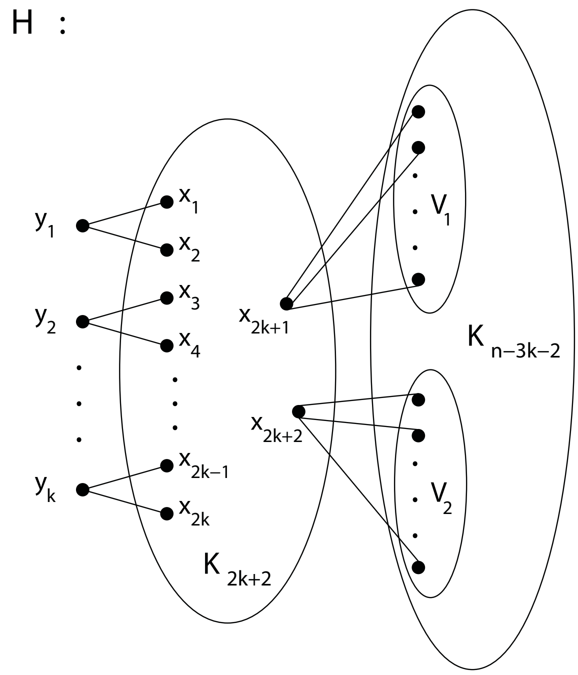

Figure 1: The set $\{y_1, y_2, \cdots, y_k\}$ is not contained in a $(3k + 3)$-cycle.

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