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Bounds for the strength of graphs

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Abstract

In this paper, we introduce the notion of strength of a graph, and establish formulas for the strength of certain classes of graphs. We also present sharp bounds for the strength of a graph and other parameters defined on graphs. Moreover, we exhibit a connection between the super magic strength and a particular type of strength, which leads us to sharp bounds for the super magic strength of super edge-magic graphs. The work conducted in this paper suggests some open problems and a new conjecture.

1 Introduction

In this paper, only finite graphs without loops or multiple edges will be considered. Terms and notation not defined below follow those used in [3] or [4]. The *vertex set* of a graph G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. As usual, the *path* of order n, the *cycle* of order n and the *complete graph* of order n are denoted by P_n, C_n and K_n, respectively. The *complete bipartite graph* with partite sets U and V, where $|U| = m$ and $|V| = n$, is denoted by $K_{m,n}$. The graph with n vertices and no edges is referred to as the *empty graph*.

The *cartesian product* $G \cong G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

$$u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)$$

or

$$u_2 = v_2 \text{ and } u_1v_1 \in E(G_1).$$

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Some important classes of graphs can be defined in terms of cartesian products. The *ladder* L_n can be defined as the graph $P_n \times K_2$. The *prism* D_n can be defined as the graph $C_n \times K_2$. The *book* B_n can be defined as the graph $K_{1,n} \times K_2$. The *hypercube* Q_n can be defined inductively as $Q_1 \cong K_2$ and $Q_n \cong Q_{n-1} \times K_2$ for any integer $n \geq 2$.

For the sake of notational convenience, we denote the interval of integers k such that $i \leq k \leq j$ by the symbol $[i, j]$. On the other hand, if $i > j$, then we treat $[i, j]$ as the empty set. If such situation occurs in particular formulas for a given vertex labeling, then we ignore the corresponding portions of the formulas.

For a graph G of order p and size q, a bijective function $f : V(G) \cup E(G) \to [1, p+q]$ is called an *edge-magic labeling* if $f(u) + f(v) + f(uv)$ is a constant $c(f)$ (called the *magic constant* or *valence*) for each $uv \in E(G)$. If such a labeling exists, then G is called an *edge-magic graph*.

The notion of edge-magic labelings was first introduced in 1970 by Kotzig and Rosa [8]. These labelings were originally called "magic valuations" by them. These were rediscovered in 1996 by Ringel and Lladó [9] who coined one of the now popular terms for them: edge-magic labelings. Afterwards, Enomoto et al. [5] defined a slightly restricted version of an edge-magic labeling f of a graph G by requiring that $f(V(G)) = [1, |V(G)|]$. Such a labeling was called by them *super edge-magic*. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. It is worth to mention that Acharya and Hegde [1] had already discovered such graphs in 1991 under the name of "strongly indexable graphs". However, they arrived at this concept from a different point view.

The concept of super magic strength was introduced by Avadayappan et al. [2]. The *super magic strength*, $sm(G)$, of a graph G is defined as the minimum of all magic constants $c(f)$, where the minimum is taken over all super edge-magic labelings f of G, that is,

$$sm(G) = \min\{c(f) \,|\, f \text{ is a super edge-magic labeling of } G\}.$$

It is an immediate consequence of the definition that if G is not a super edge-magic graph or an empty graph, then $sm(G)$ is undefined (or we could define $sm(G) = +\infty$). It is also true that G is a super edge-magic graph if and only if $sm(G) < +\infty$.

As the concept of super magic strength is effectively defined only for super edge-magic graphs, we generalize this concept in this paper for any nonempty graph as follows. A *numbering* f of a graph G of order p is a labeling that assigns distinct elements of the set $[1, p]$ to the vertices of G, where each edge uv of G is labeled $f(u) + f(v)$. The *strength*, $\mathrm{str}_f(G)$, of a numbering $f : V(G) \to [1, p]$ of G is defined by

$$\mathrm{str}_f(G) = \max\{f(u) + f(v) \,|\, uv \in E(G)\},$$

that is, $\mathrm{str}_f(G)$ is the maximum edge label of G, and the *strength*, $\mathrm{str}(G)$, of a graph G itself is

$$\mathrm{str}(G) = \min\{\mathrm{str}_f(G) \,|\, f \text{ is a numbering of } G\}.$$

A numbering f of a graph G for which $\mathrm{str}_f(G) = \mathrm{str}(G)$ is called a *strength labeling* of G. If G is an empty graph, then $\text{str}(G)$ is undefined (or we could define $\text{str}(G) = +\infty$).

The following lemma is an immediate consequence of the definition for the strength of a graph.

Lemma 1.1 *If G and H are nonempty graphs such that H is a subgraph of G, then* $\text{str}(G) \geq \text{str}(H)$.

There are other related parameters that have been studied in the area of graph labeling. Excellent sources for more information in this topic are found in the survey by Gallian [7].

2 Bounds for the strength

In this section, several bounds for the strength of a graph are presented in terms of other parameters defined on graphs.

We start with the next lower bound for the strength of a graph in terms of its order and minimum degree, which will prove to be very useful later in this paper.

Lemma 2.1 *For every graph G of order p with $\delta(G) \geq 1$,*

$$\text{str}(G) \geq p + \delta(G).$$

Proof: Let G be a graph of order p with $\delta(G) \geq 1$. If a strength labeling f of G is given, then there is a vertex v labeled p. Since $\deg v \geq \delta(G) \geq 1$, there are at least $\delta(G)$ vertices adjacent to v. Since f minimizes the greatest edge label, it follows that the vertex v must be adjacent to a vertex labeled at least $\delta(G)$. Thus, f has the property that

$$\text{str}_f(G) = \max\{f(u) + f(v) \,|\, uv \in E(G)\} \geq p + \delta(G).$$

Consequently, $\text{str}(G) \geq p + \delta(G)$. □

A graph G has no isolated vertices if and only if $\delta(G) \geq 1$. On the other hand, the bound given in the preceding result does not hold for graphs with isolated vertices. To see this, it suffices to consider the graph $K_2 \cup 2K_1$. By assigning 1 and 2 to the vertices of K_2, and 3 and 4 to the vertices of $2K_1$, we have a strength labeling of $K_2 \cup 2K_1$ with $\text{str}(K_2 \cup 2K_1) = 3$.

A *vertex-cut* in a graph G is a set S of vertices of G such that $G - S$ is disconnected. The *connectivity*, $\kappa(G)$, of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if $G \cong K_n$ for some positive integer n. Hence, $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected graph or a trivial graph.

Connectivity has an edge analogue. An *edge-cut* in a graph G is a set X of edges of G such that $G - X$ is disconnected. The *edge-connectivity*, $\kappa_1(G)$, of a graph G is the minimum cardinality of an edge-cut of G if G is nontrivial, and $\kappa_1(K_1) = 0$.

The following result, due to Whitney [10], establishes a connection between connectivity, edge-connectivity, and minimum degree of a graph.

Theorem 2.1 *For every graph G,*

$$\kappa(G) \leq \kappa_1(G) \leq \delta(G).$$

The next two lower bounds for the strength of a graph follow from the preceding result and Lemma 2.1.

Corollary 2.1 *For every graph G of order p with $\delta(G) \geq 1$,*

(1) $\operatorname{str}(G) \geq p + \kappa_1(G)$,
(2) $\operatorname{str}(G) \geq p + \kappa(G)$.

An assignment of colors (objects of a set) to the vertices of a graph G so that adjacent vertices are assigned different colors is called a *coloring* of G. A coloring in which n colors are used is an *n-coloring.* A graph G is *n-colorable* if there exists an m-coloring of G for some $m \leq n$. The minimum n for which a graph G is n-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. For an integer $n \geq 2$, we say that a graph G is *n-critical* if $\chi(G) = n$ and $\chi(H) < n$ for every proper subgraph H of G.

The following lemma taken from [3, p. 117] will prove to be useful in our study of the strength of graphs.

Lemma 2.2 *If G is an n-critical graph, then $\delta(G) \geq n - 1$.*

The next result provides a lower bound for the strength of a graph in terms of the chromatic number of n-critical graphs.

Corollary 2.2 *For every n-critical graph G of order p,*

$$\operatorname{str}(G) \geq p + n - 1.$$

Proof: Let G be an n-critical graph of order p. Since G is n-critical, it follows that $\delta(G) \geq 1$, due to the fact that isolated vertices do not affect the chromatic number. By Lemma 2.1, we have $\operatorname{str}(G) \geq p + \delta(G)$. However, we have $\delta(G) \geq n - 1$ by Lemma 2.2. Therefore, we conclude that

$$\operatorname{str}(G) \geq p + \delta(G) \geq p + n - 1,$$

which shows the desired result. □

The preceding result is the best possible in the sense that G must be n-critical. In order to see this, it suffices to consider the graph G with $V(G) = \{x_i \,|\, i \in [1,4]\}$ and $E(G) = \{x_i x_{i+1} \,|\, i \in [1,2]\} \cup \{x_1 x_i \,|\, i \in [3,4]\}$. Then $\chi(G) = \chi(G - x_4) = 3$, that is, G is not 3-critical. Since G has order 4 and $\delta(G) = 1$, it follows from Lemma 2.1 that $\operatorname{str}(G) \geq 5$. On the other hand, the labeling $f : V(G) \to [1,4]$ such that $f(x_i) = i$ $(i \in [1,4])$ has the property that $\operatorname{str}_f(G) = 5$. Thus, $\operatorname{str}(G) = 5$.

If S is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle S \rangle$ of G *induced* by S is the graph having vertex set S and whose edge set consists of those edges of G incident with two elements of S. A subgraph H of G is called *induced* if $H \cong \langle S \rangle$ for some subset S of $V(G)$.

There is a connection between the strength of a graph of order p and its induced subgraphs.

Theorem 2.2 *If G is a graph of order $p \geq 3$ containing a path of order k as an induced subgraph, then*

$$\mathrm{str}\,(G) \leq 2p - (k-1)\,,$$

where $k \in [2, p-1]$.

Proof: Let G be a graph of order $p \geq 3$ with $V(G) = \{x_i \,|i \in [1,p]\}$. Furthermore, let $S = \{x_i \,|i \in [1,k]\}$, and assume, without loss of generality, that $\langle S \rangle \cong P_k$, where $E(P_k) = \{x_i x_{i+1} \,|i \in [1,k-1]\}$. Now, consider the labeling $f : V(G) \to [1,p]$ such that

$$f(w) = \begin{cases} p+1-i & \text{if } w = x_{2i-1} \text{ and } i \in [1, \lceil k/2 \rceil], \\ p-(k-i) & \text{if } w = x_{2i} \text{ and } i \in [1, \lfloor k/2 \rfloor], \\ i & \text{if } w = x_{k+i} \text{ and } i \in [1, p-k]. \end{cases}$$

Then f has the property that

$$\begin{aligned} \mathrm{str}_f\,(G) &= \max\{f(u) + f(v) \,|uv \in E(G)\} \\ &= f(x_1) + f(x_2) = 2p - (k-1)\,. \end{aligned}$$

Thus, $\mathrm{str}\,(G) \leq 2p - (k-1)$. □

For a connected graph G and a pair $u, v \in V(G)$, the *distance* $d_G(u,v)$ between u and v is the length of a shortest $u-v$ path in G. The *diameter*, $\mathrm{diam}G$, of a connected graph G is defined as $\mathrm{diam}G = \max\{d_G(u,v) \,|u, v \in V(G)\}$.

With the aid of Theorem 2.2, it is now possible to present an upper bound for the strength of a connected graph in terms of its order and diameter.

Corollary 2.3 *For every connected graph G of order p,*

$$\mathrm{str}\,(G) \leq 2p - \mathrm{diam}G.$$

Proof: Let G be a connected graph of order p and $\mathrm{diam}G = k$. Then there is a subset $S = \{x_i \,|i \in [1, k+1]\}$ of $V(G)$ such that $\langle S \rangle \cong P_{k+1}$. Therefore, the result follows from Theorem 2.2. □

It is clear from Corollary 2.3 that every connected graph has finite strength.

In the following result, lower and upper bounds for the strength of a graph can be given in terms of the maximum degree and the order of the graph, respectively. For the purpose of proving this result, it is convenient to introduce some additional concepts and notation. The *neighborhood*, $N(v)$, of a vertex v of a graph G is the set of all vertices of G that are adjacent to v, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$.

Theorem 2.3 *For every nonempty graph G of order p,*

$$\Delta(G) + 2 \leq \mathrm{str}\,(G) \leq 2p - 1.$$

Proof: We begin with the lower bound. Let a strength labeling of a nonempty graph G be given and let v be a vertex of G for which $\deg v = \Delta(G) = k$. Furthermore, let $H \cong \langle S \rangle$, where $S = N[v]$. Then H is a subgraph of G as well as $K_{1,k}$ is a subgraph of H. By Lemma 1.1,

$$\mathrm{str}(G) \geq \mathrm{str}(H) \geq \mathrm{str}(K_{1,k}).$$

However, $K_{1,k}$ has order $k+1$ and $\delta(K_{1,k}) = 1$. It follows from Lemma 2.1 that $\mathrm{str}(K_{1,k}) \geq k+2$. Hence, $\mathrm{str}(G) \geq \Delta(G) + 2$.

Next, we show the upper bound. Since every nonempty graph G of order p is a subgraph of K_p $(p \geq 2)$, it follows from Lemma 1.1 that $\mathrm{str}(G) \leq \mathrm{str}(K_p)$ (see Theorem 3.4 for the strength of K_p). This together with Corollary 2.3 implies that $\mathrm{str}(G) \leq 2p - 1$. □

It is immediate from Theorem 2.3 that every nonempty graph has finite strength. The *join* $G \cong G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv \,|u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

With this definition in hand, it is now possible to present the next lower and upper bounds.

Theorem 2.4 *If G is a graph of order p, then*

$$p + m + \min\{p, \delta(G) + m\} \leq \mathrm{str}(G + mK_1) \leq \mathrm{str}(G) + 2m$$

for every positive integer m.

Proof: It suffices to prove the theorem for nonempty graphs. Since the join $G + mK_1$ is connected, it follows that $\delta(G + mK_1) \geq 1$. However, $G + mK_1$ has order $p + m$ and $\delta(G + mK_1) = \min\{p, \delta(G) + m\}$. Thus, the lower bound follows from Lemma 2.1.

To verify the upper bound, let $V(G + mK_1) = \{x_i \,|i \in [1,p]\} \cup \{y_i \,|i \in [1,m]\}$ and $E(G + mK_1) = E(G) \cup \{xy_i \,|x \in V(G) \text{ and } i \in [1,m]\}$. Furthermore, let $\mathrm{str}(G) = k$ for some positive integer k. Then there exists a strength labeling f for which

$$\mathrm{str}_f(G) = \max\{f(u) + f(v) \,|uv \in E(G)\} = k.$$

To complete the proof, we show that there exists a numbering g of $G + mK_1$ for which $\mathrm{str}_g(G + mK_1) = k + 2m$. Consider the labeling $g : V(G + mK_1) \to [1, p+m]$ such that

$$g(x_i) = f(x_i) + m \ (i \in [1,p]) \text{ and } g(y_i) = i \ (i \in [1,m]).$$

Then f has the property that

$$\begin{aligned} \mathrm{str}_g(G + mK_1) &= \max\{g(u) + g(v) \,|uv \in E(G + mK_1)\} \\ &= \max\{f(u) + f(v) \,|uv \in E(G)\} + 2m \\ &= k + 2m. \end{aligned}$$

Thus, $\mathrm{str}(G + mK_1) \leq \mathrm{str}(G) + 2m$. □

3 The strength of some classes of graphs

As is often the case, when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain classes of graphs. Ordinarily, the first classes to be considered are paths, cycles, complete graphs and complete bipartite graphs. Such is the case with the strength. Therefore, in this section, we present formulas for the strength of these classes of graphs. We also provide a formula for the strength of 1-regular graphs. Moreover, we examine the strength of graphs that are cartesian products and related graphs.

We begin with a formula for the strength of the path P_n.

Theorem 3.1 *For every integer* $n \geq 2$,

$$\text{str}\,(P_n) = n + 1.$$

Proof: Let n be an integer with $n \geq 2$. Then the path P_n has order n and $\delta\,(P_n) = 1$, and the inequality $\text{str}\,(P_n) \geq n + 1$ follows from Lemma 2.1. It remains to show that $\text{str}\,(P_n) \leq n + 1$. This can be easily completed by finding a numbering f of P_n for which $\text{str}_f\,(P_n) = n + 1$. If we let $V\,(P_n) = \{x_i \,| i \in [1, n]\}$ and $E\,(P_n) = \{x_i x_{i+1} \,| i \in [1, n-1]\}$, then the labeling $f : V\,(P_n) \to [1, n]$ such that

$$f\,(x_{2i-1}) = n + 1 - i \;\; (i \in [1, \lceil n/2 \rceil]\,) \text{ and } f\,(x_{2i}) = i \;\; (i \in [1, \lfloor n/2 \rfloor]\,)$$

has the property that

$$\begin{aligned} \text{str}_f\,(P_n) &= \max\{f\,(u) + f\,(v) \,| uv \in E\,(P_n)\} \\ &= f\,(x_1) + f\,(x_2) = n + 1. \end{aligned}$$

Thus, $\text{str}\,(P_n) = n + 1$. □

We next present a formula for the strength of the 1-regular graph nP_2.

Theorem 3.2 *For every positive integer* n,

$$\text{str}\,(nP_2) = 2n + 1.$$

Proof: The inequality $\text{str}\,(nP_2) \geq 2n + 1$ follows directly from Lemma 2.1, since the 1-regular graph nP_2 has order $2n$ and $\delta\,(nP_2) = 1$.

For the reverse inequality, it suffices to show the existence of a numbering f of nP_2 for which $\text{str}_f\,(nP_2) = 2n + 1$. Let $V\,(nP_2) = \{x_i \,| i \in [1, n]\} \cup \{y_i \,| i \in [1, n]\}$ and $E\,(nP_2) = \{x_i y_i \,| i \in [1, n]\}$, and consider the labeling $f : V\,(nP_2) \to [1, 2n]$ such that

$$f\,(x_i) = i \;\; (i \in [1, n]\,) \text{ and } f\,(y_i) = 2n + 1 - i \;\; (i \in [1, n]\,).$$

Then f has the property that

$$\begin{aligned} \text{str}_f\,(nP_n) &= \max\{f\,(u) + f\,(v) \,| uv \in E\,(nP_2)\} \\ &= f\,(x_1) + f\,(y_1) = 2n + 1. \end{aligned}$$

Thus, $\text{str}\,(nP_n) = 2n + 1$. □

The strength of the cycle C_n can be determined as shown below.

Theorem 3.3 *For every integer $n \geq 3$,*

$$\text{str}\,(C_n) = n + 2.$$

Proof: Since the cycle C_n has order n and $\delta\,(C_n) = 2$, it follows from Lemma 2.1 that $\text{str}\,(C_n) \geq n + 2$.

To establish that $\text{str}\,(C_n) \leq n + 2$, it suffices to show the existence of a numbering f of C_n for which $\text{str}_f\,(C_n) = n + 2$. Let $V\,(C_n) = \{x_i \,|i \in [1, n]\}$ and

$$E\,(C_n) = \{x_i x_{i+1} \,|i \in [1, n-1]\} \cup \{x_1 x_n\}.$$

Then the labeling $f : V\,(C_n) \to [1, n]$ such that

$$f\,(x_{2i-1}) = i\ (i \in [1, \lceil n/2 \rceil]) \text{ and } f\,(x_{2i}) = n + 1 - i\ (i \in [1, \lfloor n/2 \rfloor])$$

has the property that

$$\begin{aligned} \text{str}_f\,(C_n) &= \max\{f\,(u) + f\,(v) \,|uv \in E\,(C_n)\} \\ &= f\,(x_2) + f\,(x_3) = n + 2. \end{aligned}$$

Thus, $\text{str}\,(C_n) = n + 2$. □

The strength of the complete graph K_n is quite easy to determine.

Theorem 3.4 *For every integer $n \geq 2$,*

$$\text{str}\,(K_n) = 2n - 1.$$

Proof: There is only one numbering f of K_n and the label of the edge joining the vertices labeled $n - 1$ and n is $2n - 1$. Thus, $\text{str}_f\,(K_n) = \text{str}\,(K_n) = 2n - 1$. □

The strength of $K_{m,n}$ can be determined as follows.

Theorem 3.5 *For every two positive integers m and n with $n \geq m$,*

$$\text{str}\,(K_{m,n}) = 2m + n.$$

Proof: Let $X = \{x_i \,|i \in [1, m]\}$ and $Y = \{y_i \,|i \in [1, n]\}$ be the partite sets of $K_{m,n}$, where $|X| = m \leq n = |Y|$. Then $K_{m,n}$ has order $m + n$ and $\delta\,(K_{m,n}) = m$. It follows from Lemma 2.1 that $\text{str}\,(K_{m,n}) \geq 2m + n$.

For the reverse inequality, consider the labeling $f : V\,(K_{m,n}) \to [1, m + n]$ such that

$$f\,(x_i) = i\ (i \in [1, m]) \text{ and } f\,(y_i) = m + i\ (i \in [1, n]).$$

Then f has the property that

$$\begin{aligned} \text{str}_f\,(K_{m,n}) &= \max\{f\,(u) + f\,(v) \,|uv \in E\,(K_{m,n})\} \\ &= f\,(x_m) + f\,(y_n) = 2m + n. \end{aligned}$$

Thus, $\text{str}\,(K_{m,n}) \leq 2m + n$. □

For the rest of this section, we concern with graphs that are cartesian products and related graphs. We first compute the strength of the ladder L_n.

Theorem 3.6 *For every integer* $n \geq 2$,

$$\text{str}\,(L_n) = 2n + 2.$$

Proof: Let n be an integer with $n \geq 2$. Then the ladder L_n has order $2n$ and $\delta\,(L_n) = 2$. Thus, the inequality $\text{str}\,(L_n) \geq 2n + 2$ is an immediate consequence of Lemma 2.1.

For the reverse inequality, it suffices to show the existence of a numbering f of L_n for which $\text{str}_f\,(L_n) = 2n + 2$. Let $V\,(L_n) = \{x_i | i \in [1, n]\} \cup \{y_i | i \in [1, n]\}$ and

$$\begin{aligned} E\,(L_n) &= \{x_i x_{i+1} | i \in [1, n-1]\} \cup \{y_i y_{i+1} | i \in [1, n-1]\} \\ &\quad \cup \{x_i y_i | i \in [1, n]\}, \end{aligned}$$

and consider the labeling $f : V\,(L_n) \to [1, 2n]$ such that

$$f\,(w) = \begin{cases} 2i - 1 & \text{if } w = x_{2i-1} \text{ and } i \in [1, \lceil n/2 \rceil], \\ 2n + 1 - 2i & \text{if } w = x_{2i} \text{ and } i \in [1, \lfloor n/2 \rfloor], \\ 2n + 2 - 2i & \text{if } w = y_{2i-1} \text{ and } i \in [1, \lceil n/2 \rceil], \\ 2i & \text{if } w = y_{2i} \text{ and } i \in [1, \lfloor n/2 \rfloor]. \end{cases}$$

Then f has the property that

$$\begin{aligned} \text{str}_f\,(L_n) &= \max\{f\,(u) + f\,(v)\,| uv \in E\,(L_n)\} \\ &= f\,(y_1) + f\,(y_2) = 2n + 2. \end{aligned}$$

Thus, $\text{str}\,(L_n) = 2n + 2$. □

We next present a formula for the strength of the prism D_n.

Theorem 3.7 *For every integer* $n \geq 3$,

$$\text{str}\,(D_n) = 2n + 3.$$

Proof: In light of Lemma 2.1, it suffices to show that $\text{str}\,(D_n) \leq 2n + 3$, since D_n is a 3-regular graph of order $2n$. Define the ladder L_n as in the proof of Theorem 3.6, and let $V\,(D_n) = V\,(L_n)$ and $E\,(D_n) = E\,(L_n) \cup \{x_1 x_n, y_1 y_n\}$. Then the labeling $f : V\,(D_n) \to [1, 2n]$ such that

$$f\,(w) = \begin{cases} 1 & \text{if } w = x_1, \\ 2i & \text{if } w = x_{2i-1} \text{ and } i \in [2, \lceil n/2 \rceil], \\ 2n + 1 - 2i & \text{if } w = x_{2i} \text{ and } i \in [1, \lfloor n/2 \rfloor], \\ 2n + 2 - 2i & \text{if } w = y_{2i-1} \text{ and } i \in [1, \lceil (n-1)/2 \rceil], \\ 2 & \text{if } w = y_2, \\ 1 + 2i & \text{if } w = y_{2i} \text{ and } i \in [2, \lfloor (n-1)/2 \rfloor], \\ 3 & \text{if } w = y_n \end{cases}$$

has the property that

$$\begin{aligned}\mathrm{str}_f(D_n) &= \max\{f(u)+f(v)\,|uv\in E(D_n)\}\\ &= f(y_1)+f(y_n)=2n+3.\end{aligned}$$

Thus, $\mathrm{str}(D_n)\le 2n+3$. □

The *Möbius ladder* M_n is the graph obtained from the ladder L_n by joining the opposite end-vertices of the two copies of P_n. We have a formula for the strength of M_n as shown below.

Theorem 3.8 *For every integer* $n\ge 3$,

$$\mathrm{str}(M_n)=2n+3.$$

Proof: In light of Lemma 2.1, it suffices to show that $\mathrm{str}(M_n)\le 2n+3$, since M_n is a 3-regular graph of order $2n$. Define the ladder L_n as in the proof of Theorem 3.6, and let $V(M_n)=V(L_n)$ and $E(M_n)=E(L_n)\cup\{x_1y_n, x_ny_1\}$. Then the labeling $f: V(M_n)\to[1,2n]$ such that

$$f(w)=\begin{cases}2i-1 & \text{if } w=x_{2i-1} \text{ and } i\in[1,\lceil n/2\rceil],\\ 2n+2-2i & \text{if } w=x_{2i} \text{ and } i\in[1,\lfloor n/2\rfloor],\\ n+1 & \text{if } w=y_1,\\ 2n+3-2i & \text{if } w=y_{2i-1} \text{ and } i\in[2,\lceil n/2\rceil],\\ 2i & \text{if } w=y_{2i} \text{ and } i\in[1,\lfloor n/2\rfloor]\end{cases}$$

has the property that

$$\begin{aligned}\mathrm{str}_f(M_n) &= \max\{f(u)+f(v)\,|uv\in E(M_n)\}\\ &= f(x_2)+f(x_3)=2n+3.\end{aligned}$$

Thus, $\mathrm{str}(M_n)\le 2n+3$. □

The next result provides a formula for the strength of the graph $K_{m,n}\times K_2$.

Theorem 3.9 *For every two positive integers* m *and* n *with* $n\ge m$,

$$\mathrm{str}(K_{m,n}\times K_2)=3m+2n+1.$$

Proof: Let $U_1=\{u_i\,|i\in[1,m]\}$ and $U_2=\{u_i\,|i\in[m+1,m+n]\}$ be the partite sets of $K_{m,n}$ such that $\deg u_i=n$ for each $i\in[1,m]$ and $\deg u_j=m$ for each $j\in[m+1,m+n]$, where $n\ge m$. Further, let $G\cong K_{m,n}\times K_2$ and define the graph G with

$$V(G)=\{v_i\,|i\in[1,m+n]\}\cup\{w_i\,|i\in[1,m+n]\}$$

and

$$\begin{aligned}E(G) &= \{v_iw_j\,|u_iu_j\in E(K_{m,n})\}\cup\{w_iv_j\,|u_iu_j\in E(K_{m,n})\}\\ &\quad\cup\{v_iw_{m+1-i}\,|i\in[1,m]\}\cup\{v_iw_{2m+n+1-i}\,|i\in[m+1,m+n]\}.\end{aligned}$$

It follows from Lemma 2.1 that $\text{str}(G) \geq 3m + 2n + 1$, since G has order $2(m+n)$ and $\delta(G) = m + 1$.

To see that $\text{str}(G) \leq 3m + 2n + 1$, it suffices to show the existence of a numbering f of G for which $\text{str}_f(G) = 3m + 2n + 1$. Now, consider the labeling $f : V(G) \to [1, 2(m+n)]$ such that

$$f(x) = \begin{cases} i & \text{if } x = v_i \text{ and } i \in [1, m+n], \\ m+n+i & \text{if } x = w_i \text{ and } i \in [1, m+n]. \end{cases}$$

Then f has the property that

$$\begin{aligned} \text{str}_f(G) &= \max\{f(u) + f(v) \,|uv \in E(G)\} \\ &= f(v_{m+1}) + f(w_{m+n}) = 3m + 2n + 1. \end{aligned}$$

Thus, $\text{str}(G) = 3m + 2n + 1$. □

From this theorem, we obtain a formula for the strength of the book B_n.

Corollary 3.1 *For every positive integer n,*

$$\text{str}(B_n) = 2n + 4.$$

At this point, we make a remark for the classes of graphs examined in this section. For any graph G of order p and any integer m such that $p \geq \delta(G) + m$, it follows from Theorem 2.4 that $\text{str}(G + mK_1) \geq p + \delta(G) + 2m$. However, the equality $\text{str}(G) = p + \delta(G)$ holds for all the classes of graphs G considered in this section. For these classes of graphs G, the lower and upper bounds given in Theorem 2.4 coincide. Consequently, $\text{str}(G + mK_1) = p + \delta(G) + 2m = \text{str}(G) + 2m$, which produces the following result.

Corollary 3.2 *If G is a graph of order $p \geq \delta(G) + m$ with $\text{str}(G) = p + \delta(G)$, then*

$$\text{str}(G + mK_1) = p + \delta(G) + 2m = \text{str}(G) + 2m$$

for every positive integer m.

The hypercube Q_n serves as useful models for a broad range of applications such as circuit design, communication network addressing, parallel computation and computer architecture. For this reason, we next consider the strength of Q_n. It is clear that $\text{str}(Q_1) = 3$. It also follows from Theorem 3.3 that $\text{str}(Q_2) = 6$. For an integer $n \geq 3$, we have the following lower and upper bounds for the strength of Q_n.

Theorem 3.10 *For every integer $n \geq 3$,*

$$2^n + n \leq \text{str}(Q_n) \leq 2^n + 2^{n-2} + 1.$$

Proof: The lower bound for $\mathrm{str}(Q_n)$ follows immediately from Lemma 2.1, since Q_n is an n-regular graph of order 2^n.

In order to verify the upper bound, we employ induction on n. Let Q_2 be the graph with $V(Q_2) = \{x_1, x_2, x_3, x_4\}$ and $E(Q_2) = \{x_1x_3, x_1x_4, x_2x_3, x_2x_4\}$. For an integer $n \geq 2$, construct the graph Q_{n+1} by using the decomposition

$$Q_{n+1} \cong A_{n+1} \oplus B_{n+1} \oplus C_{n+1}$$

with

$$\begin{aligned}
V(A_{n+1}) &= \left\{x_i | i \in \left[1, 2^{n-1}\right]\right\} \cup \left\{x_i | i \in \left[2^n + 1, 2^n + 2^{n-1}\right]\right\}, \\
V(B_{n+1}) &= \left\{x_i | i \in \left[1, 2^{n+1}\right]\right\}, \\
V(C_{n+1}) &= \left\{x_i | i \in \left[2^{n-1} + 1, 2^n\right]\right\} \cup \left\{x_i | i \in \left[2^n + 2^{n-1} + 1, 2^{n+1}\right]\right\}, \\
E(A_{n+1}) &= \left\{x_i x_{2^n+j} | x_i x_{2^{n-1}+j} \in E(Q_n) \text{ and } i, j \in \left[1, 2^{n-1}\right]\right\}, \\
E(B_{n+1}) &= \{x_i x_{2^{n+1}+1-i} | i \in [1, 2^n]\}, \\
E(C_{n+1}) &= \left\{x_{2^{n-1}+i} x_{2^n+2^{n-1}+j} | x_i x_{2^{n-1}+j} \in E(Q_n) \text{ and } i, j \in \left[1, 2^{n-1}\right]\right\}.
\end{aligned}$$

Notice then that in this construction, A_{n+1} and C_{n+1} are isomorphic to Q_n, and also that Q_{n+1} is represented as a bipartite graph with two partite sets $\{x_i | i \in [1, 2^n]\}$ and $\{x_i | i \in [2^n + 1, 2^{n+1}]\}$ of the same cardinality 2^n.

With the aid of the preceding construction, we will prove that there exists a numbering f_n of Q_n for which $\mathrm{str}_{f_n}(Q_n) = 2^n + 2^{n-2} + 1$. For $n = 3$, the labeling f_3 of Q_3 such that $(f_3(x_i))_{i=1}^{8} = (1, 2, 3, 4, 7, 8, 5, 6)$ has the property that $\mathrm{str}_{f_n}(Q_n) = 11$. For an integer $n \geq 3$, assume that there exists a numbering f_n of Q_n for which

$$\begin{aligned}
\mathrm{str}_{f_n}(Q_n) &= \max\{f_n(u) + f_n(v) | uv \in E(Q_n)\} \\
&= f_n(x_{2^{n-2}+1}) + f_n(x_{2^{n-1}+2^{n-2}}) = 2^n + 2^{n-2} + 1,
\end{aligned}$$

and consider the labeling $f_{n+1} : V(Q_{n+1}) \to [1, 2^{n+1}]$ such that

$$\begin{aligned}
f_{n+1}(x_i) &= i, \\
f_{n+1}(x_{2^{n-1}+i}) &= 2^{n-1} + i, \\
f_{n+1}(x_{2^n+i}) &= 2^n + 2^{n-1} + i, \\
f_{n+1}(x_{2^n+2^{n-1}+i}) &= 2^n + i,
\end{aligned}$$

where $i \in [1, 2^{n-1}]$. Applying the inductive hypothesis to f_{n+1}, we obtain

$$\begin{aligned}
\mathrm{str}_{f_{n+1}}(Q_{n+1}) &= \max\{f_{n+1}(u) + f_{n+1}(v) | uv \in E(Q_{n+1})\} \\
&= f_{n+1}(x_{2^{n-1}+1}) + f_{n+1}(x_{2^n+2^{n-1}}) \\
&= \left(2^{n-1} + 1\right) + \left(2^n + 2^{n-1} + 2^{n-1}\right) = 2^{n+1} + 2^{n-1} + 1.
\end{aligned}$$

□

4 Bounds for the super magic strength

In this section, we focus on bounds for the super magic strength of super edge-magic graphs. We first state the following lemma found in [6].

Lemma 4.1 *A graph G of order p and size q is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow [1, p]$ such that the set*

$$S = \{f(u) + f(v) \,|uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $k = p + q + s$, where $s = \min(S)$ and

$$S = [k - (p + q), k - (p + 1)].$$

The preceding lemma suggests to us the next concept, which is a restriction of the concept for the strength of a graph. A *consecutive numbering* f of a graph G of order p is a labeling that assigns distinct elements of the set $[1, p]$ to the vertices of G, where each edge uv of G is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[c, c + q - 1]$ for some positive integer c. The *consecutive strength*, $\mathrm{cstr}_f(G)$, of a numbering $f : V(G) \rightarrow [1, p]$ of G is defined by

$$\mathrm{cstr}_f(G) = \max\{f(u) + f(v) \,|uv \in E(G)\},$$

that is, $\mathrm{cstr}_f(G)$ is the maximum edge label of G, and the *consecutive strength*, $\mathrm{cstr}(G)$, of a graph G itself is

$$\mathrm{cstr}(G) = \min\{\mathrm{cstr}_f(G) \,|f \text{ is a consecutive numbering of } G\}.$$

A consecutive numbering f of a graph G for which $\mathrm{cstr}_f(G) = \mathrm{cstr}(G)$ is called a *consecutive strength labeling* of G. It is clear that if G is not a super edge-magic graph or an empty graph, then $\mathrm{cstr}(G)$ is undefined (or we could define $\mathrm{cstr}(G) = +\infty$). It is also true that G is a super edge-magic graph if and only if $\mathrm{cstr}(G) < +\infty$.

For super edge-magic graphs, the notions of super edge-magic labeling and consecutive strength labeling are equivalent. From this observation, we arrive at the next connection between the super magic strength and consecutive strength of a super edge-magic graph.

Lemma 4.2 *For every super edge-magic graph G of order p,*

$$sm(G) = \mathrm{cstr}(G) + p + 1.$$

Proof: Let G be a super edge-magic graph of order p and size q, and assume that $q \geq 1$; otherwise, G is an empty graph, that is, $sm(G)$ and $\mathrm{cstr}(G)$ are undefined. To see that $sm(G) \geq \mathrm{cstr}(G) + p + 1$, let $sm(G) = k$ for some positive integer k. Then there exists a super edge-magic labeling f of G with magic constant k. By Lemma 4.1, f is a consecutive strength labeling of G such that

$$\{f(u) + f(v) \,|uv \in E(G)\} = [k - (p + q), k - (p + 1)],$$

that is, $\text{cstr}_f(G) = k-(p+1)$. Thus, $\text{cstr}(G) \leq k-(p+1)$, implying that $sm(G) \geq \text{cstr}(G)+p+1$.

To show that $sm(G) \leq \text{cstr}(G)+p+1$, let $\text{cstr}_f(G) = \text{cstr}(G) = k$ for some positive integer k. Then there exists a consecutive strength labeling f of G with consecutive strength k. By Lemma 4.1, f is a super edge-magic labeling of G with magic constant $k+p+1$ such that

$$\{f(u)+f(v)\,|uv \in E(G)\} = [k-(q-1), k].$$

Thus, $sm(G) \leq k+p+1$, implying that $sm(G) \leq \text{cstr}(G)+p+1$. □

From the proof of Lemma 4.2, it follows that $\text{cstr}(G) \geq q+2$ and thus $sm(G) \geq p+q+3$ for every super edge-magic graph G of order p and size q, since

$$\min\{f(u)+f(v)\,|uv \in E(G)\} \geq 3.$$

Furthermore, applying Lemma 4.2 to the formulas for the super magic strength of super edge-magic graphs found by Avadayappan et al. [2], we obtain formulas for the consecutive strength of the same classes of graphs.

It follows immediately from the definitions that $\text{cstr}(G) \geq \text{str}(G)$ for any super edge-magic graph G. This together with Lemma 4.2 gives us the next lower bound for the super magic strength of a super edge-magic graph in terms of its order and strength.

Corollary 4.1 *For every super edge-magic graph G of order p,*

$$sm(G) \geq \text{str}(G)+p+1.$$

We have used the fact that $\text{cstr}(G) \geq \text{str}(G)$ for any super edge-magic graph G to obtain the preceding result. On the other hand, it is not difficult to construct a super edge-magic graph G such that $\text{cstr}(G) - \text{str}(G) = +\infty$. Indeed, more is true as given a positive integer n, it is always possible to construct a super edge-magic graph G such that $\text{cstr}(G) - \text{str}(G) = n-1$. For example, Avadayappan et al. [2] showed that $sm(P_{2n}) = 5n+1$ for any positive integer n. Applying Lemma 4.2 with $G \cong P_{2n}$, we have $\text{cstr}(P_{2n}) = 3n$; however, we know from Theorem 3.1 that $\text{str}(P_{2n}) = 2n+1$.

As consequences of Corollary 4.1, the super magic strength analogues to the strength results are now presented below.

Corollary 4.2 *For every super edge-magic graph G of order p with $\delta(G) \geq 1$,*

(1) $sm(G) \geq 2p+\delta(G)+1$,
(2) $sm(G) \geq 2p+\kappa_1(G)+1$,
(3) $sm(G) \geq 2p+\kappa(G)+1$.

Corollary 4.3 *For every super edge-magic graph G of order p that is n-critical,*

$$sm(G) \geq 2p+n.$$

Corollary 4.4 *For every super edge-magic graph G of order p,*

$$sm\,(G) \geq \Delta\,(G) + p + 3.$$

We conclude this section with the remark that the sharpness of all the bounds stated in the corollaries above follow from Theorem 3.5 and the result established by Avadayappan et al. [2] that $sm\,(K_{1,n}) = 2n + 4$ for any positive integer n.

5 Conclusions

In this paper, we have extended the notion of super magic strength by introducing the concept of strength. This naturally arises from the observation that the super magic strength can be defined only for super edge-magic graphs. In the following, we summarize the work conducted in this paper, and propose new lines of research by introducing some open problems and a new conjecture.

In Section 2, we have established several bounds for the strength of a graph in terms of other parameters studied in graph theory. All of these bounds are sharp in the sense that there are infinitely many graphs that attain the bounds (consult the results contained in Section 3). However, some of these bounds are not particularly good for certain classes of graphs. For example, the bound provided by Corollary 2.3 for the complete bipartite graph $K_{m,n}$ of order $m + n$, where $n \geq m \geq 3$, differs from its strength by $n - 2$ (consult with Theorem 3.5). Also, the bound presented in Theorem 2.3 may be strict as illustrated by the complete bipartite graph. This motivates us to propose the next problem.

Problem 1 *Find good bounds for the strength of a graph.*

In Section 3, we have established formulas for the strength of certain classes of graphs, and lower and upper bounds for the strength of the hypercube. All these classes of graphs except possibly for the hypercube attain the bound given in Lemma 2.1. However, since $2^n + n = 2^n + 2^{n-2} + 1$ for $n \in [2, 3]$, it follows that the lower and upper bounds provided in Theorem 3.10 coincide for $n \in [2, 3]$. These lead us to propose the next two problems.

Problem 2 *For every integer $n \geq 4$, determine the exact value of* $\mathrm{str}\,(Q_n)$*.*

Problem 3 *Find sufficient conditions for a graph G of order p with $\delta\,(G) \geq 1$ to ensure that* $\mathrm{str}\,(G) = p + \delta\,(G)$*.*

In Section 4, we have introduced the concept of consecutive strength as a possible restriction of the concept for the strength of a graph. We then have shown a connection between the super magic strength of a super edge-magic graph and its consecutive strength. From this connection, it follows that the problems of determining the super magic strength and consecutive strength are equivalent. As corollaries of the bounds obtained in Section 2, we supply several lower bounds for the super magic strength and show that all of them are sharp by considering the star $K_{1,n}$ of order $n+1$. Consulting the results found by Avadayappan et al. [2] and contained in this paper, one can verify that these lower bounds are not particularly good in general. Therefore, we propose the next problem.

Problem 4 *Find good lower bounds for the super magic strength of a super edge-magic graph.*

Unfortunately, an upper bound is not known for the super magic strength thus far; hence, the most natural question is whether one can find at least upper bounds for certain classes of super edge-magic graphs as stated in the next problem.

Problem 5 *Find good upper bounds for the super magic strength of some classes of super edge-magic graphs.*

It was mentioned earlier that G is a super edge-magic graph if and only if $sm(G) < +\infty$. It has been conjectured by Enomoto et al. [5] that every nontrivial tree is super edge-magic. Analogously, we state the next conjecture.

Conjecture 1 *For every nontrivial tree T, $sm(T) < +\infty$.*

We have constructed a super edge-magic graph G such that $\text{cstr}(G) - \text{str}(G) = n$ for a given nonnegative integer n. This implies that $\text{cstr}(G) \geq \text{str}(G)$ for a super edge-magic graph G in general. Therefore, we propose the next problem.

Problem 6 *Find sufficient conditions for a super edge-magic graph G to ensure that $\text{cstr}(G) = \text{str}(G)$.*

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