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Upper bounds on the smallest size of a complete cap in $\mathrm{PG}(N,q)$, $N \geq 3$, under a certain probabilistic conjecture

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Abstract

In the N-dimensional projective space $\mathrm{PG}(N,q)$ over the Galois field of order q, $N \geq 3$, an iterative step-by-step construction of complete caps by adding a new point in every step is considered. It is proved that uncovered points are uniformly distributed in the space. A natural conjecture on an estimate of the number of new covered points in every step is done. For a part of the iterative process, this estimate is proved rigorously. Under the conjecture mentioned, new upper bounds on the smallest size $t_2(N,q)$ of a complete cap in $\mathrm{PG}(N,q)$, $N \geq 3$, are obtained, in particular,

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2 \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}.$$

A connection with the Birthday problem is noted. The effectiveness of the new bounds is illustrated by comparison with sizes of complete caps obtained by computer in wide regions of q.

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1 Introduction

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the Galois field $\mathbb{F}_q$ of order q. A k-cap in $\mathrm{PG}(N,q)$ is a set of k points no three of which are collinear. A k-cap is complete if it is not contained in a $(k+1)$-cap. Caps in $\mathrm{PG}(2,q)$ are also called arcs and they have been widely studied; see [4,5,7,8,20,28,30–33,39]. Let $\mathrm{AG}(N,q)$ be the N-dimensional affine space over $\mathbb{F}_q$. If $N>2$ only a few constructions and bounds are known for small complete caps in $\mathrm{PG}(N,q)$ and $\mathrm{AG}(N,q)$; see [1–3,6,10–14,20–32,35,36,38,39] for surveys and results.

Caps have been intensively studied for their connection with coding theory [30, 31,34]. A linear q-ary code of length n, dimension k, and minimum distance d is denoted by $[n,k,d]_q$. If a parity-check matrix of a linear q-ary code is obtained by taking as columns the homogeneous coordinates of the points of a cap in $\mathrm{PG}(N,q)$, then the code has minimum distance $d=4$ (with the exceptions of the complete 5-cap in $\mathrm{PG}(3,2)$ and 11-cap in $\mathrm{PG}(4,3)$ giving rise to the $[5,1,5]_2$ and $[11,6,5]_3$ codes). Complete n-caps in $\mathrm{PG}(N,q)$ correspond to non-extendable $[n,n-N-1,4]_q$ quasi-perfect codes of covering radius 2 [17,19]. If $N=2$ these codes are Minimum Distance Separable (MDS); for $N=3$ they are Almost MDS since their Singleton defect is equal to 1. For fixed N, the covering density of the mentioned codes decreases with decreasing n. Thus small complete caps have a better covering quality than the big ones.

Note also that caps are connected with quantum codes; see, for instance, [15,40].

In general, a central problem concerning caps is to determine the spectrum of the possible sizes of complete caps in a given space; see [30,31] and the references therein. Of particular interest for applications to coding theory is the lower part of the spectrum, as small complete caps correspond to short quasi-perfect linear codes with small covering density.

Let $t_2(N,q)$ be the *smallest size* of a complete cap in $\mathrm{PG}(N,q)$.

A hard open problem in the study of projective spaces is the determination of $t_2(N,q)$. The exact values of $t_2(N,q)$, $N\geq 3$, are known only for very small q. For instance, $t_2(3,q)$ is known only for $q\leq 7$; see [20, Table 3].

This work is devoted to *upper bounds* on $t_2(N,q)$, $N\geq 3$.

The trivial lower bound for $t_2(N,q)$ is $\sqrt{2}q^{\frac{N-1}{2}}$. Constructions of complete caps whose sizes are close to this lower bound are known for the following cases: $q=2$ and N arbitrary; $q=2^m>2$ and N odd; q is even square [14,20,21,25,27,35,38]. Using a modification of the probabilistic approach of [33] for the projective plane, the upper bound

$$t_2(N,q)<cq^{\frac{N-1}{2}}\log^{300}q,$$

where c is a constant independent of q, has been obtained in [13]. Computer assisted results on small complete caps in $\mathrm{PG}(N,q)$ and $\mathrm{AG}(N,q)$ are given in [6,10–12,20, 22,24,36].

The main result of this paper is given by Theorem 1.1 based on Theorem 4.5.

Theorem 1.1 (***The main result.***) *Let* $t_2(N,q)$ *be the* smallest size *of a complete cap in the projective space* $\mathrm{PG}(N,q)$. *Let* $D \geq 1$ *be a constant independent of* q.

(i) *Under Conjecture* 3.3(i), *in* $\mathrm{PG}(N,q)$ *with* $N \geq 3$, *we have*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{D}\sqrt{(N+1)\ln q}+1\right)+2 \sim \sqrt{D}q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}. \quad (1.1)$$

(ii) *Under Conjecture* 3.3(ii), *the bound* (1.1) *with* $D=1$ *holds, i.e.*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2 \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}, \quad N \geq 3. \quad (1.2)$$

Conjecture 1.2 *In* $\mathrm{PG}(N,q)$, $N \geq 3$, *the upper bound* (1.2) *holds for all* q *without any extra conditions and conjectures.*

This work can be treated as a development of the paper [4].

Some results of this work were briefly presented in [9].

The paper is organized as follows. In Section 2, we describe the iterative step-by-step process for constructing caps. In Section 3, probabilities of events, that points of $\mathrm{PG}(N,q)$ are not covered by a current cap, are considered. It is proved that uncovered points are uniformly distributed in the space. A natural Conjecture 3.3 on the estimate of the number of new covered points in every step of the iterative process is done. In Section 4, under the conjecture of Section 3 we give new upper bounds on $t_2(N,q)$. In Section 5, we illustrate the effectiveness of the new bounds comparing them with the results of computer search from [10, 11]. For a part of the iterative process, a rigorous proof of Conjecture 3.3 is given in Section 6. In Section 7, the reasonableness of Conjecture 3.3 is discussed. It is shown that in the steps of the iterative process where the rigorous estimates give bad results, these estimates do not reflect the real situation effectively. The reason is that the rigorous estimates assume that the number of uncovered points on unisecants is the same for all unisecants. However, in fact, there is a dispersion of the number of uncovered points on unisecants. Moreover, this dispersion grows in the iterative process. In Section 8, the results are discussed.

2 An iterative step-by-step process

Assume that in $\mathrm{PG}(N,q)$, $N \geq 3$, a complete cap is constructed by a step-by-step algorithm (*Algorithm* for short) which adds one new point to the cap in each step. As an example, we can mention the greedy algorithm that in every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points; see [7, 8, 20, 22].

Recall that a *point* of $\mathrm{PG}(N,q)$ is *covered by a cap* if the point lies on a bisecant of the cap. Clearly, all points of the cap are covered.

The space $\mathrm{PG}(N,q)$ contains $\theta_{N,q} = \frac{q^{N+1}-1}{q-1}$ points.

Assume that after the w-th step of Algorithm, a w-cap is obtained that does not cover exactly U_w points. Let $\mathbf{S}(U_w)$ be the set of all w-caps in $\mathrm{PG}(N,q)$ each of which does not cover exactly U_w points. Evidently, the group of collineations $P\Gamma L(N+1,q)$ preserves $\mathbf{S}(U_w)$.

Consider the $(w+1)$-st step of Algorithm. This step starts from a w-cap $\mathcal{K}_w$ with $\mathcal{K}_w \in \mathbf{S}(U_w)$. The choice $\mathcal{K}_w$ from $\mathbf{S}(U_w)$ can be done in distinct ways.

One way is to choose randomly a w-cap of $\mathbf{S}(U_w)$ so that for every cap of $\mathbf{S}(U_w)$ the probability to be chosen is equal to $\frac{1}{\#\mathbf{S}(U_w)}$. In this case, the set $\mathbf{S}(U_w)$ is considered as an ensemble of random objects with the uniform probability distribution. Anywhere where we depend on probabilities and mathematical expectations, such a random choice is supposed.

On the other hand, sometimes we study values that are average or maximum for all caps of $\mathbf{S}(U_w)$ without a random choice. Also, we can consider some properties that hold for all caps of $\mathbf{S}(U_w)$.

Finally, for practice calculations (e.g. for the illustration of investigations) we use the same cap adding to it one new point in each step of the iterative process.

Denote by $\mathcal{U}(\mathcal{K})$ the set of points of $\mathrm{PG}(N,q)$ that are not covered by a cap $\mathcal{K}$. By definition,

$$\#\mathcal{U}(\mathcal{K}_w) = U_w.$$

Let the cap $\mathcal{K}_w$ consist of w points $A_1, A_2, \ldots, A_w$. Let $A_{w+1} \in \mathcal{U}(\mathcal{K}_w)$ be the point that will be included into the cap in the $(w+1)$-st step.

Remark 2.1 Below we introduce a few point subsets, depending on A_{w+1}, for which we use the notation of the form $\mathcal{M}_w(A_{w+1})$. Any uncovered point may be added to $\mathcal{K}_w$. So, there exist U_w distinct subsets $\mathcal{M}_w(A_{w+1})$. When a particular point A_{w+1} is not relevant, one may use the short notation $\mathcal{M}_w$. The same concerns the quantities $\Delta_w(A_{w+1})$ and Δ_w introduced below.

A point A_{w+1} defines a bundle $\mathcal{B}(A_{w+1})$ of w unisecants of $\mathcal{K}_w$ which are denoted as $\overline{A_1A_{w+1}}, \overline{A_2A_{w+1}}, \ldots, \overline{A_wA_{w+1}}$, where $\overline{A_iA_{w+1}}$ is the unisecant connecting A_{w+1} with the cap point A_i. Every unisecant contains $q+1$ points. Except for $A_1, \ldots, A_w$, all the points on the unisecants in the bundle are candidates to be new covered points in the $(w+1)$-st step. Denote by $\mathcal{C}_w(A_{w+1})$ the point set of the candidates. By definition,

$$\mathcal{C}_w(A_{w+1}) = \mathcal{B}(A_{w+1}) \setminus \mathcal{K}_w, \quad \#\mathcal{C}_w = w(q-1)+1.$$

We call $\{A_{w+1}\}$ and $\mathcal{B}(A_{w+1}) \setminus (\mathcal{K}_w \cup \{A_{w+1}\})$, respectively, the *head* and the *basic part* of the bundle $\mathcal{B}(A_{w+1})$. For a given cap $\mathcal{K}_w$, in total, there are $\#\mathcal{U}(\mathcal{K}_w) = U_w$ distinct bundles and, respectively, U_w distinct sets of candidates.

Let $\Delta_w(A_{w+1})$ be the number of new covered points in the $(w+1)$-st step, i.e.

$$\Delta_w(A_{w+1}) = \#\mathcal{U}(\mathcal{K}_w) - \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = \#\{\mathcal{C}_w(A_{w+1}) \cap \mathcal{U}(\mathcal{K}_w)\}. \tag{2.1}$$

In the future, we consider continuous approximations of the discrete functions $\#\mathcal{U}(\mathcal{K}_w)$, $\Delta_w(A_{w+1})$, $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\})$, and other ones keeping the same notations.

3 Probabilities of uncovering and key conjectures

Let $n_w(H)$ be the number of caps of $\mathbf{S}(U_w)$ that do not cover a point H of $\mathrm{PG}(N,q)$. Each point $H \in PG(N,q)$ will be considered as a random object that is not covered by a randomly chosen w-cap $\mathcal{K}_w$ with some probability $p_w(H)$ defined as

$$p_w(H) = \frac{n_w(H)}{\#\mathbf{S}(U_w)}.$$

Lemma 3.1 *The value $n_w(H)$ is the same for all points $H \in \mathrm{PG}(N,q)$.*

Proof: Let $\mathbf{K}_w(H) \subseteq \mathbf{S}(U_w)$ be the subset of w-caps in $\mathbf{S}(U_w)$ that do not cover H. By the definition, $n_w(H) = \#\mathbf{K}_w(H)$. Let H_i and H_j be two distinct points of $\mathrm{PG}(N,q)$. In the group $P\Gamma L(N+1,q)$, denote by $\Psi(H_i,H_j)$ the subset of collineations sending H_i to H_j. Clearly, $\Psi(H_i,H_j)$ embeds the subset $\mathbf{K}_w(H_i)$ in $\mathbf{K}_w(H_j)$. Therefore, $\#\mathbf{K}_w(H_i) \le \#\mathbf{K}_w(H_j)$. Vice versa, $\Psi(H_j,H_i)$ embeds $\mathbf{K}_w(H_j)$ in $\mathbf{K}_w(H_i)$, and we have $\#\mathbf{K}_w(H_j) \le \#\mathbf{K}_w(H_i)$. Thus, $\#\mathbf{K}_w(H_i) = \#\mathbf{K}_w(H_j)$, i.e. $n_w(H_i) = n_w(H_j)$. □

So, $n_w(H)$ can be considered as n_w. This means that the probability $p_w(H)$ is the same for all points H; it may be considered as

$$p_w = \frac{n_w}{\#\mathbf{S}(U_w)}.$$

In turn, since the probability to be uncovered is independent of a point, we conclude that, for a w-cap $\mathcal{K}_w$ randomly chosen from $\mathbf{S}(U_w)$, the fraction $\#\mathcal{U}_w(\mathcal{K}_w)/\theta_{N,q}$ of uncovered points of $\mathrm{PG}(N,q)$ is equal to the probability p_w that a point of $\mathrm{PG}(N,q)$ is not covered. In other words,

$$p_w = \frac{\#\mathcal{U}_w(\mathcal{K}_w)}{\theta_{N,q}} = \frac{U_w}{\theta_{N,q}}. \tag{3.1}$$

Equality (3.1) can also be explained as follows. By Lemma 3.1, the multiset consisting of all points that are not covered by all caps of $\mathbf{S}(U_w)$ has cardinality $n_w \cdot \#PG(N,q)$, where $\#PG(N,q) = \theta_{N,q}$. This cardinality can also be written as $U_w \cdot \#\mathbf{S}(U_w)$. Thus, $n_w\theta_{N,q} = U_w \cdot \#\mathbf{S}(U_w)$, whence

$$\frac{n_w}{\#\mathbf{S}(U_w)} = \frac{U_w}{\theta_{N,q}}.$$

Let $s_w(h)$ be the number of ones in a sequence of h random and independent $1/0$ trials each of which yields 1 with the probability p_w. For the random variable $s_w(h)$ we have the binomial probability distribution; the expected value of $s_w(h)$ is

$$\mathbf{E}[s_w(h)] = hp_w = h\frac{U_w}{\theta_{N,q}}. \tag{3.2}$$

Remark 3.2 We can also consider the hypergeometric probability distribution, which describes the probability of $s'_w(h)$ successes in h random and independent draws without replacement from a finite population of size $\theta_{N,q}$ containing exactly U_w successes. The expected value of $s'_w(h)$ again is

$$\mathbf{E}[s'_w(h)] = h\frac{U_w}{\theta_{N,q}} = \mathbf{E}[s_w(h)].$$

Note also that the average number of uncovered points among h points of $\mathrm{PG}(N,q)$ calculated over all $\binom{\theta_{N,q}}{h}$ combinations of h points is

$$\frac{1}{\binom{\theta_{N,q}}{h}}\sum_{i=1}^{h} i\binom{\theta_{N,q}-U_w}{h-i}\binom{U_w}{i} = \frac{U_w}{\binom{\theta_{N,q}}{h}}\sum_{i=1}^{h}\binom{\theta_{N,q}-U_w}{h-i}\binom{U_w-1}{i-1} = \frac{U_w\binom{\theta_{N,q}-1}{h-1}}{\binom{\theta_{N,q}}{h}}$$
$$= h\frac{U_w}{\theta_{N,q}} = \mathbf{E}[s_w(h)].$$

We denote by $\mathbf{E}_{w,q}$ the expected value of the number of uncovered points among $w(q-1)+1$ randomly taken points in $\mathrm{PG}(N,q)$, if the events to be uncovered are independent. By Lemma 3.1, taking into account (3.1), (3.2), we have

$$\mathbf{E}_{w,q} = \mathbf{E}[s_w(w(q-1)+1)] = (w(q-1)+1)p_w = \frac{(w(q-1)+1)U_w}{\theta_{N,q}}. \tag{3.3}$$

In (2.1), we defined $\Delta_w(A_{w+1})$ as the number of new covered points in the $(w+1)$-st step. Since all candidates to be new covered points lie on some bundle, they cannot be considered as randomly taken points for which the events to be uncovered are independent. Thus, in the general case, the expected value $\mathbf{E}[\Delta_w]$ is not equal to $\mathbf{E}_{w,q}$.

On the other hand, there is a large number of random factors affecting the process, for instance, the relative positions and intersections of bisecants and unisecants. These factors especially act for growing q, when the volume of the ensemble $\mathbf{S}(U_w)$ and the number of distinct bundles $\mathcal{B}(A_{w+1})$ are relatively large. Therefore, the variance of the random variable Δ_w, in principle, implies the existence of bundles $\mathcal{B}(A_{w+1})$ providing the inequality $\Delta_w(A_{w+1}) > \mathbf{E}[\Delta_w]$. By these arguments (see also Section 7) Conjecture 3.3 seems to be reasonable and founded.

Conjecture 3.3 (i) (***The generalized conjecture.***) *For q large enough, in every $(w+1)$-st step of the iterative process in $\mathrm{PG}(N,q)$ considered in Section 2, there exists a w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$ such that one can find an uncovered point A_{w+1} providing the inequality*

$$\Delta_w(A_{w+1}) \geq \frac{\mathbf{E}_{w,q}}{D} = \frac{1}{D}\cdot\frac{(w(q-1)+1)U_w}{\theta_{N,q}}, \tag{3.4}$$

where $D \geq 1$ is a constant independent of q.

(ii) (***the basic conjecture***) *In (3.4), we have $D = 1$.*

4 Upper bounds on $t_2(N,q)$

We denote

$$Q = \frac{\theta_{N,q}}{q-1} = \frac{q^{N+1}-1}{(q-1)^2}. \tag{4.1}$$

By Conjecture 3.3, taking into account (2.1), (3.3), (3.4), we obtain

$$\begin{aligned} &\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = \#\mathcal{U}(\mathcal{K}_w) - \Delta_w(A_{w+1}) \\ &\le U_w\left(1 - \frac{w(q-1)+1}{D\theta_{N,q}}\right) < U_w\left(1 - \frac{w(q-1)}{D\theta_{N,q}}\right) < U_w\left(1 - \frac{w}{DQ}\right). \end{aligned} \tag{4.2}$$

Clearly, $\#\mathcal{U}(\mathcal{K}_1) = U_1 = \theta_{N,q} - 1$. Using (4.2) iteratively, we have

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \le (\theta_{N,q} - 1) f_q(w;D) < \theta_{N,q} f_q(w;D) \tag{4.3}$$

where

$$f_q(w;D) = \prod_{i=1}^{w}\left(1 - \frac{i}{DQ}\right). \tag{4.4}$$

Remark 4.1 The function $f_q(w;D)$ and its approximations, including (4.8), appear in distinct tasks of probability theory, for example, in the Birthday problem *[16, 18, 37]*. Actually, let the year contain DQ days and let all birthdays occur with the same probability. Then $P^{\neq}_{DQ}(w+1) = f_q(w;D)$ where $P^{\neq}_{DQ}(w+1)$ is the probability that no two persons from $w+1$ random persons have the same birthday. Moreover, if birthdays occur with different probabilities we have $P^{\neq}_{DQ}(w+1) < f_q(w;D)$ *[18]*.

In the following discussion, we consider a *truncated iterative process*. The iterative process ends when $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \le \xi$ where $\xi \ge 1$ is some value chosen to improve estimates. Then, to obtain a complete k-cap, it is sufficient to add to $\mathcal{K}_w$ at most ξ points. The size k of the complete cap obtained is as follows:

$$w+1 \le k \le w+1+\xi \text{ under condition } \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \le \xi. \tag{4.5}$$

Theorem 4.2 *Let* $f_q(w;D)$ *be as in* (4.4). *Let* ξ *be a constant independent of* w *with* $\xi \ge 1$. *Under Conjecture* 3.3, *in* PG(N,q),

$$t_2(N,q) \le w+1+\xi \tag{4.6}$$

where the value w *satisfies the inequality*

$$f_q(w;D) \le \frac{\xi}{\theta_{N,q}}. \tag{4.7}$$

Proof: By (4.3), to provide the inequality $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \le \xi$ it is sufficient to find w such that $\theta_{N,q} f_q(w;D) \le \xi$. Now (4.6) follows from (4.5). □

We find an upper bound on the smallest possible solution of inequality (4.7).

The Taylor series of $e^{-\alpha}$ implies $1-\alpha < e^{-\alpha}$ for $\alpha \neq 0$, whence

$$\prod_{i=1}^{w}\left(1-\frac{i}{DQ}\right) < \prod_{i=1}^{w} e^{-i/DQ} = e^{-(w^2+w)/2DQ} < e^{-w^2/2DQ}. \tag{4.8}$$

Lemma 4.3 *Let ξ be a constant independent of w with $\xi \geq 1$. The value*

$$w \geq \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + 1 \tag{4.9}$$

satisfies the inequality (4.7).

PROOF: By (4.4),(4.8), to provide (4.7) it is sufficient to find w such that

$$e^{-w^2/2DQ} \leq \frac{\xi}{\theta_{N,q}}.$$

As w should be an integer, in (4.9) one is added. □

Theorem 4.4 *Let $D \geq 1$ be a constant independent of q. Under Conjecture* 3.3(i),

$$t_2(N,q) \leq \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + \xi + 2, \quad \xi \geq 1, \tag{4.10}$$

where ξ is an arbitrarily chosen constant independent of w.

PROOF: The assertion follows from (4.6) and (4.9). □

We should choose ξ to obtain a relatively small value in the right part of (4.10). We consider the function of ξ of the form

$$\phi(\xi) = \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + \xi + 2.$$

Its derivative by ξ is

$$\phi'(\xi) = 1 - \frac{1}{\xi}\sqrt{\frac{DQ}{2\ln\frac{\theta_{N,q}}{\xi}}}\ .$$

Put $\phi'(\xi) = 0$. Then

$$\xi^2 = \frac{DQ}{2\ln\theta_{N,q} - 2\ln\xi} = \frac{D\theta_{N,q}}{2(q-1)(\ln\theta_{N,q} - \ln\xi)}\ . \tag{4.11}$$

We find ξ in the form $\xi = \sqrt{\frac{\theta_{N,q}}{c \ln \theta_{N,q}}}$. By (4.11),

$$c = \frac{q-1}{D \ln \theta_{N,q}} (\ln \theta_{N,q} + \ln c + \ln \ln \theta_{N,q}) = \frac{q-1}{D} \left(1 + \frac{\ln c + \ln \ln \theta_{N,q}}{\ln \theta_{N,q}} \right).$$

So, for q large enough, one could take

$$c = \frac{q-1}{D}, \quad \xi = \sqrt{\frac{D\theta_{N,q}}{(q-1)\ln \theta_{N,q}}} = \sqrt{\frac{D(q^{N+1}-1)}{(q-1)^2 \ln \theta_{N,q}}}\ .$$

For simplicity, we put

$$\xi = \frac{\sqrt{q^{N+1}}}{q-1}\,. \tag{4.12}$$

Theorem 4.5 *Let $D \geq 1$ be a constant independent of q. Under Conjecture* 3.3(i), *the following upper bound on the smallest size $t_2(N,q)$ of a complete cap in* $\mathrm{PG}(N,q)$, *$N \geq 3$, holds:*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1} \left(\sqrt{D}\sqrt{(N+1)\ln q} + 1 \right) + 2 \sim \sqrt{D} q^{\frac{N-1}{2}} \sqrt{(N+1)\ln q}. \tag{4.13}$$

PROOF: In (4.10), we take Q and ξ from (4.1) and (4.12) and obtain

$$t_2(N,q) < \sqrt{2D\frac{q^{N+1}-1}{(q-1)^2} \cdot \ln \frac{\frac{q^{N+1}-1}{q-1}}{\frac{q^{\frac{N+1}{2}}}{q-1}}} + \frac{\sqrt{q^{N+1}}}{q-1} + 2$$

whence the relation (4.13) follows directly, as $q^{N+1} - 1 < q^{N+1}$. □

From Theorem 4.5 we obtain Theorem 1.1.

5 Illustration of the effectiveness of the new bounds

In [10, 11], for $\mathrm{PG}(N,q)$, $N = 3, 4$, $q \in L_N$, complete caps are obtained by computer search. Here

$$L_3 := \{q \leq 4673,\ q \text{ prime}\} \cup \{5003, 6007, 7001, 8009\},$$
$$L_4 := \{q \leq 1361,\ q \text{ prime}\} \cup \{1409\}.$$

All the complete caps obtained satisfy the bound (4.13) with $D = 1$ (equivalently, the bound (1.2)).

Let $\bar{t}_2(N,q)$ be the smallest *known* size of a complete cap in $\mathrm{PG}(N,q)$; the sizes $\bar{t}_2(N,q)$ can be found in [10].

In Figure 1, we compare the upper bound (1.2) with the sizes $\bar{t}_2(N,q)$. The top dashed-dotted curve, corresponding to (1.2), is strictly higher than the bottom curve $\bar{t}_2(N,q)$.

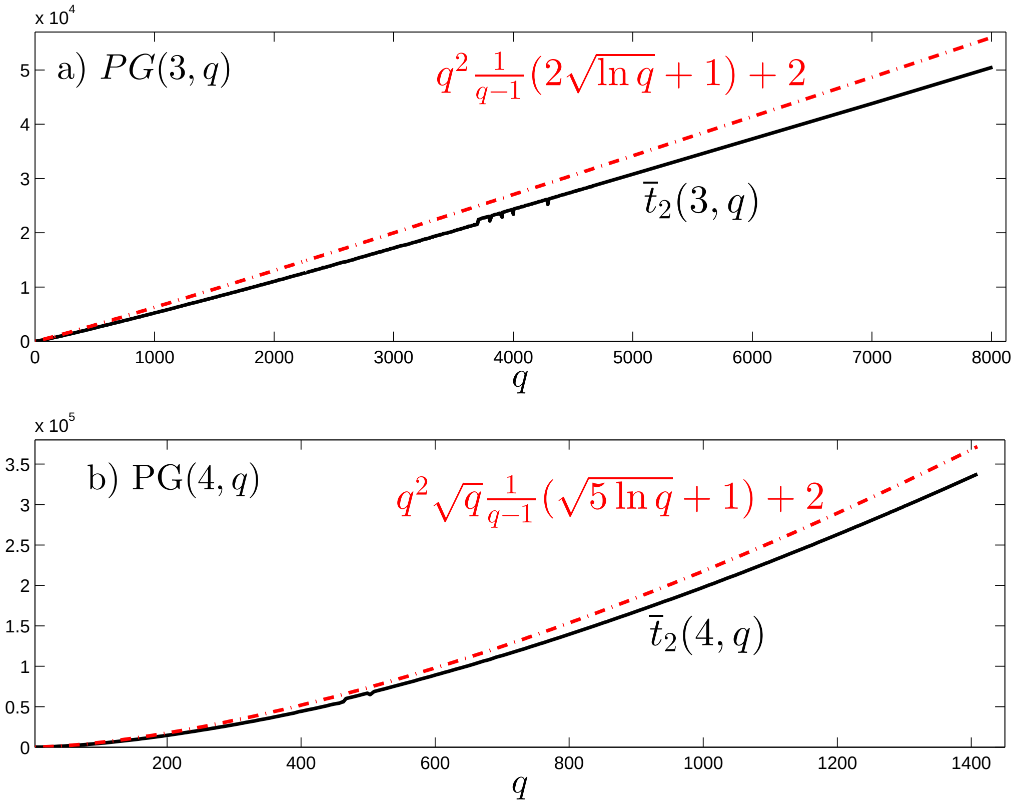

Figure 1: Bound $t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2$ (top dashed-dotted curve) vs. the smallest known sizes $\bar{t}_2(N,q)$ of complete caps, $q \in L_N$, $N = 3,4$ (bottom curve). a) PG$(3,q)$ b) PG$(4,q)$

6 A rigorous proof of Conjecture 3.3 for a part of the iterative process

In the following discussion, we take into account the fact that all points not covered by a cap lie on unisecants of the cap.

There are $\theta_{N-1,q}$ lines through every point of PG(N,q). Therefore, through every point A_i of $\mathcal{K}_w$ there is a pencil $\mathcal{P}(A_i)$ of $\theta_{N-1,q}-(w-1)$ unisecants of $\mathcal{K}_w$, where $i = 1,2,\dots,w$. The total number T_w^Σ of the unisecants of $\mathcal{K}_w$ is

$$T_w^\Sigma = w(\theta_{N-1,q}+1-w). \tag{6.1}$$

Let $\gamma_{w,j}$ be the number of uncovered points on the j-th unisecant $\mathcal{T}_j$, $j = 1,2,\dots,T_w^\Sigma$.

Observation 6.1 Every unisecant of $\mathcal{K}_w$ belongs to one and only one pencil $\mathcal{P}(A_i)$, $i \in \{1,2,\dots,w\}$. Every uncovered point belongs to one and only one unisecant from every pencil $\mathcal{P}(A_i)$, $i = 1,2,\dots,w$. Every uncovered point A lies on exactly w unisecants which form the bundle $\mathcal{B}(A)$ with the head $\{A\}$. All unisecants from the same bundle belong to distinct pencils. A unisecant $\mathcal{T}_j$ belongs to $\gamma_{w,j}$ distinct bundles.

Every uncovered point lies on exactly w unisecants; due to this multiplicity, in total, on all unisecants there are Γ_w^Σ uncovered points, where

$$\Gamma_w^\Sigma = \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w. \tag{6.2}$$

By (6.1) and (6.2), the average number γ_w^{aver} of uncovered points on a unisecant is

$$\gamma_w^{\text{aver}} = \frac{\Gamma_w^\Sigma}{T_w^\Sigma} = \frac{U_w}{\theta_{N-1,q} + 1 - w}. \tag{6.3}$$

A unisecant $\mathcal{T}_j$ belongs to $\gamma_{w,j}$ distinct bundles, because every uncovered point on $\mathcal{T}_j$ may be the head of a bundle. Moreover, $\mathcal{T}_j$ provides $\gamma_{w,j}(\gamma_{w,j} - 1)$ uncovered points to the basic parts of all these bundles. The noted points are counted with multiplicity.

Taking into account the multiplicity, in all U_w the bundles there are

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}(\gamma_{w,j} - 1) \tag{6.4}$$

uncovered points, where U_w is the total numbers of all the heads. By (6.2) and (6.4),

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 - \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = U_w(1-w) + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2.$$

For a cap $\mathcal{K}_w$, we denote by $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ the average value of $\Delta_w(A_{w+1})$ by all $\#\mathcal{U}(\mathcal{K}_w)$ uncovered points A_{w+1}, i.e.

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{\sum\limits_{A_{w+1}} \Delta_w(A_{w+1})}{\#\mathcal{U}(\mathcal{K}_w)} = \frac{\sum\limits_{A_{w+1}} \Delta_w(A_{w+1})}{U_w} = \frac{\sum\limits_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w} - w + 1 \geq 1. \tag{6.5}$$

Also note that

$$\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \geq \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w. \tag{6.6}$$

We denote a lower estimate of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$, see Lemma 6.2 below, as follows:

$$\Delta_w^{\text{rigor}}(\mathcal{K}_w) := \max\left\{1, \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1\right\} = \tag{6.7}$$
$$= \begin{cases} \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 & \text{if} \quad U_w \geq \theta_{N-1,q} + 1 - w, \\ 1 & \text{if} \quad U_w < \theta_{N-1,q} + 1 - w. \end{cases}$$

Lemma 6.2 *For any w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$, the following assertions hold:*

- *This inequality always holds*

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w). \tag{6.8}$$

- *In (6.8), we have the equality*

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1 \tag{6.9}$$

if and only if every unisecant contains the same number $\frac{U_w}{\theta_{N-1,q}+1-w}$ of uncovered points where $\frac{U_w}{\theta_{N-1,q}+1-w}$ is an integer.

- *In (6.8), the equality*

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = 1 \tag{6.10}$$

holds if and only if each unisecant contains at most one uncovered point.

PROOF: By the Cauchy–Schwarz–Bunyakovsky inequality, we have

$$\left(\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} \right)^2 \leq T_w^\Sigma \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \tag{6.11}$$

where equality holds if and only if all $\gamma_{w,j}$ coincide. In this case, $\gamma_{w,j} = \frac{U_w}{\theta_{N-1,q}+1-w}$ for all j and, moreover, the ratio $\frac{U_w}{\theta_{N-1,q}+1-w}$ is an integer. Now, by (6.1) and (6.2),

$$\frac{wU_w}{\theta_{N-1,q} + 1 - w} \leq \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w}$$

that together with (6.2), (6.5), (6.6) and (6.7) gives (6.8)–(6.10). □

Remark 6.3 One can treat the estimates (6.8) and (6.9) as follows. A bundle contains w unisecants having a common point, its head. Therefore the average number of uncovered points in a bundle is $w\gamma_w^{\text{aver}} - (w-1)$ where γ_w^{aver} is defined in (6.3) and the term $w-1$ takes into account the common point.

It is clear that for any w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$, we have

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) \geq \left\lceil \Delta_w^{\text{aver}}(\mathcal{K}_w) \right\rceil . \tag{6.12}$$

Corollary 6.4 *The following inequality holds:*

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) \geq \max\left\{1, \left\lceil \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 \right\rceil\right\}.$$

Let $D \geq 1$ be a constant independent of q. Throughout the paper we denote

$$\Phi_{w,q}(D) = \frac{D(w-1)\theta_{N,q}(\theta_{N-1,q}+1-w)}{Dw\theta_{N,q} - (\theta_{N-1,q}+1-w)(w(q-1)+1)},$$
$$\Upsilon_{w,q}(D) = \frac{D\theta_{N,q}}{w(q-1)+1}.$$

Lemma 6.5 *Let $D \geq 1$ be a constant independent of q. Let either of the following two conditions hold:*

$$U_w \geq \Phi_{w,q}(D), \quad \Upsilon_{w,q}(D) \geq U_w.$$

Then, for any cap $\mathcal{K}_w$ of $\mathbf{S}(U_w)$, we have

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \frac{\mathbf{E}_{w,q}}{D}.$$

Proof: By (6.7) and (6.8),

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w) \geq \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1.$$

It is easy to see that under condition $U_w \geq \Phi_{w,q}(D)$, we have

$$\frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 - \frac{(w(q-1)+1)U_w}{D\theta_{N,q}} \geq 0.$$

If $U_w \leq \Upsilon_{w,q}(D)$ then $\frac{\mathbf{E}_{w,q}}{D} \leq 1$. On the other hand, the inequalities $\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w) \geq 1$ always hold. □

From Lemmas 6.2 and 6.5 we obtain the following corollary.

Corollary 6.6 *Let $D \geq 1$ be a constant independent of q. Let either of the following two conditions hold:*

$$U_w \geq \Phi_{w,q}(D), \quad \Upsilon_{w,q}(D) \geq U_w.$$

Then, for any cap $\mathcal{K}_w$ of $\mathbf{S}(U_w)$, there exists an uncovered point A_{w+1} yielding the inequality

$$\Delta_w(A_{w+1}) \geq \frac{\mathbf{E}_{w,q}}{D} = \frac{(w(q-1)+1)U_w}{D\theta_{N,q}}.$$

Proof: By definition of the average value (6.5), there is always an uncovered point A_{w+1} yielding the inequality $\Delta_w(A_{w+1}) \geq \Delta_w^{\text{aver}}(\mathcal{K}_w)$; see also (6.12). □

7 On reasonableness of Conjecture 3.3

In this section we show (by reflections, calculations and figures) that in the steps of the iterative process where the rigorous estimates give bad results, these estimates do not reflect the real situation effectively.

(i) First we will illustrate the following: when the rigorous bound (6.7)–(6.8) is smaller than the expectation $\mathbf{E}_{w,q}$, the average value $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ of (6.5) is greater (and the maximum value $\max\limits_{A_{w+1}} \Delta_w(A_{w+1})$ is essentially greater) than $\mathbf{E}_{w,q}$.

We have calculated the values $\Delta_w(A_{w+1})$, defined in (2.1), for numerous iterative processes in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$. It is important that for all the calculations which have been done, we have

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) > \mathbf{E}_{w,q}.$$

Moreover, the ratio $\max\limits_{A_{w+1}} \Delta_w(A_{w+1})/\mathbf{E}_{w,q}$ has an increasing trend when w grows. Thus, the variance of the random value Δ_w helps to get good results.

The existence of points A_{w+1} providing $\Delta_w(A_{w+1}) > \mathbf{E}_{w,q}$ is used by the greedy algorithms to obtain complete caps smaller than the bounds following from Conjecture 3.3.

An illustration of the aforesaid is shown in Figure 2 where for a complete k-cap in $\mathrm{PG}(3,101)$, $k = 415$, obtained by the greedy algorithm, the values

$$\delta_w^{\min} = \frac{\min\limits_{A_{w+1}} \Delta_w(A_{w+1})}{\mathbf{E}_{w,q}}, \quad \delta_w^{\max} = \frac{\max\limits_{A_{w+1}} \Delta_w(A_{w+1})}{\mathbf{E}_{w,q}},$$
$$\delta_w^{\text{aver}} = \frac{\Delta_w^{\text{aver}}(\mathcal{K}_w)}{\mathbf{E}_{w,q}}, \quad \delta_w^{\text{rigor}} = \frac{\Delta_w^{\text{rigor}}(\mathcal{K}_w)}{\mathbf{E}_{w,q}},$$

are presented. The horizontal axis shows the values of $\frac{w}{k}$. The final region of the iterative process when $U_w \le \Upsilon_{w,q}(D)$ and $\frac{\mathbf{E}_{w,q}}{D} \le 1$ is not completely shown. The lines $y = 1$ and $y = \frac{1}{5}$ correspond, respectively, to Conjecture 3.3(ii), where $D = 1$, and Conjecture 3.3(i) with $D = 5$. The signs $\bullet$ correspond to the values $\Phi_{w,q}(D)$ and $\Upsilon_{w,q}(D)$ with $D = 1$ and $D = 5$. It is important that, for all the steps of the iterative process, we have $\Delta_w^{\text{aver}}(\mathcal{K}_w) > \mathbf{E}_{w,q}$, i.e. $\delta_w^{\text{aver}} > 1$.

In Figure 2, the region where we rigorously prove Conjecture 3.3 lies on the left of $\Phi_{w,q}(D)$ and on the right of $\Upsilon_{w,q}(D)$. This region takes $\sim 35\%$ of the whole iterative process for $D = 1$ and $\sim 75\%$ for $D = 5$.

Note that the forms of curves $\delta_w^{\max}$ and δ_w^{aver} are similar for all q and N for which we calculated these values.

(ii) Now we consider the dispersion of the number of uncovered points on unisecants.

The lower estimate in (6.8), based on (6.11), is attained in two cases: either every unisecant contains the same number of uncovered points or each unisecant contains at most one uncovered point.

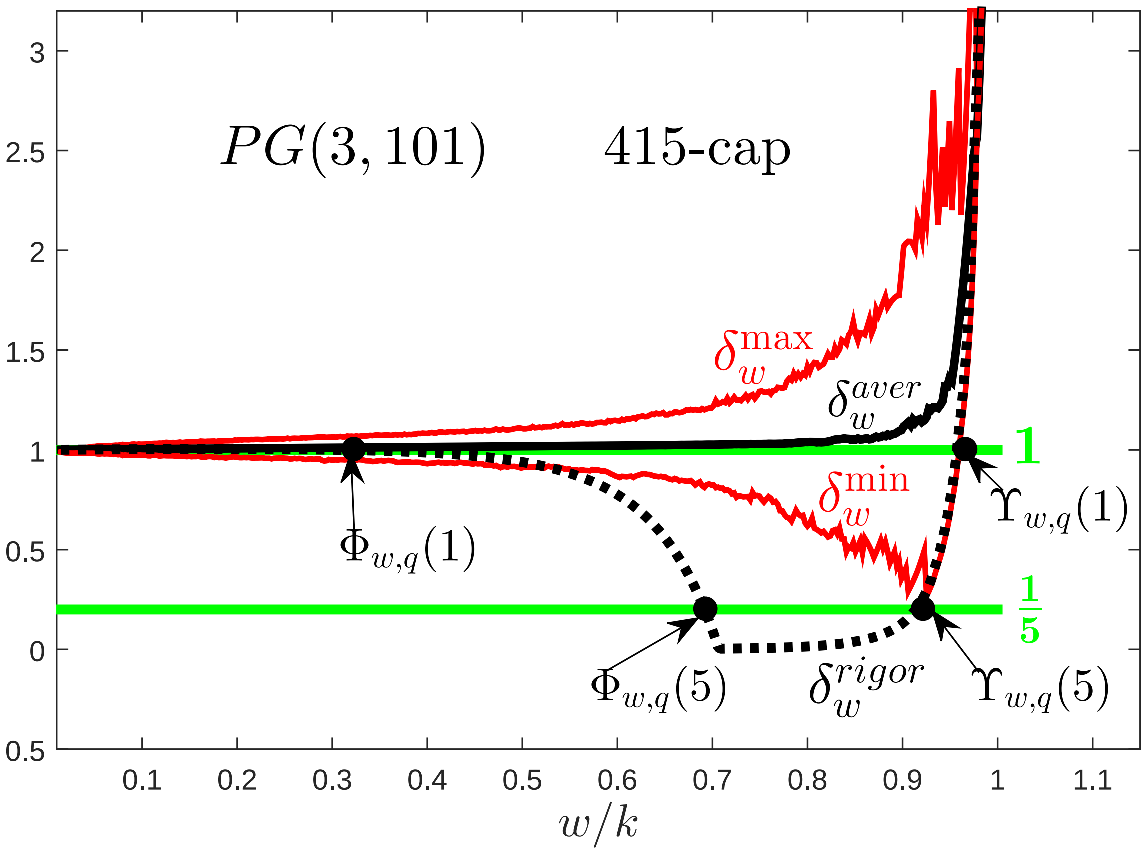

Figure 2: Illustration of reasonableness of Conjecture 3.3. Values $\delta_w^{\bullet}$ for a complete k-cap in PG$(3,q)$, $k = 415$: $\delta_w^{\max}$ (top curve), δ_w^{aver} (the second curve), $\delta_w^{\min}$ (the third curve), δ_w^{rigor} (bottom dotted curve); lines $y = 1$ (for $D = 1$) and $y = \frac{1}{5}$ (for $D = 5$)

The first situation holds in the starting steps of the iterative process only. Then the differences $\gamma_{w,j} - \gamma_{w,i}$ become nonzero. But while the inequality $U_w(D) \geq \Phi_{w,q}(D)$ holds, these differences are relatively small and the estimate (6.8) works "well". When U_w decreases, the differences relatively increase, and the estimate becomes worse in the sense that actually $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\Delta_w^{\text{rigor}}(\mathcal{K}_w)$.

The second situation is possible, in principle, when $U_w \leq \theta_{N-1,q} + 1 - w$ and the average number γ_w^{aver} of uncovered points on a unisecant is smaller than one; see (6.3). But in this stage of the iterative process variations in the values $\gamma_{w,j}$ are relatively large; and again $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\Delta_w^{\text{rigor}}(\mathcal{K}_w)$.

In the final region of the iterative process, where $U_w \leq \Upsilon_{w,q}(D)$ and $\frac{\mathbf{E}_{w,q}}{D} \leq 1$, the estimate (6.8) becomes reasonable once more. Thus, in the region

$$\Phi_{w,q}(D) > U_w > \Upsilon_{w,q}(D)$$

the lower estimate (6.8) does not reflect the real situation effectively. This leads the necessity to formulate Conjecture 3.3 as a (plausible) hypothesis.

Let γ_w^{aver} be defined in (6.3). Let $\gamma_w^{\max}$ and $\gamma_w^{\min}$ be, respectively, the maximum and minimum of the number $\gamma_{w,j}$ of uncovered points on a unisecant, i.e.

$$\gamma_w^{\max} = \max_j \gamma_{w,j}, \quad \gamma_w^{\min} = \min_j \gamma_{w,j}.$$

An illustration of the fact that the numbers $\gamma_{w,j}$ of uncovered points on unisecants lie in a relatively wide region is shown in Figure 3, where for a complete k-cap in $\mathrm{PG}(3,101)$, $k = 415$, obtained by the greedy algorithm, the values $\gamma_w^{\max}/\gamma_w^{aver}$ and $\gamma_w^{\min}/\gamma_w^{aver}$ are presented. The horizontal axis shows the values of $\frac{w}{k}$. Such curves were obtained for numerous iterative processes in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$. It is important that for all the calculations have been done, the forms of the curves are similar. Moreover, the value $\gamma_w^{\max}/\gamma_w^{aver}$ increases when the ratio $\frac{w}{k}$ grows; in the region $0.78 < \frac{w}{k} < 0.95$ (it is not shown in Figure 3), the value $\gamma_w^{\max}/\gamma_w^{aver}$ increases from 20 to 590 for the 415-cap in $\mathrm{PG}(3,101)$.

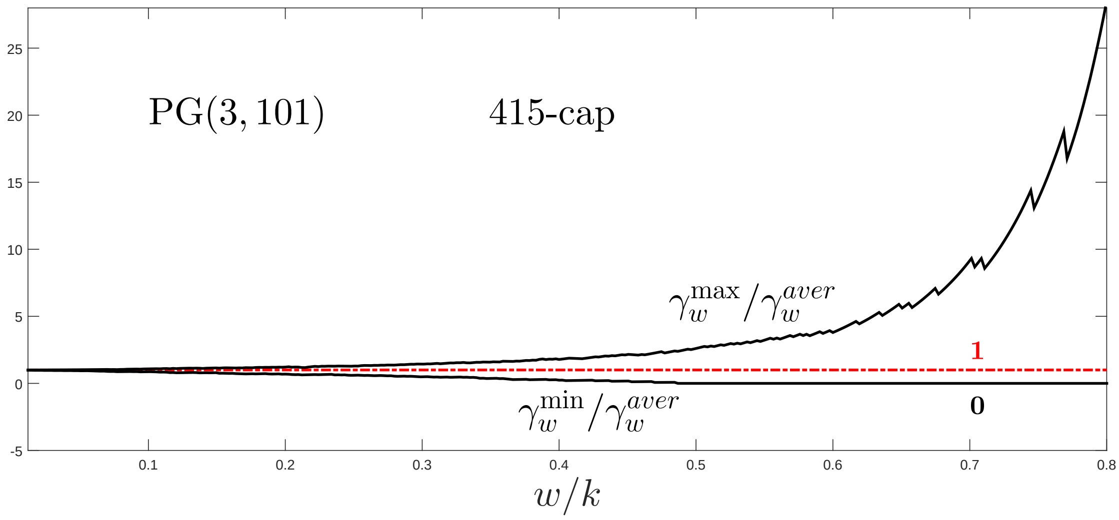

Figure 3: Dispersion of the number $\gamma_{w,j}$ of uncovered points on unisecants. Values $\gamma_w^{\max}/\gamma_w^{aver}$ (top solid curve), $\gamma_w^{\min}/\gamma_w^{aver}$ (bottom solid curve) and line $y = 1$ for a complete k-cap in $\mathrm{PG}(3,q)$, $k = 415$

Remark 7.1 It can be proved rigorously (using Observation 6.1) that if in some step of the iterative process every unisecant contains the same number of uncovered points then in the next step this situation does not hold.

The calculations mentioned in this section and Figures 2, 3 illustrate the soundness of the key Conjecture 3.3.

8 Conclusion

In the present paper, we have made an attempt to obtain a theoretical upper bound on $t_2(N,q)$ with the main term of the form $cq^{\frac{N-1}{2}}\sqrt{\ln q}$, where c is a small constant independent of q. The bound is based on explaining the mechanism of a step-by-step greedy algorithm for constructing complete caps in $\mathrm{PG}(N,q)$ and on quantitative estimations of the algorithm. For a part of the steps of the iterative process, these estimations have been proved rigorously. We made the natural (and well-founded) conjecture that they hold for other steps too. Under this conjecture we have given new upper bounds on $t_2(N,q)$ in the required form; see (1.1) and (1.2). We have illustrated the effectiveness of the new bounds, comparing them with the results of a computer search from [10, 11]; see Figure 1.

We have not obtained a rigorous proof for precisely the part of the process where the variance of the random variable $\Delta_w(A_{w+1})$ determining the estimates implies the existence of points A_{w+1} which are considerably better than what is necessary for fulfillment of the conjecture (see the curve $\delta_w^{\max}$ in Figure 2). In other words, in the steps of the iterative process where the rigorous estimates give bad results, these estimates do not reflect the real situation effectively. The reason is that the rigorous estimates assume that the number of uncovered points on unisecants is the same for all unisecants. However, there is a dispersion of the number of uncovered points on unisecants; see Section 7. Moreover, this dispersion grows in the iterative process. So Conjecture 3.3 seems to be reasonable.

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References

[1] N. Anbar, D. Bartoli, M. Giulietti and I. Platoni, Small complete caps from singular cubics, *J. Combin. Des.* **22** (2014), 409–424.

[2] N. Anbar, D. Bartoli, M. Giulietti and I. Platoni, Small complete caps from singular cubics, II, *J. Algebraic Combin.* **41** (2015), 185–216.

[3] N. Anbar and M. Giulietti, Bicovering arcs and small complete caps from elliptic curves, *J. Algebraic. Combin.* **38** (2013), 371–392.

[4] D. Bartoli, A. A. Davydov, G. Faina, A. A. Kreshchuk, S. Marcugini and F. Pambianco, Upper bounds on the smallest size of a complete arc in $\mathrm{PG}(2, q)$ under a certain probabilistic conjecture, *Problems Inform. Transmission* **50** (2014), 320–339.

[5] D. Bartoli, A. A. Davydov, G. Faina, A. A. Kreshchuk, S. Marcugini and F. Pambianco, Upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane based on computer search, *J. Geom.* **107** (2016), 89–117.

[6] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, New upper bounds on the smallest size of a complete cap in the space $\mathrm{PG}(3, q)$, in *Proc. VII Int. Workshop on Optimal Codes and Related Topics, OC2013*, Albena, Bulgaria, 2013, 26–32. `http://www.moi.math.bas.bg/oc2013/a4.pdf`

[7] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, On sizes of complete arcs in $\mathrm{PG}(2, q)$, *Discrete Math.* **312** (2012), 680–698.

[8] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, New types of estimates for the smallest size of complete arcs in a finite Desarguesian projective plane, *J. Geom.* **106** (2015), 1–17.

[9] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, Conjectural upper bounds on the smallest size of a complete cap in $\mathrm{PG}(N, q)$, $N \geq 3$, *Electron. Notes Discrete Math.* **57** (2017), 15–20.

[10] D. Bartoli, A. A. Davydov, A. A. Kreshchuk, S. Marcugini and F. Pambianco, Tables, bounds and graphics of the smallest known sizes of complete caps in the spaces PG(3,q) and PG(4,q), arXiv:1610.09656 [math.CO] (2016).

[11] D. Bartoli, A. A. Davydov, A. A. Kreshchuk, S. Marcugini and F. Pambianco, Upper bounds on the smallest size of a complete cap in $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$, *Electron. Notes Discrete Math.* **57** (2017), 21–26.

[12] D. Bartoli, G. Faina and M. Giulietti, Small complete caps in three-dimensional Galois spaces, *Finite Fields Appl.* **24** (2013), 184–191.

[13] D. Bartoli, G. Faina, S. Marcugini and F. Pambianco, A construction of small complete caps in projective spaces, *J. Geom.* **108** (2017), 215–246.

[14] D. Bartoli, M. Giulietti, G. Marino and O. Polverino, Maximum scattered linear sets and complete caps in Galois spaces, *Combinatorica* **38** (2018), 255–278.

[15] D. Bartoli, S. Marcugini and F. Pambianco, New quantum caps in $\mathrm{PG}(4, 4)$, *J. Combin. Des.* **20** (2012), 448–466.

[16] D. Brink, A (probably) exact solution to the birthday problem, *Ramanujan J.* **28** (2012), 223–238.

[17] R. A. Brualdi, S. Litsyn and V. S. Pless, Covering radius, in *Handbook of Coding Theory* (Eds.: V.S. Pless, W.C. Huffman and R.A. Brualdi), Vol. 1, Elsevier, Amsterdam, The Netherlands, 1998, 755–826.

[18] M. L. Clevenson and W. Watkins, Majorization and the Birthday inequality, *Math. Mag.* **64** (1991), 183–188.

[19] G. D. Cohen, I. S. Honkala, S. Litsyn and A. C. Lobstein, *Covering Codes*, Elsevier, Amsterdam, The Netherlands, 1997.

[20] A. A. Davydov, G. Faina, S. Marcugini and F. Pambianco, On sizes of complete caps in projective spaces $\mathrm{PG}(n, q)$ and arcs in planes $\mathrm{PG}(2, q)$, *J. Geom.* **94** (2009), 31–58.

[21] A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco, New inductive constructions of complete caps in $\mathrm{PG}(n, q)$, q even, *J. Combin. Des.* **18** (2010), 177–201.

[22] A. A. Davydov, S. Marcugini and F. Pambianco, Complete caps in projective spaces $\mathrm{PG}(n, q)$, *J. Geom.* **80** (2004), 23–30.

[23] A. A. Davydov and P. R. J. Östergård, Recursive constructions of complete caps, *J. Statist. Plann. Inference* **95** (2001), 167–173.

[24] G. Faina, F. Pasticci and L. Schmidt, Small complete caps in Galois spaces, *Ars Combin.* **105** (2012), 299–303.

[25] E. M. Gabidulin, A. A. Davydov and L. M. Tombak, Linear codes with covering radius 2 and other new covering codes, *IEEE Trans. Inform. Theory* **37** (1991), 219–224.

[26] M. Giulietti, Small complete caps in Galois affine spaces, *J. Algebraic Combin.* **25** (2007), 149–168.

[27] M. Giulietti, Small complete caps in $\mathrm{PG}(n, q)$, q even, *J. Combin. Des.* **15** (2007), 420–436.

[28] M. Giulietti, The geometry of covering codes: small complete caps and saturating sets in Galois spaces, in *Surveys in Combinatorics* 2013, London Math. Soc., Lec. Note Ser. 409, Cambridge University Press, Cambridge, 2013, 51–90.

[29] M. Giulietti and F. Pasticci, Quasi-perfect linear codes with minimum distance 4, *IEEE Trans. Inform. Theory* **53** (2007), 1928–1935.

[30] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *J. Statist. Plann. Inference* **72** (1998), 355–380.

[31] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: update 2001, in *Finite geometries*, 201-246, Dev. Math., 3, Kluwer Acad. Publ., Dordrecht, 2001.

[32] J. W. P. Hirschfeld and J. A. Thas, Open problems in finite projective spaces, *Finite Fields Appl.* **32** (2015), 44–81.

[33] J. H. Kim and V. Vu, Small complete arcs in projective planes, *Combinatorica* **23** (2003), 311–363.

[34] I. Landjev and L. Storme, Galois geometry and coding theory, in *Current Research Topics in Galois geometry* (Eds.: J. De Beule and L. Storme), Chapter 8, Nova Science Publisher, 2011, 185–212.

[35] F. Pambianco and L. Storme, Small complete caps in spaces of even characteristic, *J. Combin. Theory Ser. A* **75** (1996), 70–84.

[36] I. Platoni, Complete caps in AG$(3, q)$ from elliptic curves, *J. Alg. Appl.* **13** (2014), 1450050 (8 pp.).

[37] M. Sayrafiezadeh, The Birthday problem revisited, *Math. Mag.* **67** (1994), 220–223.

[38] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, *Acta Arith.* **5** (1959), 315–332.

[39] T. Szőnyi, Arcs, caps, codes and 3-independent subsets, in *Giornate di Geometrie Combinatorie* (Eds.: G. Faina and G. Tallini), Università degli Studi di Perugia, Perugia, Italy, 1993, 57–80.

[40] V. D. Tonchev, Quantum codes from caps, *Discrete Math.* **308** (2008), 6368–6372.

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