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Monochromatic sinks in 3-switched tournaments

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Abstract

Let T be a tournament whose arcs are colored using at least three colors. A cycle C in T is called q-switched if there are at most q vertices in C whose incident arcs in C receive two distinct colors. We prove that if every cycle in T of length at least four is 3-switched and every cycle of length three is 2-switched, then T contains a monochromatic sink.

1 Introduction

A *tournament* is an oriented complete graph. That is, if u and v are vertices in T, exactly one of uv and vu is an arc in T. A *k-arc-colored* tournament is one in which each arc is assigned one of k distinct colors. Such a coloring need not be proper. A *monochromatic path* in a k-arc-colored tournament is a directed path, all of whose arcs are colored the same. In their 1982 work on arc-colored digraphs, Sands, Sauer, and Woodrow [7] showed, when specialized to tournaments, that every 2-arc-colored tournament contains a vertex v such that for any other vertex w, there is a monochromatic path from w to v. Such a vertex in an arc-colored tournament is called a *monochromatic sink*. It is well-known that in any 1-arc-colored tournament, there exists a monochromatic sink. Consider any Hamiltonian path of such a tournament (at least one is guaranteed to exist) and the final vertex of this path.

When attempting to extend this idea to k-arc-colored tournaments for $k \geq 3$, one runs into obstacles. A *rainbow cycle* is a directed cycle in which no two arcs share the same color. In particular, a *rainbow triangle*, denoted Δ_3, is a rainbow 3-cycle.

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Consider the vertex set R of a rainbow triangle together with a set X of an arbitrary number of other vertices such that for all $x \in X$ and $r \in R$, there is an arc directed from x to r. No matter the coloring and directions of the remaining arcs, the resulting tournament, having vertex set $R \cup X$, does not contain a monochromatic sink. Hence, a rainbow triangle can be a road block for the existence of a monochromatic sink. This leads to the following question:

Question 1.1 *Let T be a k-arc-colored tournament without rainbow triangles. Must T contain a monochromatic sink?*

We call a tournament without rainbow triangles Δ_3*-free*. Shen [8] provided the answer to Question 1.1 by constructing a 5-arc-colored Δ_3-free tournament without a monochromatic sink. Later, Galeana-Sánchez and Rojas-Monroy [4] constructed a 4-arc-colored Δ_3-free tournament without a monochromatic sink. However, these examples contain a rainbow cycle of size five and four, respectively. In light of this work, the following question remains open:

Question 1.2 *Let T be a k-arc-colored Δ_3-free tournament without rainbow k-cycles. Must T contain a monochromatic sink?*

We refer the reader to [1], [3], [4], [5], [6], and [8] for background on the work during the past three decades pertaining to monochromatic sinks in tournaments. Much of this progress comes in the form of assuming the tournament has an additional property, usually concerning the coloring of the arcs or the orientations of the arcs. In this paper, we make an additional assumption of this type. In particular, we restrict the coloring of the arcs with respect to cycles. Such a restriction is natural considering rainbow cycles are potentially problematic.

Let C be a directed cycle in an arc-colored tournament T. We will refer to directed cycles simply as cycles. Similarly, directed paths will be referred to as paths. Also, since we will only be considering arc colorings of tournaments and never vertex colorings, it will not be ambiguous to say T is simply a k-colored tournament instead of a k-arc-colored tournament. We call a vertex v a *switch vertex with respect to C* if the arc in C incident from v and the arc in C incident to v have distinct colors. If v is a switch vertex with respect to C, we will say C *contains the switch vertex v*. We call C q*-switched* if it contains at most q switch vertices. We call a tournament q*-switched* if every cycle in the tournament is q-switched. The main result of this paper is the following.

Theorem 1.3 *Let T be a k-colored 3-switched Δ_3-free tournament. Then T has a monochromatic sink.*

We will also prove an analogous result for 2-switched tournaments. We leave it open to consider tournaments that contain cycles having more than three switch vertices.

Note that T has a monochromatic sink if T has fewer than three vertices. Furthermore, a tournament on three vertices is either transitive or a cycle. In either case, it is easily checked that a Δ_3-free tournament on three vertices necessarily contains a monochromatic sink. For this reason, we may assume throughout this paper that the tournaments we are dealing with have at least four vertices.

This paper is organized as follows. In Section 2, we introduce some notation and prove a lemma that will be used extensively to establish Theorem 1.3. This lemma motivates the definition of a special type of cycle called a dominating cycle. In Sections 3 and 4, we prove Theorem 1.3, dividing our argument into two main cases depending on the number of switch vertices present in a Hamiltonian dominating cycle.

2 Preliminaries

In the subsequent sections, we will assume a counterexample to Theorem 1.3 exists and arrive at a contradiction in each case we consider. Thus, it is important to develop the structure that is forced when we assume a k-colored tournament is Δ_3-free and has no monochromatic sink, and that is what we do here. We begin with some useful notation. For general concepts, we refer the reader to [2].

Let i and j be integers with $i < j$. We use the notation $[i, j]$ to denote the set of all integers k such that $i \leq k \leq j$. When $i = 0$, we will abbreviate $[i, j]$ to $[j]$. Throughout this paper, we assume the tournament T we are dealing with has n vertices, and we let the vertex set of T be $\{v_0, v_1, \ldots, v_{n-1}\}$. The sub-tournament of T formed by deleting a single vertex v_i will be denoted as $T-v_i$. As previously stated, we are interested in considering k-colorings of the arcs in a tournament. Thus, we will be partitioning the set of arcs of T, which we denote by $A(T)$, into k color classes, that we will label from the set of distinct colors $\{c_1, c_2, \ldots, c_k\}$. We will often refer to the color of a monochromatic path by the color of the arcs in the path. It will be convenient to have notation to indicate the direction of an arc between two vertices, as well as the color (or possible colors) of the arc. To this end, we let $v_i \longrightarrow v_j$ indicate there is an arc between v_i and v_j that is directed from v_i to v_j. Also, if we know the color of such an arc, say it is c_1, we write $v_i \xrightarrow{c_1} v_j$ to indicate that there is an arc from v_i to v_j colored c_1. When we know the arc from v_i to v_j is one of two colors, say c_1 or c_2, we will write $v_i \xrightarrow{c_1,c_2} v_j$. If there exists a monochromatic path from a vertex u to a vertex v, we write $u \rightsquigarrow v$. The color, or possible colors, of the monochromatic path will be notated similarly to that of the arcs. For two vertices u and v of a path P, we write uPv to indicate the subpath of P from u to v. Such a path can be empty and as a result, uPu represents the single vertex u. If u and v are distinct vertices in a k-colored cycle C, we write $\widehat{u,v}$ to indicate that at least one of u and v is a switch vertex in C. We use the common notation $|P|$ to denote the number of arcs in a path P.

Definition 2.1 *A cycle $C = u_0u_1u_2 \ldots u_{\ell-1}u_0$ in an arc-colored tournament T is a dominating cycle if for all i in $[\ell - 1]$, where i is considered modulo ℓ, the vertex u_i is a monochromatic sink in $T - u_{i+1}$, but there is no monochromatic path from u_{i+1} to u_i in T.*

We call $\mathcal{P}$ a *hereditary property* of an arc-colored tournament T if whenever T has property $\mathcal{P}$, then every sub-tournament of T has property $\mathcal{P}$. Note that each of the properties k-colored, q-switched, and Δ_3-free are hereditary properties of arc-colored tournaments. The following lemma by Melcher and Reid [6] was originally used implicitly by Shen [8]. We omit the proof, which can be found in [6].

Lemma 2.2 *Let $\mathcal{P}$ be a hereditary property of arc-colored tournaments. Suppose there exists an arc-colored tournament with property $\mathcal{P}$, but no monochromatic sink. If T is a smallest arc-colored tournament such that $\mathcal{P}$ is a property of T and T contains no monochromatic sink, then T has a Hamiltonian dominating cycle.*

Lemma 2.2 implies that a smallest counterexample to Theorem 1.3 must contain a Hamiltonian dominating cycle. Consequently, when we prove Theorem 1.3, we will do so by contradiction and consider T to be a smallest counterexample, gaining the usefulness of the Hamiltonian dominating cycle structure. For this reason, when referring to a vertex v_i in a tournament T, the subscript i will be interpreted modulo n.

Before further developing the tools needed to prove Theorem 1.3, we prove an analogous result for k-colored 2-switched tournaments.

Theorem 2.3 *Let T be a k-colored 2-switched tournament of order $n \geq 2$. Then T has a monochromatic sink.*

Proof. Suppose, to the contrary, the result does not hold. Let T be a smallest counterexample. Since T does not contain a monochromatic sink, Lemma 2.2 implies that T contains a Hamiltonian dominating cycle $C = v_0v_1 \dots v_{n-1}v_0$. We see that C must contain exactly two switch vertices. Let v_s be a switch vertex in C. Up to a relabeling of colors, we may assume without loss of generality that $v_{s-1} \xrightarrow{c_1} v_s$ and $v_s \xrightarrow{c_2} v_{s+1}$. By Definition 2.1, there exists a monochromatic path P from v_{s+1} to v_{s-1}. We know P cannot be colored c_1 or c_2, since this would imply $v_{s+1} \overset{c_1}{\rightsquigarrow} v_s$ or $v_s \overset{c_2}{\rightsquigarrow} v_{s-1}$, respectively, contradicting Definition 2.1. Now the cycle $v_sv_{s+1}Pv_{s-1}v_s$ has three switch vertices, v_{s-1}, v_s, and v_{s+1}, contradicting our assumption that T is 2-switched. □

Naturally, one may wonder about such a result for k-colored 1-switched tournaments. Since it is impossible for a cycle to contain exactly one switch vertex, in such a tournament T, all cycles (if any) are monochromatic. Thus, any vertex in the terminal strong component of T is a monochromatic sink in T.

The remainder of this paper will focus on the proof of Theorem 1.3. As we have observed, if T is a smallest counterexample to Theorem 1.3, then T is a k-colored, Δ_3-free, 3-switched tournament without a monochromatic sink, and by Lemma 2.2, T has a Hamiltonian dominating cycle. Furthermore, by definition, any dominating cycle cannot be 0-switched, and as stated above, it is impossible for a cycle to be 1-switched. Thus to prove Theorem 1.3 by contradiction, we need only consider the two cases when a smallest counterexample has a 2-switched or a 3-switched Hamiltonian dominating cycle. These are the topics of the next two sections. In the various situations that arise, we will commonly arrive at two main contradictions. Namely, we will often either produce a cycle that contains at least four switch vertices, contradicting the assumption that our tournament is 3-switched, or we will determine that $v_i \rightsquigarrow v_{i-1}$ for some $i \in [n-1]$, contradicting Definition 2.1.

3 2-switched dominating cycle

The goal of this section is to rule out the case that a smallest counterexample to Theorem 1.3 contains a 2-switched Hamiltonian dominating cycle.

Lemma 3.1 *If T is a smallest counterexample to Theorem 1.3, then T does not have a Hamiltonian 2-switched dominating cycle.*

Before proving Lemma 3.1, it will be useful to investigate the structure of a k-colored Δ_3-free tournament that contains a Hamiltonian dominating cycle.

Lemma 3.2 *If T has a Hamiltonian dominating cycle $C = v_0v_1 \dots v_{n-1}v_0$ that contains a switch vertex v_i for some $i \in [n-1]$, then $v_{i-1} \longrightarrow v_{i+1}$.*

Proof. Suppose to the contrary that $v_{i+1} \longrightarrow v_{i-1}$. Without loss of generality, we may assume $v_{i-1} \xrightarrow{c_1} v_i$ and $v_i \xrightarrow{c_2} v_{i+1}$. Now, if the arc $v_{i+1}v_{i-1}$ is not colored c_1 or c_2, then T contains a Δ_3, a contradiction. If $v_{i+1} \xrightarrow{c_1} v_{i-1}$, then $v_{i+1} \overset{c_1}{\rightsquigarrow} v_i$ and v_i is a monochromatic sink in T, while if $v_{i+1} \xrightarrow{c_2} v_{i-1}$, then $v_i \overset{c_2}{\rightsquigarrow} v_{i-1}$ and v_{i-1} is a monochromatic sink in T. In either case a contradiction results. □

Lemma 3.3 *If T has a Hamiltonian dominating cycle $C = v_0v_1 \dots v_{n-1}v_0$ and there exist $i, j \in [n-1]$ such that $j \notin \{i, i-1\}$, $v_i \longrightarrow v_j$, and $v_j \longrightarrow v_{i-1}$, then exactly one of the arcs v_iv_j and v_jv_{i-1} receives the same color as the arc $v_{i-1}v_i$.*

Proof. First, note that if the arcs v_iv_j and v_jv_{i-1} are the same color, then $v_i \rightsquigarrow v_{i-1}$, contradicting the assumption that C is a dominating cycle. Thus, v_iv_j and v_jv_{i-1} differ in color, and if neither arc is the same color as $v_{i-1}v_i$, we have a Δ_3. From this, the result follows. □

Lemma 3.4 *If T has a Hamiltonian dominating cycle $C = v_0v_1 \dots v_{n-1}v_0$ that contains a switch vertex v_i for some $i \in [n-1]$, then there exists a monochromatic path P from v_{i+1} to v_{i-1}. Moreover, the color of P is distinct from the colors of the arcs $v_{i-1}v_i$ and v_iv_{i+1}.*

Proof. Since v_{i-1} is a monochromatic sink in $T - v_i$ by the definition of C, and v_{i+1} is a vertex in $T - v_i$, such a path P exists in T. If P shares the color of arc $v_{i-1}v_i$, then there is a monochromatic path from v_{i+1} to v_i. But, by the definition of C, v_i is a monochromatic sink in $T - v_{i+1}$, so v_i is a monochromatic sink in T, a contradiction. If P shares the color of arc v_iv_{i+1}, then there is a monochromatic path from v_i to v_{i-1}. But, by the definition of C, v_{i-1} is a monochromatic sink in $T - v_i$, so v_{i-1} is a monochromatic sink in T, a contradiction. □

Lemma 3.5 *Let T be a q-switched tournament, $q \geq 3$, and suppose T has a Hamiltonian dominating cycle $C = v_0v_1 \ldots v_{n-1}v_0$ containing at least $q-1$ switch vertices. Let v_s and v_t be switch vertices of C where $s < t$ and let v_iv_j be any arc in T with $s < i < j < t$. If either*

(i) C has $q-1$ switch vertices and v_s and v_t are consecutive switch vertices in C, or

(ii) C has q switch vertices and there exists at most one switch vertex v_y of C with $i < y < j$,

then the color of v_iv_j must either equal the color of $v_{i-1}v_i$ or it must equal the color of v_jv_{j+1}.

Proof. Let S be the set of all switch vertices in C. By hypothesis, $|S| \geq q-1$. Let arc v_iv_j be as is described in the statement, and suppose that v_iv_j differs in color from both $v_{i-1}v_i$ and v_jv_{j+1}. Then, no matter whether (i) or (ii) is assumed, the cycle $v_iv_jCv_i$ contains the switch vertices v_i and v_j as well as $q-1$ of the vertices of S, for a total of $q+1$ switch vertices. This contradicts the hypothesis that T is q-switched. □

Proof of Lemma 3.1. By Lemma 2.2, such a tournament T has a Hamiltonian dominating cycle $C = v_0v_1 \ldots v_{n-1}v_0$. Suppose, to the contrary, that C is 2-switched. Let the switch vertices in C be v_0 and v_s, for some $s \in [1, n-1]$. Up to relabeling of colors, we may assume without loss of generality that the path v_0Cv_s is colored c_1, the path v_sCv_0 is colored c_2, and that $v_s \longrightarrow v_0$. Thus, $s \neq 1$. If $s = n-1$, then $v_0 \rightsquigarrow v_{n-1}$, contradicting Definition 2.1. By Lemma 3.2, it must be that $v_{n-1} \longrightarrow v_1$ and $v_{s-1} \longrightarrow v_{s+1}$. This implies $n \neq 4$ and thus either $|v_0Cv_s| \geq 3$ or $|v_sCv_0| \geq 3$; otherwise T is a Δ_3. It follows that $n \geq 5$. The definition of C implies that v_0 is a monochromatic sink of $T - v_1$. If arc v_sv_0 is colored c_1, then the path v_1Cv_s followed by the arc v_sv_0 is a monochromatic path from v_1 to v_0, making v_0 a monochromatic sink of T, a contradiction. So, arc v_sv_0 is not colored c_1.

Now, by Lemma 3.4, there must exist a monochromatic path P from v_1 to v_{n-1}. If P were colored c_1, then v_{n-1} would be a monochromatic sink of T, and if P were colored c_2, then v_0 would be a monochromatic sink of T. It follows that P is not colored c_1 or c_2. Thus we may assume the color of P is c_3. Moreover, P cannot contain the vertex v_0 since, if it did, there would exist a monochromatic path from v_0 to v_{n-1}, making v_{n-1} a monochromatic sink in T, which contradicts the definition of C. Consider the first arc v_1v_i in P. By Lemma 3.5, we know that $i \in [s, n-2]$. Suppose $i = s$. Then $v_1 \xrightarrow{c_3} v_s$, and since v_sv_0 is not colored c_1 or c_3 (else we have a path from v_1 to v_0 colored c_3), we have a contradiction to Lemma 3.3. Thus, it must be that $i \in [s+1, n-2]$.

Now, by Lemma 3.5, there must exist a first arc v_jv_k in P, such that $j \in [s+1, i]$ and $k \in [2, s]$. Moreover, if $k < s$, then the cycle $v_1v_iPv_kCv_sv_0v_1$ contains the switch vertices v_1, v_k, v_s, and v_0. Hence T is not 3-switched. If $k = s$, we see that

v_sv_0 is not colored c_3 since otherwise, the path v_1Pv_s followed by the arc v_sv_0 is a monochromatic path from v_1 to v_0 in T, which contradicts the definition of C. But now the cycle $v_0Cv_{s-1}v_{s+1}Cv_jv_sv_0$ is not 3-switched, as it contains the switch vertices v_s, v_0, v_j, and $\widehat{v_{s-1}, v_{s+1}}$. We conclude that T does not have a 2-switched Hamiltonian dominating cycle. □

4 3-switched dominating cycle

Having proven that a smallest counterexample to Theorem 1.3 cannot have a 2-switched Hamiltonian dominating cycle, we now show such a tournament cannot have a 3-switched Hamiltonian dominating cycle. Before proceeding, we state a result that we consider quite important as it requires very few assumptions about the structure of T. For a vertex v, we let $A^+(v)$ denote the set of arcs incident from v. We call $A^+(v)$ the *out-arc set* of v. If all of the arcs in $A^+(v)$ are the same color, then we say that v has a *monochromatic out-arc set*. Analogously, we let $A^-(v)$ denote the *in-arc set* of v and if all of the arcs in $A^-(v)$ are colored the same, then we say that v has a *monochromatic in-arc set*.

Lemma 4.1 *Let T be a smallest k-colored Δ_3-free tournament on $n \geq 2$ vertices that does not contain a monochromatic sink. Then for any vertex v in T, neither $A^+(v)$ nor $A^-(v)$ is monochromatic.*

Proof. We prove that for any vertex v in T, the set $A^+(v)$ is not monochromatic. A symmetric argument can be used to prove that $A^-(v)$ is not monochromatic. Suppose, to the contrary, that T does not have a monochromatic sink and there exists a vertex in T with a monochromatic out-arc set. Then by Lemma 2.2, T has a Hamiltonian dominating cycle $C = v_0v_1v_2 \ldots v_{n-1}v_0$. Without loss of generality, we may assume that $A^+(v_1)$ is monochromatic and that each arc in $A^+(v_1)$ is colored c_1. By Definition 2.1, we know there exist monochromatic paths from v_1 to v_i for all $i \in [2, n-1]$ and since each arc in $A^+(v_1)$ is colored c_1, we know that all such monochromatic paths from v_1 must be colored c_1. If $v_0 \xrightarrow{c_1} v_1$, then since $v_1 \overset{c_1}{\rightsquigarrow} v_{n-1}$, we see that $v_0 \overset{c_1}{\rightsquigarrow} v_{n-1}$, a contradiction. Therefore, $v_0 \xrightarrow{c_j} v_1$ for some $j \neq 1$. Since each arc in $A^+(v_1)$ is colored c_1, we have $v_1 \xrightarrow{c_1} v_2$. As arc v_0v_1 is not colored c_1, we now see that v_1 is a switch vertex in C. Thus, by Lemma 3.4, there exists a monochromatic path $P : v_2 = u_0u_1u_2 \ldots u_t = v_0$ from v_2 to v_0. Let the color of the path P be c_i. If $i = 1$, then v_0 is a monochromatic sink in T, while if $i = j$, then v_1 is a monochromatic sink in T. Thus, by the definition of C, we know that $i \notin \{1, j\}$. Since $v_1 \longrightarrow v_2$ and $v_0 \longrightarrow v_1$, there exists a smallest index $\ell \in [t-1]$ such that $u_{\ell+1} \xrightarrow{c_s} v_1$ and thus $v_1 \xrightarrow{c_1} u_\ell$, for some $s \in [1, k]$. If $s \notin \{1, i\}$, then $v_1u_\ell u_{\ell+1}v_1$ is a Δ_3. If $s = i$, then $v_2Pu_{\ell+1}v_1$ is a monochromatic path from v_2 to v_1, which, by the definition of C, yields a contradiction. It follows that $s = 1$. Since C is Hamiltonian, there exists $\alpha \in [3, n-1]$ such that $u_{\ell+1} = v_\alpha$. Now, since $v_1 \overset{c_1}{\rightsquigarrow} v_{\alpha-1}$ and $v_\alpha \xrightarrow{c_1} v_1$, we see that $v_\alpha \overset{c_1}{\rightsquigarrow} v_{\alpha-1}$, which again contradicts the definition of C. We conclude that T has a monochromatic sink. □

We are now ready to prove Theorem 1.3.

Proof.[Proof of Theorem 1.3] Suppose T is a smallest counterexample to Theorem 1.3. Then, by Lemma 2.2, T has a Hamiltonian dominating cycle $C = v_0v_1 \dots v_{n-1}v_0$. We know C is not 0- or 1-switched, and by Lemma 3.1, C is not 2-switched. We will now show that C is not 3-switched, which will contradict the fact that T is a 3-switched tournament.

To the contrary, suppose that C contains three switch vertices. Let the switch vertices in C be v_0, v_s, and v_t, for some $s, t \in [1, n-1]$ where $s < t$. Without loss of generality, we may assume the path v_tCv_0 is colored c_1, the path v_0Cv_s is colored c_2, and the path v_sCv_t is colored c_3. By Lemma 3.2, we have $v_{n-1} \longrightarrow v_1$, $v_{s-1} \longrightarrow v_{s+1}$, and $v_{t-1} \longrightarrow v_{t+1}$. We separate our proof into four cases determined by the size of $A(T) \cap E'$, where $E' = \{v_0v_s, v_sv_t, v_tv_0\}$.

Case 1: Let $|A(T) \cap E'| = 0$. Then $v_0v_t, v_tv_s, v_sv_0 \in A(T)$. It follows that $|v_0Cv_s|, |v_sCv_t|, |v_tCv_0| \geq 2$. If arc v_sv_0 is colored c_2, then the path v_1Cv_s followed by the arc v_sv_0 is a monochromatic path from v_1 to v_0, which contradicts the definition of C. Thus arc v_sv_0 cannot be colored c_2. Similarly, arcs v_tv_s and v_0v_t cannot be colored c_1, and c_3, respectively. If the triangle $v_0v_tv_sv_0$ is not monochromatic, we may assume, by symmetry, that the color of the arc v_tv_s is different from the color of the arc v_sv_0. Then the cycle $v_0Cv_{s-1}v_{s+1}Cv_tv_sv_0$ is not 3-switched, since it has switch vertices v_t, v_s, v_0, and $\widehat{v_{s-1}, v_{s+1}}$. This contradicts our hypothesis that T is 3-switched, and from this contradiction, we see the triangle $v_0v_tv_sv_0$ must be monochromatic and it cannot be colored c_1, c_2, or c_3.

Now, since $v_0 \longrightarrow v_t$ and $v_t \longrightarrow v_s$, there exists a consecutive pair of vertices, v_i and v_{i+1} where $i \in [0, s-1]$, such that $v_i \longrightarrow v_t$ and $v_t \longrightarrow v_{i+1}$. In the event the color of the arc v_iv_t is different from the color of the arc v_tv_{i+1}, the cycle $v_iv_tv_{i+1}Cv_sCv_{t-1}v_{t+1}Cv_0Cv_i$ is not 3-switched, having switch vertices v_t, v_s, $\widehat{v_{t-1}, v_{t+1}}$, and v_0. Note that this is even the case if $i = s-1$, for in this event, the colors of the arcs v_tv_s and v_sv_{s+1} must be different, since v_tv_s cannot be colored c_3. Thus the arcs v_iv_t and v_tv_{i+1} must be colored the same. If this color is not c_2, then the cycle $v_iv_tv_{i+1}Cv_sCv_{t-1}v_{t+1}Cv_0Cv_i$ once again is not 3-switched, having switch vertices v_i, v_{i+1}, v_s, and v_0. Hence $v_i \xrightarrow{c_2} v_t$ and $v_t \xrightarrow{c_2} v_{i+1}$. Now since the triangle $v_0v_tv_sv_0$ is monochromatic and cannot be colored c_2 or c_3, we see that in particular, the color of the arc v_tv_s is not c_2 or c_3. Thus the cycle $v_iv_tv_sCv_{t-1}v_{t+1}Cv_0Cv_i$ is not 3-switched, having switch vertices v_t, v_s, $\widehat{v_{t-1}, v_{t+1}}$, and v_0 . With this contradiction, we conclude $|A(T) \cap E'| \geq 1$.

Case 2: Let $|A(T) \cap E'| = 3$. Then $v_0v_s, v_sv_t, v_tv_0 \in A(T)$. Since $|V(T)| \geq 4$, we may assume at least one of v_0Cv_s, v_sCv_t, and v_tCv_0 has size at least two. Suppose exactly one of these paths, without loss of generality say v_0Cv_s, has size at least two. Then by Lemma 3.2, $v_s \longrightarrow v_0$, which is a contradiction. So we may assume at least two of v_0Cv_s, v_sCv_t, and v_tCv_0 has size at least two. Without loss of generality, assume $|v_sCv_t| \geq 2$ and $|v_0Cv_s| \geq 2$.

Suppose $|v_tCv_0| = 1$. Then, by Lemma 3.2, $v_t \longrightarrow v_1$ and $v_{t-1} \longrightarrow v_0$. Also, if arcs v_0v_s and v_sv_t share the same color, then since $t = n-1$, $v_0v_sv_t$ is a monochromatic path from v_0 to v_{n-1}, contradicting the definition of C. Thus arcs v_0v_s and v_sv_t must have distinct colors. If $|v_0Cv_s| \geq 3$ and $|v_sCv_t| \geq 3$, then by Lemma 3.2, $v_{s-1} \longrightarrow v_{s+1}$ and so the cycle $v_sv_tv_1Cv_{s-1}v_{s+1}Cv_{t-1}v_0v_s$ is not 3-switched, having switch vertices v_s, $\widehat{v_t, v_1}$, $\widehat{v_{s-1}, v_{s+1}}$, and $\widehat{v_{t-1}, v_0}$. Thus at least one of $|v_0Cv_s|$ and $|v_sCv_t|$ is equal to two. If both are equal to two, then $s = 2$ and $t = 4$, and by Lemma 3.2, $v_1 \longrightarrow v_3$ and $v_3 \longrightarrow v_0$. By Lemma 3.3, either $v_1 \xrightarrow{c_2} v_3$ or $v_3 \xrightarrow{c_2} v_0$. If the latter is true, then the arc v_3v_0 followed by the path v_0Cv_2 is a monochromatic path from v_3 to v_2. This contradicts the definition of C, from which it follows that $v_1 \xrightarrow{c_2} v_3$. Now, by Lemma 3.3, $v_4 \xrightarrow{c_3} v_1$, which now implies that v_2Cv_4 followed by the arc v_4v_1 is a monochromatic path from v_2 to v_1, a contradiction to the definition of C. From this, we conclude that exactly one of $|v_0Cv_s|$ and $|v_sCv_t|$ is at least three while the other is equal to two. If $|v_0Cv_s| \geq 3$ and $|v_sCv_t| = 2$, then $t-1 = s+1$ and Lemma 3.2 implies $v_{s-1} \longrightarrow v_{t-1}$. Hence the cycle $v_sv_tv_1Cv_{s-1}v_{t-1}v_0v_s$ has at least four switch vertices: $\widehat{v_t, v_1}$, $\widehat{v_{s-1}, v_{t-1}}$, v_0, and v_s. If $|v_0Cv_s| = 2$ and $|v_sCv_t| \geq 3$, then $s = 2$ and by Lemma 3.2, $v_1 \longrightarrow v_{s+1}$. Hence the cycle $v_sv_tv_1v_{s+1}Cv_{t-1}v_0v_s$ is not 3-switched having switch vertices v_t, $\widehat{v_1, v_{s+1}}$, $\widehat{v_{t-1}, v_0}$, and v_s. With this contradiction, we conclude that each of $|v_0Cv_s|$, $|v_sCv_t|$, and $|v_tCv_0|$ is at least two.

Since $v_0 \longrightarrow v_s$ and $v_t \longrightarrow v_0$, there exists i in $[s, t-1]$ such that $v_0 \longrightarrow v_i$ and $v_{i+1} \longrightarrow v_0$. Furthermore, by Lemma 3.3, exactly one of these arcs is colored c_3. First suppose $v_0 \xrightarrow{c_3} v_i$ (implying $v_{i+1}v_0$ is not colored with c_3). Then, by Definition 2.1, we see that v_tv_0 is not colored c_3. If the arc $v_{s-1}v_{s+1}$ is colored something other than c_2 or c_3, then the cycle $v_{s-1}v_{s+1}Cv_{s-1}$ is not 3-switched. Therefore $v_{s-1} \xrightarrow{c_2,c_3} v_{s+1}$. If $i = s$, then $v_{i+1}v_0$ is not colored c_3 so the cycle $v_0v_sv_tCv_{n-1}v_1Cv_{s-1}v_{i+1}v_0$ has switch vertices v_0, $\widehat{v_s, v_t}$, $\widehat{v_{n-1}, v_1}$, and v_{i+1}. Thus we assume $i > s$. Consider the cycle $C' = v_0v_iCv_{t-1}v_{t+1}Cv_{n-1}v_1Cv_sv_tv_0$. It is easily seen v_0 and $\widehat{v_s, v_t}$ are switch vertices of C'. If $|v_tCv_0| \geq 3$, then $\widehat{v_{t-1}, v_{t+1}}$ and $\widehat{v_{n-1}, v_1}$ are also switch vertices of C'. If $|v_tCv_0| = 2$, then by Definition 2.1, $v_{t-1}v_{t+1}$ is not colored c_3. If $v_{t-1} \xrightarrow{c_2} v_{t+1}$, then the cycle $v_{t-1}v_{t+1}Cv_{t-1}$ has switch vertices v_{t+1}, v_0, v_s, and v_{t-1}, so it is not 3-switched. It follows that v_{t-1} and $\widehat{v_{t+1}, v_1}$ are also switch vertices of C' when $|v_tCv_0| = 2$. In each scenario, C' is not 3-switched, contradicting the hypothesis that T is 3-switched. Thus, we may now assume $v_{i+1} \xrightarrow{c_3} v_0$. Then by Definition 2.1, v_0v_s is not colored c_3. If $|v_0Cv_s| = 2$, then $v_1 \longrightarrow v_{s+1}$. By Lemma 3.5, $v_1 \xrightarrow{c_2,c_3} v_{s+1}$. However, if $v_1 \xrightarrow{c_3} v_{s+1}$, then $v_1 \overset{c_3}{\rightsquigarrow} v_0$. With this contradiction, we conclude $v_1 \xrightarrow{c_2} v_{s+1}$. Now the cycle $v_0v_sv_tCv_{n-1}v_1v_{s+1}Cv_{i+1}v_0$ has switch vertices v_0, v_{s+1}, $\widehat{v_{n-1}, v_1}$, and $\widehat{v_s, v_t}$. If $|v_0Cv_s| \geq 3$, then the cycle $v_0v_sv_tCv_{n-1}v_1Cv_{s-1}v_{s+1}Cv_{i+1}v_0$ has switch vertices v_0, $\widehat{v_s, v_t}$, $\widehat{v_{n-1}, v_1}$, and $\widehat{v_{s-1}, v_{s+1}}$, a contradiction.

Case 3: Let $|A(T) \cap E'| = 1$. Without loss of generality, assume v_0v_s, v_tv_s, and v_0v_t are all arcs in T. Note that these arcs imply each of $|v_sCv_t|$ and $|v_tCv_0|$ is at least two. It follows from Definition 2.1 that v_0v_t is not colored c_1 and v_tv_s is not colored c_3. If v_0v_t and v_tv_s are colored differently, then the cycle $v_0v_tv_sCv_{t-1}v_{t+1}Cv_0$ has switch vertices v_0, v_t, v_s, and $\widehat{v_{t-1}, v_{t+1}}$. So both arcs must be colored the same and since v_0v_t cannot be colored c_1 and v_tv_s cannot be colored c_3, this color cannot be c_1 or c_3. We break this case into two subcases depending on $|v_0Cv_s|$.

Subcase 3.1: Assume $|v_0Cv_s| = 1$. By Lemma 3.4, there exists a monochromatic path P from v_{t+1} to v_{t-1} and the color of P cannot be c_1 or c_3. Note that v_t is not a vertex in the path P, since this would imply the existence of a monochromatic path from v_t to v_{t-1}, contradicting the definition of C. Let $v_{t+1}v_k$ and $v_\ell v_{t-1}$, respectively, be the first and last arcs of P. Then by Lemma 3.5, we have $k \in [0, t-2]$ and $\ell \in [t+2, s]$. However, if $k = 0$, then since the color of P is not c_1, neither $v_{t+1}v_0$ nor v_0v_t are colored c_1, which contradicts Lemma 3.3. Similarly, if $\ell = s$, neither v_tv_s nor v_sv_{t-1} are colored c_3, again contradicting Lemma 3.3. It follows that $k \in [s, t-2]$ and $\ell \in [t+2, 0]$. Therefore, there must be an arc v_xv_y contained in the $v_k - v_\ell$ subpath of P such that $x \in [s, t-2]$ and $y \in [t+2, 0]$. Now, if $y = 0$, then $x \in [s+1, t-2]$ and the arc v_0v_t must have a color different than the color of P. Hence, the cycle $v_xv_0v_tCv_{n-1}v_sCv_x$ is not 3-switched since it contains the switch vertices v_x, v_0, v_t, and $\widehat{v_{n-1}, v_s}$. Similarly, if $x = s$, then $y \in [t+2, n-1]$ and the arc v_tv_s cannot be the same color as P. Thus, the cycle $v_xv_yCv_0v_{s+1}Cv_tv_x$ is not 3-switched since it contains the switch vertices v_y, v_t, v_x, and $\widehat{v_0, v_{s+1}}$. Therefore $x \in [s+1, t-2]$ and $y \in [t+2, n-1]$, so the cycle $v_xv_yCv_x$ is not 3-switched since it has switch vertices v_x, v_y, v_0, and v_s. With this contradiction, we conclude $|v_0Cv_s| \neq 1$.

Subcase 3.2: Now assume $|v_0Cv_s| \geq 2$. By Definition 2.1, there exists a monochromatic path P' from v_s to v_0. Let v_sv_k and $v_\ell v_0$ be the first and final arcs, respectively, of P'. First consider the possible values of k. If $k \in [1, s-2]$, then the cycle $v_sv_kCv_{s-1}v_{s+1}Cv_{t-1}v_{t+1}Cv_0v_tv_s$ has switch vertices v_k, v_0, and at least one of v_{s-1} and v_{t+1}. Moreover, at least one of v_t and v_s is a switch vertex as well, unless the path $v_0v_tv_sv_k$ is monochromatic. However, if $v_0v_tv_sv_k$ is a monochromatic path, the color of this path cannot be c_1, c_2, or c_3. Then the cycle $v_sv_kCv_{s-1}v_{s+1}Cv_0v_s$ is not 3-switched since it has switch vertices v_k, $\widehat{v_{s-1}, v_{s+1}}$, v_t, and $\widehat{v_0, v_s}$. Hence $k \in [1, s-2]$ leads to a contradiction of our hypothesis that T is 3-switched.

Suppose $k \in [t+1, n-1]$. Recall the arc v_tv_s is not colored c_1 or c_3. Then the cycle $v_sv_kCv_0Cv_{s-1}v_{s+1}Cv_tv_s$ is not 3-switched as it has as switch vertices v_0, $\widehat{v_{s-1}, v_{s+1}}$, $\widehat{v_s, v_k}$, and v_t. We conclude that $k \in [s+1, t-1]$. From this, and the observation that the color of P' cannot be c_2, we find that in order for the cycle $v_sv_kCv_0Cv_s$ to be 3-switched, the color of P' must be c_3. In particular, v_sv_k and $v_\ell v_0$ are both colored c_3.

We now consider the possible values of ℓ. If $\ell \in [t+1, n-2]$, then the cycle $v_\ell v_0Cv_\ell$ is not 3-switched as it has switch vertices v_ℓ, v_0, v_s, and v_t. Recall the arc v_0v_t is not colored c_3 or c_1. If $\ell \in [2, s-1]$, then the cycle $v_\ell v_0v_tCv_{n-1}v_1Cv_\ell$ is not 3-switched as it has switch vertices v_ℓ, v_t, v_0, and $\widehat{v_{n-1}, v_1}$. We conclude that $\ell \in [s+1, t-1]$. But then the cycle $v_\ell v_0v_tCv_{n-1}v_1Cv_\ell$ is not 3-switched having switch vertices v_0, v_t, $\widehat{v_{n-1}, v_1}$, and v_s. This contradiction concludes our argument for Case 3.

Case 4: Let $|A(T) \cap E'| = 2$. Without loss of generality, we may assume that $A(T) \cap E' = \{v_0v_s, v_sv_t\}$. It follows that $|v_tCv_0| \geq 2$, and so there exists a vertex

v_{n-1}, where $n-1>t$. Moreover, if v_0v_t is c_1, then $v_0v_tCv_{n-1}$ is a monochromatic path from v_0 to v_{n-1}. Hence, the definition of C guarantees that v_0v_t is not colored c_1.

Suppose $s=1$. Then by Lemma 3.2, $v_{n-1}\longrightarrow v_s$ and $v_0\longrightarrow v_{s+1}$. Since $v_{n-1}\longrightarrow v_s$ and $v_s\longrightarrow v_t$, there exists $i\in[t,n-2]$ such that $v_s\longrightarrow v_i$ and $v_{i+1}\longrightarrow v_s$. By Lemma 3.3, one of these arcs is colored c_1 and to avoid a monochromatic path from v_s to v_0, the arc $v_{i+1}v_s$ must be colored c_1. By Definition 2.1, this implies $|v_sCv_t|\geq 2$. By Lemma 3.4, there is a monochromatic path Q from v_{s+1} to v_0 that is not colored c_2 nor c_3. Consider the first arc $v_{s+1}v_p$ of path Q. By Lemma 3.5, we have that $p\in[t+1,n-1]$. If this arc is not colored c_1, then $v_{s+1}v_pCv_{s+1}$ is not 3-switched since it has switch vertices v_0, v_s, v_{s+1}, and v_p. Thus path Q is colored c_1. If $p\leq i+1$, then $v_{s+1}\rightsquigarrow v_s$, a contradiction. Thus $p>i+1$. Then $v_{s+1}v_pCv_0v_tCv_{i+1}v_sv_{s+1}$ is not 3-switched since it has switch vertices v_0, v_t, v_s, and v_{s+1}. We may therefore assume $s>1$.

By Lemma 3.4, there exists a v_1-v_{n-1} monochromatic path. Let P be such a path and let v_1v_f and $v_\ell v_{n-1}$ be the first and last arcs in P, respectively. In what follows, our goal will be to deduce that there are no possible values for f and ℓ, thus arriving at a contradiction. We know by Lemma 3.4 that the color of P cannot be c_1 or c_2. Hence Lemma 3.5 implies that f is not in $[2,s-1]$ and ℓ is not in $[t+1,n-2]$. Eliminating other values for f and ℓ will require further analysis of the structure of T.

We first show that $v_0\longrightarrow v_i$, for all i in $[s,t]$. Certainly, $v_0\longrightarrow v_s$ and $v_0\longrightarrow v_t$. Toward a contradiction, suppose there exists i in $[s+1,t-1]$ such that $v_i\longrightarrow v_0$. Note that in the event $v_{s-1}v_{s+1}$ is not colored c_2 or c_3, the cycle $v_0Cv_{s-1}v_{s+1}Cv_0$ is not 3-switched. Thus, we know $v_{s-1}v_{s+1}$ must be colored c_2 or c_3. Moreover, by Definition 2.1, it cannot be the case that v_0v_s and v_sv_t are both colored c_1. From these observations, we deduce that if v_iv_0 is colored c_3 for such a vertex v_i, then since v_0v_s could not be colored c_3 (according to Definition 2.1), the cycle $v_iv_0v_sv_tCv_{n-1}v_1Cv_{s-1}v_{s+1}Cv_i$ is not 3-switched since the vertices v_0, $\widehat{v_s,v_t}$, and at least two vertices from the path $v_{n-1}v_1Cv_{s-1}v_{s+1}Cv_i$ are switch vertices. If, on the other hand, v_iv_0 is not colored c_3, then the cycle $v_iv_0v_tCv_{n-1}v_1Cv_sCv_i$ is not 3-switched, having switch vertices v_i, v_t, $\widehat{v_{n-1},v_1}$, and v_s. It follows that $v_0\longrightarrow v_i$ for all $i\in[s,t]$.

Next, we show that $v_0\longrightarrow v_j$, for all $j\in[1,s-1]$. Suppose this is not the case and let j be the smallest integer in $[2,s-1]$ such that $v_j\longrightarrow v_0$. We know that v_jv_0 cannot be colored c_2. If v_jv_0 is not colored c_3, then the cycle $v_jv_0v_sCv_tCv_{n-1}v_1Cv_j$ is not 3-switched, since it has switch vertices v_s, v_t, $\widehat{v_{n-1},v_1}$, and $\widehat{v_0,v_1}$. Therefore $v_j\xrightarrow{c_3}v_0$. A similar argument shows that for all $q\in[j+1,s]$, if $v_q\longrightarrow v_0$, then it is colored c_3. Moreover, for all such q where $v_0\longrightarrow v_q$, it must be that $v_0\xrightarrow{c_3}v_q$, otherwise the cycle $v_jv_0v_qCv_{n-1}v_1Cv_j$ is not 3-switched. Now, if the path P is colored c_3, then P must avoid every vertex v_i where $i\in[j,t]$. If this were not the case, then either $v_0\overset{c_3}{\rightsquigarrow}v_{n-1}$ or $v_1\overset{c_3}{\rightsquigarrow}v_0$, contradicting Definition 2.1. Therefore $f\in[t+1,n-2]$ and $\ell\in[2,j-1]$. Now, as P avoids v_i for all i in $[j,t]$, there must exist an arc v_xv_y in P such that $x\in[t+1,n-2]$ and $y\in[2,j-1]$. Since P is colored c_3, this produces the cycle $v_xv_yCv_x$, which is not 3-switched. We conclude from this contradiction that the color of the path P cannot be c_3.

With no loss in generality, we may assume P is colored c_4, where $c_4 \notin \{c_1, c_2, c_3\}$. Then, by Lemma 3.5, it must be that $f \in [t, n-2]$ and $\ell \in [2, s]$. However, if $\ell = s$, then $v_{n-1}v_0v_sv_{n-1}$ is a Δ_3. Hence $\ell < s$. It now follows from this and Lemma 3.5 that P contains an arc of the form $v_\alpha v_\beta$, where $\alpha \in [s, t]$ and $\beta \in [2, s-1]$. If $\alpha = s$, then $\beta < s-1$ and the cycle $v_\alpha v_\beta C v_{s-1} v_{s+1} C v_t C v_0 v_\alpha$ is not 3-switched since it contains switch vertices v_β, $\widehat{v_{s-1}, v_{s+1}}$, v_t, and $\widehat{v_0, v_\alpha}$. Thus $\alpha > s$. If $\alpha = t$, then, since $v_\alpha v_\beta$ is colored c_4, the cycle $v_\alpha v_\beta C v_{t-1} v_{t+1} C v_0 v_\alpha$ has switch vertices $\widehat{v_{t-1}, v_{t+1}}$, v_s, v_β, and $\widehat{v_0, v_\alpha}$, and is therefore not 3-switched. With this contradiction, we may assume $\alpha \in [s+1, t-1]$. Now, if $\alpha > s+1$, then the cycle $v_\alpha v_\beta C v_s v_t C v_0 v_{\alpha-1} v_\alpha$ is not 3-switched, having switch vertices v_β, $\widehat{v_s, v_t}$, $\widehat{v_0, v_{\alpha-1}}$, and v_α. Hence, it must be that $\alpha = s+1$. Lemma 3.2 now implies that $\beta < s-1$.

We have already shown that $v_0 \longrightarrow v_\alpha$. If $v_0 \xrightarrow{c_4} v_\alpha$, then $v_0 v_\alpha P v_{n-1}$ is a monochromatic path from v_0 to v_{n-1}, contradicting Definition 2.1. Now, if the color of arc $v_0 v_\alpha$ is not c_1, then the cycle $v_0 v_\alpha v_\beta C v_s v_t C v_0$ is not 3-switched, having switch vertices v_α, v_β, $\widehat{v_s, v_t}$, and v_0. We conclude from this contradiction that $v_0 \xrightarrow{c_1} v_\alpha$.

We now consider the location of v_β relative to v_j. If $\beta < j$, then the cycle $v_\alpha v_\beta C v_j v_0 v_\alpha$ has switch vertices v_β, v_j, v_0, and v_α, and is not 3-switched. If $\beta = j$, then $v_0 v_\alpha v_\beta v_0$ is a Δ_3. From these contradictions, we may now conclude that $\beta \in [j+1, s-2]$. However, in this case, the switch vertices in the cycle $v_j v_0 v_\alpha v_\beta C v_s v_t C v_{n-1} v_1 C v_j$ include the vertices v_0, v_α, v_β, and v_j, which contradicts our hypothesis that T is 3-switched. This contradiction now allows us to conclude that $v_0 \longrightarrow v_j$, for all $j \in [1, s-1]$.

Now, by Lemma 4.1, $A^-(v_0)$ is not monochromatic. Hence there exists an arc $v_\gamma v_0$ that is not colored c_1. Since $v_0 \longrightarrow v_i$, for all i in $[1, t]$, it must be that $\gamma \in [t+1, n-2]$. Furthermore, we have that $v_\gamma \xrightarrow{c_2} v_0$, for otherwise, $v_\gamma v_0 C v_\gamma$ is a cycle that is not 3-switched. Now, we know $v_0 \longrightarrow v_j$ for all j in $[1, s-1]$. If such an arc $v_0 v_j$ is not colored c_2, then the cycle $v_\gamma v_0 v_j C v_\gamma$ is not 3-switched. Hence $v_0 \xrightarrow{c_2} v_j$, for all $j \in [1, s-1]$.

We claim that for all $r \in [s, t-1]$, the arc $v_0 v_r$ cannot be colored c_1. Indeed, if such an arc is colored c_1, then the cycle $v_0 v_r C v_\gamma v_0$ is not 3-switched since it contains the switch vertices v_0, v_r, v_t, and v_γ. It is also clear by Definition 2.1 that $v_0 v_t$ cannot be colored c_1. Thus, for all $j \in [1, t]$, the arc $v_0 v_j$ cannot be colored c_1. Since the $v_1 - v_{n-1}$ monochromatic path P is not colored c_1, if the last arc $v_\ell v_{n-1}$ of P is such that $\ell \in [1, t]$, then we arrive at a contradiction to Lemma 3.3. Hence the last arc $v_\ell v_{n-1}$ of P is such that $\ell \in [t+1, n-3]$. We have stated that by Lemma 3.5, this cannot be the case. With this final contradiction, we have now considered all possibilities, each leading to a contradiction. Therefore, it must be that C has more than three switch vertices. From this, it follows that no counterexample to Theorem 1.3 exists, thus establishing the result. □

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