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Pandiagonal type-p Franklin squares

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Abstract

For prime p we define magic squares of order kp^3, called type-p Franklin squares, whose properties specialize to those of classical Franklin squares in the case $p = 2$. We construct type-p Franklin squares in prime-power orders.

1 Introduction

1.1 Purpose, Briefly Stated

For prime p we define magic squares of order kp^3, called type-p Franklin squares, whose properties specialize to those of classical Franklin squares in the case $p = 2$. We construct such squares in prime power orders. Our construction is motivated by a relationship, first noted in [11] and further explored in [6], between classical most-perfect magic squares of triply even order and pandiagonal classical Franklin squares.

1.2 Franklin Squares

Classical **Franklin squares** are natural semi-magic squares of doubly even order first constructed by Benjamin Franklin in the mid 1730's (two in order 8, one in order 16) to fend off boredom while clerking in the Pennsylvania Assembly. They have the following additional magic properties:

(i) Half-rows and half-columns add to half of the magic sum.

(ii) The symbols in any 2×2 subsquare formed from consecutive rows and columns (allowing toric wraparound) sum to $2(n^2 - 1)$.

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(iii) Entries in each set of **bent diagonals** add to the magic sum. Bent diagonals come in four varieties: up, right, down, and left. An up-diagonal is formed by half of a broken main diagonal (allowing vertical wraparound) beginning at the left edge of the square, together with its reflection across the vertical midline. The right, down, and left varieties are obtained from the up-diagonal locations by 90°, 180°, and 270° clockwise rotations of the ambient square, respectively.

Item (ii) above assumes, as we do throughout, that the symbol set for an order-n natural magic square is $\{0, 1, \ldots, n^2 - 1\}$. Franklin's famous order-8 square is shown in Figure 1.

51	60	3	12	19	28	35	44
13	2	61	50	45	34	29	18
52	59	4	11	20	27	36	43
10	5	58	53	42	37	26	21
54	57	6	9	22	25	38	41
8	7	56	55	40	39	24	23
49	62	1	14	17	30	33	46
15	0	63	48	47	32	31	16

			12	19			
13			50	45			18
	59					36	
		58			37		
			9	22		38	
						24	
49	62	1	14			33	
						31	

Figure 1: Left: Franklin's famous order-8 square with symbols 0 through 63. Right: An indication of its properties. Numbers $13, 59, \ldots, 36, 18$ form an up-diagonal.

Investigation of classical Franklin squares largely fits into three categories. The first is historical: Franklin's method of constructing his squares remains unknown. His correspondence makes only brief mention of them, including a lament concerning the time he wasted in such activities. Pasles' article [8] and book [9] contain a thorough historical account of Franklin's squares and a survey of methods he may have used to construct them. The most plausible of these methods appears to be the one conjectured in [3]. Another category is existential: The definition of classical Franklin squares allows for doubly even orders, but the only Franklin squares that have been discovered thus far are of triply even order. Franklin squares exist in orders $8k$ for each $k \in \mathbb{Z}^+$ (e.g., [3] and [6]). Meanwhile, there are no Franklin squares of order 4 or 12 (see [2]), and the existential question is unresolved for other orders of the form $8k+4$. The third category concerns construction and enumeration: One example is [1], in which Hilbert bases for polyhedral cones are used to place an upper bound on the number of Franklin squares. Another example is [11], in which an involution on arrays is used to define an injection from the set of most-perfect squares of order 8 to the set of pandiagonal Franklin squares of order 8, thus giving a reasonable lower bound on the number of order-8 Franklin squares. Importantly, this latter work was generalized in [6] to squares of order $8k$ for any $k \in \mathbb{Z}^+$.

1.3 Most-perfect Squares

This article makes vital use of most-perfect squares. Let n be a natural number divisible by p. A natural pandiagonal magic square R of order n is said to be a **most-perfect square of type-p** if the following two properties hold:

(i) (**Complementary property**) Starting from any location in R, consider the symbol in that location together with the $p-1$ other symbols lying in the same broken main-diagonal n/p units apart from one another. The sum of these symbols is $\dfrac{p(n^2-1)}{2}$.

(ii) ($p \times p$ **property**) The symbols in any $p \times p$ subsquare formed from consecutive rows and columns (allowing toric wraparound) sum to $\dfrac{p^2(n^2-1)}{2}$.

Examples of type-2 and type-3 most-perfect squares are given in Figure 2.

0	31	48	47	56	39	8	23
59	36	11	20	3	28	51	44
6	25	54	41	62	33	14	17
61	34	13	18	5	26	53	42
7	24	55	40	63	32	15	16
60	35	12	19	4	27	52	43
1	30	49	46	57	38	9	22
58	37	10	21	2	29	50	45

0	16	23	63	79	59	45	34	41
64	80	57	46	35	39	1	17	21
47	33	40	2	15	22	65	78	58
7	14	18	70	77	54	52	32	36
71	75	55	53	30	37	8	12	19
51	31	38	6	13	20	69	76	56
5	9	25	68	72	61	50	27	43
66	73	62	48	28	44	3	10	26
49	29	42	4	11	24	67	74	60

Figure 2: Left: A type-2 (classical) most-perfect square of order-8. Right: A type-3 most-perfect square of order 9. The gridlines serve as an aid in locating complementary entries.

Type-p most-perfect squares specialize to classical most-perfect squares when $p = 2$, in which case n must be doubly even [10]. The tasks of counting and constructing classical most-perfect squares were first approached by McClintock [5] and culminate in the work of Ollerenshaw and Bree [7], which gives a count of the classical most-perfect squares for any doubly even order n, along with a construction method for all such squares. As mentioned above, classical most-perfect squares are used in [11] and [6] for constructing Franklin squares. When $p \geq 2$, a linear construction of type-p most-perfect squares of order p^r ($r \geq 2$) is given in [4].

1.4 Type-p Franklin Squares

Let p be prime. We say that a natural square S of order $n = kp^3$ is a **Franklin square of type** p if it has the following properties:

- ($p \times p$ **property**): This is as described above for type-p most-perfect squares.
- ($1/p$**-property for both rows and columns**): We say that S possesses the $1/p$ column property if upon splitting a column of R naturally into p parts, the entries in each part add to $\frac{1}{p}$ times the magic sum, or rather $\dfrac{n(n^2-1)}{2p}$. The $1/p$ **row property** is defined similarly.
- (**Franklin pattern property**): The numbers in every Franklin pattern in S add to the magic sum $\dfrac{n(n^2-1)}{2}$. Franklin patterns specialize to bent diagonals in the case $p = 2$. A detailed description of these patterns is given in Section 3.

An example of a type-3 Franklin square of order 27 is given in Figure 3. The following discussion assumes that Figure 3 has been rotated 90° clockwise, so that the square is viewed in its ordinary orientation. In the lower region of this square the boxed entries indicate the 1/3-row and column properties and the 3×3 property. In the upper portion of the square we observe a collection of boxed entries sitting within a frame of 3×3 subsquares. These boxed entries, when taken together, look like the letter "W." This collection of entries is a Franklin-up pattern. These entries add to the magic sum and can be translated vertically throughout the square (with vertical wraparound). There are also analogous downward Franklin patterns, as well as left and right versions. A detailed description of Franklin patterns is given in Section 3. In Sections 4 and 5 we show that type-p Franklin squares exist in orders p^r with $r \geq 3$. The appendix contains a larger rendition of this square (Figure 6.)

Inspiration for these results comes chiefly from [11] and [6], where the authors introduce an involution θ that maps classical most-perfect squares into pandiagonal classical Franklin squares. This involution may be generalized (see Section 2) so that it applies to type-p most-perfect squares, examples of which exist in all orders p^r with $r \geq 2$ by [4]. Therefore, in searching for a reasonable definition for type-p Franklin squares, one could do worse than studying $\theta(R)$ where R is a type-p most-perfect square. One readily finds that $\theta(R)$ is pandiagonal, has the $p \times p$ property, and has the $1/p$-row and column properties (see Section 2). Determining reasonable Franklin patterns is considerably harder, but we are guided by the complementary property of R and Lemma 4.2 (see Sections 4 and 5). The type-3 order-27 Franklin square given above has the form $\theta(R)$, where R is a (linear) most-perfect square constructed using the method of [4].

2 An Involution and its Application to Most-Perfect Squares of Type-p

Let $n = kp^r$ with $r \geq 2$ and let R be an array of order n. We may view R as an order-p^2 array

$$R = (R_{i,j}) \quad \text{with} \quad 0 \leq i, j \leq p^2 - 1, \tag{1}$$

where each $R_{i,j}$ is an array of order $\frac{n}{p^2}$. We define an involution θ on arrays of order n by

$$[\theta(R)]_{i,j} = R_{\bar{i},\bar{j}} \tag{2}$$

where, if $i = \ell p + m$ with $\ell, m \in \{0, 1, \ldots, p-1\}$ then $\bar{i} = mp + \ell$. We emphasize that θ depends on p.

By way of illustration, if $p = 2$ then

$$R = \begin{array}{|c|c||c|c|} \hline R_{0,0} & R_{0,1} & R_{0,2} & R_{0,3} \\ \hline R_{1,0} & R_{1,1} & R_{1,2} & R_{1,3} \\ \hline \hline R_{2,0} & R_{2,1} & R_{2,2} & R_{2,3} \\ \hline R_{3,0} & R_{3,1} & R_{3,2} & R_{3,3} \\ \hline \end{array} \quad \Longrightarrow \theta(R) = \begin{array}{|c|c||c|c|} \hline R_{0,0} & R_{0,2} & R_{0,1} & R_{0,3} \\ \hline R_{2,0} & R_{2,2} & R_{2,1} & R_{2,3} \\ \hline \hline R_{1,0} & R_{1,2} & R_{1,1} & R_{1,3} \\ \hline R_{3,0} & R_{3,2} & R_{3,1} & R_{3,3} \\ \hline \end{array}.$$

Likewise, if $p = 3$ then

0	691	401	432	151	509	621	259	212	567	286	239	27	718	347	459	97	536	405	124	563	594	313	185	54	664	374
457	140	495	646	248	198	25	680	387	52	707	333	484	86	522	592	275	225	619	302	171	79	653	360	430	113	549
635	261	196	14	693	385	446	153	493	473	99	520	581	288	223	41	720	331	68	666	358	419	126	547	608	315	169
16	698	378	448	158	486	637	266	189	583	293	216	43	725	324	475	104	513	421	131	540	610	320	162	70	671	351
437	144	511	626	252	214	5	684	403	32	711	349	464	90	538	572	279	241	599	306	187	59	657	376	410	117	565
639	250	203	18	682	392	450	142	500	477	88	527	585	277	230	45	709	338	72	655	365	423	115	554	612	304	176
23	675	394	455	135	502	644	243	205	590	270	232	50	702	340	482	81	529	428	108	556	617	297	178	77	648	367
441	160	491	630	268	194	9	700	383	36	727	329	468	106	518	576	295	221	603	322	167	63	673	356	414	133	545
628	257	207	7	689	396	439	149	504	466	95	531	574	284	234	34	716	342	61	662	369	412	122	558	601	311	180
21	676	395	453	136	503	642	244	206	588	271	233	48	703	341	480	82	530	426	109	557	615	298	179	75	649	368
442	161	489	631	269	192	10	701	381	37	728	327	469	107	516	577	296	219	604	323	165	64	674	354	415	134	543
629	255	208	8	687	397	440	147	505	467	93	532	575	282	235	35	714	343	62	660	370	413	120	559	602	309	181
1	692	399	433	152	507	622	260	210	568	287	237	28	719	345	460	98	534	406	125	561	595	314	183	55	665	372
458	138	496	647	246	199	26	678	388	53	705	334	485	84	523	593	273	226	620	300	172	80	651	361	431	111	550
633	262	197	12	694	386	444	154	494	471	100	521	579	289	224	39	721	332	66	667	359	417	127	548	606	316	170
17	696	379	449	156	487	638	264	190	584	291	217	44	723	325	476	102	514	422	129	541	611	318	163	71	669	352
435	145	512	624	253	215	3	685	404	30	712	350	462	91	539	570	280	242	597	307	188	57	658	377	408	118	566
640	251	201	19	683	390	451	143	498	478	89	525	586	278	228	46	710	336	73	656	363	424	116	552	613	305	174
15	697	380	447	157	488	636	265	191	582	292	218	42	724	326	474	103	515	420	130	542	609	319	164	69	670	353
436	146	510	625	254	213	4	686	402	31	713	348	463	92	537	571	281	240	598	308	186	58	659	375	409	119	564
641	249	202	20	681	391	452	141	499	479	87	526	587	276	229	47	708	337	74	654	364	425	114	553	614	303	175
22	677	393	454	137	501	643	245	204	589	272	231	49	704	339	481	83	528	427	110	555	616	299	177	76	650	366
443	159	490	632	267	193	11	699	382	38	726	328	470	105	517	578	294	220	605	321	166	65	672	355	416	132	544
627	256	209	6	688	398	438	148	506	465	94	533	573	283	236	33	715	344	60	661	371	411	121	560	600	310	182
2	690	400	434	150	508	623	258	211	569	285	238	29	717	346	461	96	535	407	123	562	596	312	184	56	663	373
456	139	497	645	247	200	24	679	389	51	706	335	483	85	524	591	274	227	618	301	173	78	652	362	429	112	551
634	263	195	13	695	384	445	155	492	472	101	519	580	290	222	40	722	330	67	668	357	418	128	546	607	317	168

Figure 3: A type-3 Franklin square of order 27.

$$R = \begin{array}{|ccc||ccc||ccc|}\hline R_{0,0} & R_{0,1} & R_{0,2} & R_{0,3} & R_{0,4} & R_{0,5} & R_{0,6} & R_{0,7} & R_{0,8} \\ R_{1,0} & R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} & R_{1,5} & R_{1,6} & R_{1,7} & R_{1,8} \\ R_{2,0} & R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} & R_{2,5} & R_{2,6} & R_{2,7} & R_{2,8} \\ \hline\hline R_{3,0} & R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} & R_{3,7} & R_{3,8} \\ R_{4,0} & R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & R_{4,5} & R_{4,6} & R_{4,7} & R_{4,8} \\ R_{5,0} & R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & R_{5,5} & R_{5,6} & R_{5,7} & R_{5,8} \\ \hline\hline R_{6,0} & R_{6,1} & R_{6,2} & R_{6,3} & R_{6,4} & R_{6,5} & R_{6,6} & R_{6,7} & R_{6,8} \\ R_{7,0} & R_{7,1} & R_{7,2} & R_{7,3} & R_{7,4} & R_{7,5} & R_{7,6} & R_{7,7} & R_{7,8} \\ R_{8,0} & R_{8,1} & R_{8,2} & R_{8,3} & R_{8,4} & R_{8,5} & R_{8,6} & R_{8,7} & R_{8,8} \\ \hline \end{array}$$

implies

$$\theta(R) = \begin{array}{|ccc||ccc||ccc|}\hline R_{0,0} & R_{0,3} & R_{0,6} & R_{0,1} & R_{0,4} & R_{0,7} & R_{0,2} & R_{0,5} & R_{0,8} \\ R_{3,0} & R_{3,3} & R_{3,6} & R_{3,1} & R_{3,4} & R_{3,7} & R_{3,2} & R_{3,5} & R_{3,8} \\ R_{6,0} & R_{6,3} & R_{6,6} & R_{6,1} & R_{6,4} & R_{6,7} & R_{6,2} & R_{6,5} & R_{6,8} \\ \hline\hline R_{1,0} & R_{1,3} & R_{1,6} & R_{1,1} & R_{1,4} & R_{1,7} & R_{1,2} & R_{1,5} & R_{1,8} \\ R_{4,0} & R_{4,3} & R_{4,6} & R_{4,1} & R_{4,4} & R_{4,7} & R_{4,2} & R_{4,5} & R_{4,8} \\ R_{7,0} & R_{7,3} & R_{7,6} & R_{7,1} & R_{7,4} & R_{7,7} & R_{7,2} & R_{7,5} & R_{7,8} \\ \hline\hline R_{2,0} & R_{2,3} & R_{2,6} & R_{2,1} & R_{2,4} & R_{2,7} & R_{2,2} & R_{2,5} & R_{2,8} \\ R_{5,0} & R_{5,3} & R_{5,6} & R_{5,1} & R_{5,4} & R_{5,7} & R_{5,2} & R_{5,5} & R_{5,8} \\ R_{8,0} & R_{8,3} & R_{8,6} & R_{8,1} & R_{8,4} & R_{8,7} & R_{8,2} & R_{8,5} & R_{8,8} \\ \hline \end{array}.$$

The mapping θ specializes to the involution given in [11] in the case $p = 2$ and $n = 8$; the reader may check that if R is the square in the left portion of Figure 2, then $\theta(R)$ is a Franklin square of order 8.

It is our intention to provide examples of type-p Franklin squares by applying θ to most-perfect squares of type p. We begin this process over the next several results, culminating in Proposition 2.6.

Proposition 2.1 *Suppose n is triply divisible by p and that R is a square of order n possessing the $p \times p$ property. Then $\theta(R)$ has the $p \times p$ property.*

Proof: Observe that R has the $p \times p$ property if and only if for any $(p+1) \times (p+1)$-subsquare A of R formed from consecutive rows and columns (allowing wraparound), with

$$A = \begin{array}{|cccc|c|}\hline a_{11} & a_{12} & \dots & a_{1p} & a_{1,p+1} \\ a_{21} & a_{22} & \dots & a_{2p} & a_{2,p+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \dots & a_{pp} & a_{p,p+1} \\ \hline a_{p+1,1} & a_{p+1,2} & \dots & a_{p+1,p} & a_{p+1,p+1} \\ \hline \end{array},$$

we have $\sum_{j=1}^{p} a_{1j} = \sum_{j=1}^{p} a_{p+1,j}$ and $\sum_{j=1}^{p} a_{j1} = \sum_{j=1}^{p} a_{j,p+1}$. Also, we may define variants θ_{row} and θ_{col} of θ by

$$[\theta_{row}(R)]_{i,j} = R_{\bar{i},j} \qquad \text{and} \qquad [\theta_{col}(R)]_{i,j} = R_{i,\bar{j}},$$

where $\bar{i}$ and $\bar{j}$ are as in (2).

We first show that $\theta_{row}(R)$ possesses the $p \times p$ property. We may view obtaining $\theta_{row}(R)$ from R by swapping one pair of rows at a time. Let r be a row of R lying in the band $R_{i,0}, R_{i,1}, \dots, R_{i,p^2-1}$ of subsquares. According to the definition of θ_{row}, we swap r with a row $\bar{r}$ in R that lies in the same relative position in the band of subsquares $R_{\bar{i},0}, R_{\bar{i},1}, \dots, R_{\bar{i},p^2-1}$. Therefore r is being swapped with a row $\bar{r}$ that lies $|i - \bar{i}|(n/p^2)$ units distant from r. Because $\frac{n}{p^2}$ is a multiple of p, the characterization of the $p \times p$ property given at the beginning of this proof indicates that the $p \times p$ property remains intact after this row swap. It follows that $\theta_{row}(R)$ possesses the $p \times p$ property. A similar argument shows that $\theta_{col}(R)$ possesses the $p \times p$ property, and a combination of these two results gives that $\theta(R) = \theta_{col}(\theta_{row}(R))$ possesses the $p \times p$ property. $\square$

Proposition 2.2 *Let n be triply divisible by a prime p and let R be a type-p most-perfect square of order n. Then $\theta(R)$ has the $1/p$ row and column properties.*

Proof: It suffices to show that $\theta_{row}(R)$ has the $1/p$ column property. First we establish some notation: Fix $k \in \{0, \dots, \frac{n}{p^2} - 1\}$ and let $\sigma_{i,j}$ denote the sum of the entries in the k-th column of $R_{i,j}$. This sum has n/p^2 terms, a fact that will be important later in the proof. Similarly let $\tilde{\sigma}_{i,j}$ denote the sum of the entries in the k-th column of $[\theta_{row}(R)]_{i,j}$. Recall throughout that $i, j \in \{0, 1, \dots, p^2 - 1\}$.

Observe that $\tilde{\sigma}_{0,j} + \cdots + \tilde{\sigma}_{p-1,j}$ is the sum of the first n/p entries of the $j \cdot \frac{n}{p^2} + k$ column of $\theta_{row}(R)$. (We could address another collection of n/p entries in this same column by replacing $\tilde{\sigma}_{0,j}$ with $\tilde{\sigma}_{i+0,j}$, etc., but this clutters the indices so we consider the top n/p entries only.) Applying Equation (2), the $p \times p$ property of R (actually the characterization given at the beginning of the proof of Proposition 2.1), and the complementary property of R in succession, we obtain

$$\begin{aligned}\tilde{\sigma}_{0,j} + \tilde{\sigma}_{1,j} + \cdots + \tilde{\sigma}_{p-1,j} &= \sigma_{0,j} + \sigma_{p,j} + \sigma_{2p,j} + \cdots + \sigma_{(p-1)p,j} \\ &= \sigma_{0,j} + \sigma_{p,j+p} + \sigma_{2p,j+2p} + \cdots + \sigma_{(p-1)p,j+(p-1)p} \\ &= \frac{n}{p^2} \cdot \frac{p(n^2-1)}{2} = \frac{n(n^2-1)}{2p},\end{aligned}$$

as desired. The use of the complementary property to obtain the last line of the displayed equation requires a bit more explanation: Gather the first terms of each sum $\sigma_{\ell p, j+\ell p}$. These add to $\frac{p(n^2-1)}{2}$ by the complementary property, as does the collection of second terms, etc. Since each $\sigma_{\ell p,j+\ell p}$ has $\frac{n}{p^2}$ terms, we obtain $\frac{p(n^2-1)}{2}$ exactly $\frac{n}{p^2}$ times. $\square$

Next we go about showing that if R is a type-p most-perfect square of order n, then $\theta(R)$ is pandiagonal. We begin with a pair of lemmas.

Lemma 2.3 *Let $m, n \in \mathbb{N}$ and consider a nonnegative integer array A of size $(mp+1) \times (np+1)$ with*

$$A = \begin{array}{c|c|c} a & v & b \\ \hline u & D & w \\ \hline c & z & d \end{array}\ .$$

Here $a, b, c, d \in \mathbb{Z}$, u, w are lists of length $mp - 1$, v, z are lists of length $np - 1$, and D is an $(mp-1) \times (np-1)$ array. If A possesses the $p \times p$ property then $a+d = c+b$.

Proof: By the $p \times p$ property

$$a + u + v + D = b + v + w + D = c + u + z + D = d + z + w + D,$$

where the additions indicate the total sums of symbols in each type of list. It follows that

$$(a + u + v + D) + (d + z + w + D) = (b + v + w + D) + (c + u + z + D),$$

and cancellation gives the result. □

Lemma 2.4 *Let $m \in \mathbb{Z}^+$ and $A = (a_{i,j})$ be an $m \times m$ array such that if $\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} \\ a_{i_2,j_1} & a_{i_2,j_2} \end{pmatrix}$ is a 2×2 subarray of A, then $a_{i_1,j_1} + a_{i_2,j_2} = a_{i_1,j_2} + a_{i_2,j_1}$. Then all transversals of A have the same sum.*

Proof: Let $T = \{a_{1,j_1}, a_{2,j_2}, \ldots, a_{m,j_m}\}$ be a transversal for A. We show that the sum of the elements of T equals the sum of the main diagonal elements of A. This is done by constructing a chain of transversals, culminating in the diagonal transversal, each of which has the same sum. We form a new transversal T_1 from T as follows: if $a_{1,j_1} = a_{1,1}$, then $T_1 = T$. If $a_{1,j_1} \neq a_{1,1}$, then, because T is a transversal, there exists $1 < k \leq m$ with $j_k = 1$. Using the the fact that $j_k = 1$ and the array property in the hypothesis, we have that

$$a_{k,j_1} + a_{1,1} = a_{k,j_1} + a_{1,j_k} = a_{1,j_1} + a_{k,j_k}.$$

So if we declare T_1 to be the set we obtain from T by replacing a_{1,j_1} and a_{k,j_k} by $a_{1,1}$ and a_{k,j_1}, then T_1 and T have the same sum, and, importantly, $a_{1,1} \in T_1$. Furthermore, T_1 is a transversal of A because all rows and columns of A are still accounted for in T_1.

Observe that if we eliminate the first row and column from A and remove $a_{1,1}$ from T_1, then the remaining elements of T_1 form a transversal of the new array, and we can repeat the process above to obtain a transversal $T_2 = \{a_{1,1}, a_{2,2}, \ldots, a_{m,j_m}\}$ of A that has the same sum as T_1, with $a_{1,1}$ and $a_{2,2}$ in T_2. Continuing in this fashion, we see that the sum of T is equal to the sum of T_m, which is the main diagonal transversal of A. □

Proposition 2.5 *Let p be prime and n triply divisible by p. If R is a type-p most-perfect square of order n then $\theta(R)$ is pandiagonal.*

Proof: Let $d_0, \ldots, d_{n-1}$ denote the elements of a broken diagonal in $\theta(R)$ with d_j lying in the j-th column of $\theta(R)$. Let $k \in \{0, 1, \ldots, \frac{n}{p^2} - 1\}$ and put

$$a_i = d_{i \cdot \frac{n}{p^2} + k} \quad (0 \leq i \leq p^2 - 1).$$

We claim that $a_0 + a_1 + \cdots + a_{p^2-1} = \frac{p^2(n^2-1)}{2}$. If this is true then

$$\sum_{j=0}^{n-1} d_j = \frac{n}{p^2} \cdot \frac{p^2(n^2-1)}{2} = \frac{n(n^2-1)}{2},$$

as desired.

We set about proving the claim. Due to their construction, all of the a_k's lie in the same (relative) location within an $[\theta(R)]_{i,j}$. Because the mapping $R \mapsto \theta(R)$ is of order two and merely permutes the $R_{i,j}$'s without altering the relative locations of entries within $R_{i,j}$'s (see Equation (2)), we also know that if $B = (b_{i,j})$ is the $p^2 \times p^2$ subarray of R consisting of *all* entries lying in this same relative location within some $R_{i,j}$, then $\{a_0, a_1, \ldots, a_{p^2-1}\}$ is a transversal of B. Because R has the $p \times p$ property and n is triply divisible by p, we may apply Lemma 2.3 to the various 2×2 subarrays of B, and so the hypotheses of Lemma 2.4 are satisfied for B. Therefore $a_0 + a_1 + \cdots + a_{p^2-1}$ is equal to the sum $b_{0,0} + b_{1,1} + \cdots b_{p^2-1,p^2-1}$ of the diagonal transversal of B.

Observe that adjacent terms of the sum $b_{0,0} + b_{1,1} + \cdots b_{p^2-1,p^2-1}$ are actually $\frac{n}{p^2}$ units apart on the main diagonal of R. Therefore if we rewrite this sum as

$$\begin{aligned} b_{0,0} + b_{1,1} + \cdots + b_{p^2-1,p^2-1} &= (b_{0,0} + b_{p,p} + b_{2p,2p} + \cdots + b_{(p-1)p,(p-1)p}) \\ &+ (b_{1,1} + b_{1+p,1+p} + b_{1+2p,1+2p} + \cdots + b_{1+(p-1)p,1+(p-1)p}) \\ &+ \cdots + (b_{p-1,p-1} + b_{2p-1,2p-1} + \cdots + b_{p^2-1,p^2-1}) \end{aligned}$$

then within each parenthetical summand there are p terms and adjacent terms are n/p units apart in R. Because R possesses the complementary property, we then know that each parenthetical summand adds to $\frac{p(n^2-1)}{2}$. Because there are p parenthetical summands, we may then conclude that

$$a_0 + a_1 + \cdots + a_{p^2-1} = b_{0,0} + b_{1,1} + \cdots b_{p^2-1,p^2-1} = p \cdot \frac{p(n^2-1)}{2} = \frac{p^2(n^2-1)}{2}.$$

Therefore the claim is proved. □

We may summarize the previous results as follows:

Proposition 2.6 *Let n be triply divisible by p and suppose R is a type-p most-perfect square of order n. Then $\theta(R)$ is semi-magic, possesses the $p \times p$ property, possesses the $1/p$ row and column properties, and is pandiagonal.*

3 Defining Type-p Franklin Squares: Bent Diagonals

In the introduction we established precise characteristics of type-p Franklin squares, with the exception of the bent diagonals, which we address presently. We will refer to the type-p analogs of bent diagonals as **Franklin patterns**. In the interest of simplicity we describe Franklin patterns first in the special case $n = p^3$ before addressing the general case $n = kp^3$ (Section 5). These squares, except for the smallest few primes, are large, so we will be using the special cases $p = 2, 3, 5$ to illustrate several key points. Also, we will first focus our attention on the construction of a particular Franklin pattern, called a **Franklin-up pattern**, an example of which is given in Section 1.4. These patterns specialize to classical Franklin "V" patterns when $p = 2$.

Consider a collection of $n/p = p^2$ consecutive rows of S, which we intend to serve as a **frame** for a Franklin-up pattern W. This frame can be partitioned into a $p \times p^2$ array T whose entries are subsquares $T_{i,j}$, each of size $p \times p$, where $0 \leq i \leq p$ and $0 \leq j \leq p^2 - 1$. Square $T_{i,j}$, which we occasionally refer to as a **block**, lies in the i-row and j-column of T. We describe which subsquares of T have non-trivial intersection with W. The array T can be partitioned into $p \times p$ subarrays $B_0, \ldots, B_{p-1}$ (called **bands**), each containing p columns of T, where B_0 contains the leftmost p columns of T, B_1 contains the next p columns of T, and so on. For $0 \leq j < \frac{p-1}{2}$, the Franklin-up pattern W intersects each entry of the main diagonal of B_j when j is even, and each entry of the off-diagonal of B_j when j is odd. The locations of these intersections reflect across the central band $B_{(p-1)/2}$, so that W intersects each entry of the off-diagonal of $B_{(p-1)-j}$ when j is even, and each entry of the main diagonal of $B_{(p-1)-j}$ when j is odd. When p is odd there will be a central band $B_{(p-1)/2}$, in which intersection with W will rise to a central peak when $(p-1)/2$ is odd and fall to central valley when $(p-1)/2$ is even. These intersections of W with T are indicated below in cases $p = 2, 3, 5$; double vertical lines separate bands.

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In the figure above, we emphasize that each small rectangle represents some $p \times p$ array $T_{i,j}$ in T, not an individual entry in S.

In case the description above is not sufficiently specific, the Franklin-up pattern we construct in this frame will intersect the following subsquares:

$$T_{j,2mp+j} \text{ and } T_{j,(p^2-1)-(2mp+j)} \text{ for } 0 \leq j \leq p-1 \text{ and } 0 \leq m < \frac{p-1}{4},$$

and

$$T_{j,(2mp-1)-j} \text{ and } T_{j,(p^2-1)-((2mp-1)-j)} \text{ for } 0 \leq j \leq p-1 \text{ and } 0 < m \leq \frac{p-1}{4}.$$

Further, if p is odd, then W will also intersect the following subsquares, depending on the parity of $(p-1)/2$: If $(p-1)/2$ is even then W intersects $T_{p-1,\frac{p^2-1}{2}}$ and

$$T_{j,\frac{p(p-1)}{2}+\lfloor\frac{j}{2}\rfloor} \text{ and } T_{j,\frac{p(p-1)}{2}+(p-1)-\lfloor\frac{j}{2}\rfloor} \text{ for } 0 \le j < p-1.$$

On the other hand, if $(p-1)/2$ is odd then W intersects $T_{0,\frac{p^2-1}{2}}$ and

$$T_{j,\frac{p^2-1}{2}-\lceil\frac{j}{2}\rceil} \text{ and } T_{j,\frac{p^2-1}{2}-\lceil\frac{j}{2}\rceil} \text{ for } 0 < j \le p-1.$$

We've seen which of the arrays $T_{i,j}$ intersect W non-trivially, and we now need to determine those intersections precisely. For $0 \le j \le p-1$ with $j \ne (p-1)/2$, we let B_j^i denote the $p \times p$ square in the i-th row of B_j that intersects W. Further, when p is odd, we let $B^{i,0}_{\frac{p-1}{2}}$ and $B^{i,1}_{\frac{p-1}{2}}$ denote the left and right squares, respectively, in the i-th row of $B_{\frac{p-1}{2}}$. These squares will coincide exactly when $i=0$ and $\frac{p-1}{2}$ is odd or when $i = p-1$ and $\frac{p-1}{2}$ is even. (Each B_j^i is a $T_{k,\ell}$ for some k, ℓ, and while we can make this connection explicitly, it seems unnecessary and perhaps counterproductive.) Below we indicate the positions of the B_j^i in cases $p = 2, 3, 5$:

B_0^0			B_1^0
	B_0^1	B_1^1	

B_0^0				$B_1^{0,0} = B_1^{0,1}$				B_2^0
	B_0^1		$B_1^{1,0}$		$B_1^{1,1}$		B_2^1	
		B_0^2	$B_1^{2,0}$		$B_1^{2,1}$	B_2^2		

B_0^0									B_1^0	$B_2^{0,0}$				$B_2^{0,1}$	B_3^0									B_4^0
	B_0^1							B_1^1		$B_2^{1,0}$				$B_2^{1,1}$		B_3^1							B_4^1	
		B_0^2					B_1^2				$B_2^{2,0}$		$B_2^{2,1}$				B_3^2					B_4^2		
			B_0^3			B_1^3					$B_2^{3,0}$		$B_2^{3,1}$					B_3^3			B_4^3			
				B_0^4	B_1^4							$B_2^{4,0} = B_2^{4,1}$							B_3^4	B_4^4				

Let $1 \le \alpha, \beta < p$ with $\alpha + \beta = p$, and let $0 \le j < (p-1)/2$. Recall that each B_j^i is a $p \times p$ array. The Franklin-up pattern W will intersect the B_j^i as follows, where in each instance $1 \le i \le p$.

- If j is even then $B_j^i \cap W$ consists of the first α entries in row $2j$ and the last β entries in row $2j+1$ of B_j^i.
- If j is even then $B_{p-1-j}^i \cap W$ consists of the last β entries in row $2j$ and the first α entries in row $2j+1$ of B_{p-1-j}^i.
- If j is odd then $B_j^i \cap W$ consists of the last β entries in row $2j$ and the first α entries in row $2j+1$ of B_j^i.
- If j is odd then $B_{p-1-j}^i \cap W$ consists of the first α entries of row $2j$ and the last β entries of row $2j+1$.

A pictorial representation of these intersections is given in Figure 4.

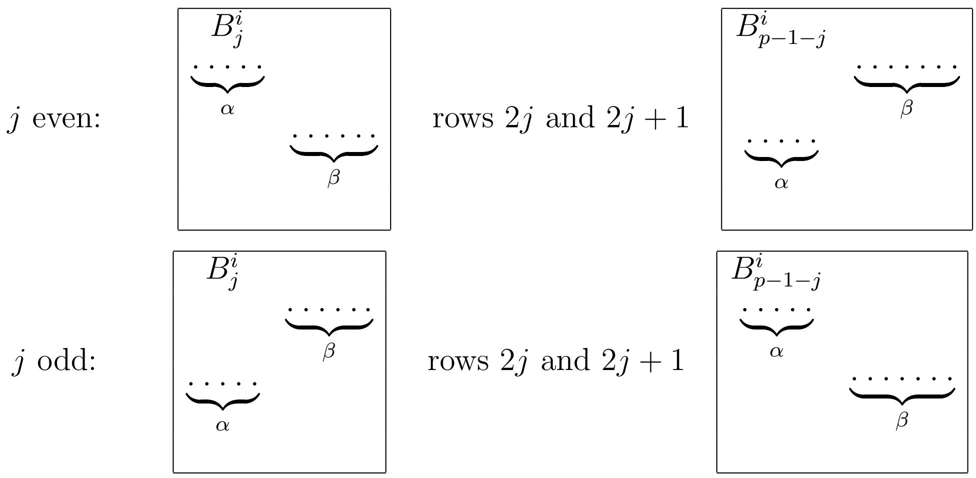


Figure 4: Intersections of B^i_j and B^i_{p-1-j} with W when $0 \leq j < (p-1)/2$.

It remains to see how, when p is odd, the squares $B^{i,k}_{\frac{p-1}{2}}$ in the central band will intersect W:

- If i is even then $B^{i,0}_{\frac{p-1}{2}} \cap W$ consists of the first α entries in the bottom row of $B^{i,0}_{\frac{p-1}{2}}$.
- If i is even then $B^{i,1}_{\frac{p-1}{2}} \cap W$ consists of the last β entries in the bottom row of $B^{i,1}_{\frac{p-1}{2}}$.
- If i is odd then $B^{i,0}_{\frac{p-1}{2}} \cap W$ consists of the last β entries in the bottom row of $B^{i,0}_{\frac{p-1}{2}}$.
- If i is odd then $B^{i,1}_{\frac{p-1}{2}} \cap W$ consists of the first α entries in the bottom row of $B^{i,1}_{\frac{p-1}{2}}$.
- In the special case that $B^{i,0}_{\frac{p-1}{2}} = B^{i,1}_{\frac{p-1}{2}}$, their intersection with W consists of the entire bottom row of $B^{i,0}_{\frac{p-1}{2}}$.

Below is a pictorial representation of these intersections:

In the case $p = 3$, $\alpha = 1$, and $\beta = 2$, the intersections described above, which characterize a Franklin-up W pattern, are illustrated in the order-27 square shown in Section 1.4.

We observe that within its frame, a Franklin-up pattern W intersects each column of S exactly once, and each row exactly p times, so W has $n = p^3$ entries. Also, while W does not have vertical midline symmetry when $p > 2$, the blocks containing W do possess this symmetry. Finally, we can obtain Franklin-right, Franklin-down, and Franklin-left patterns from a Franklin-up pattern via clockwise rotations of the ambient square S through 90°, 180°, and 270°, respectively. These constitute the

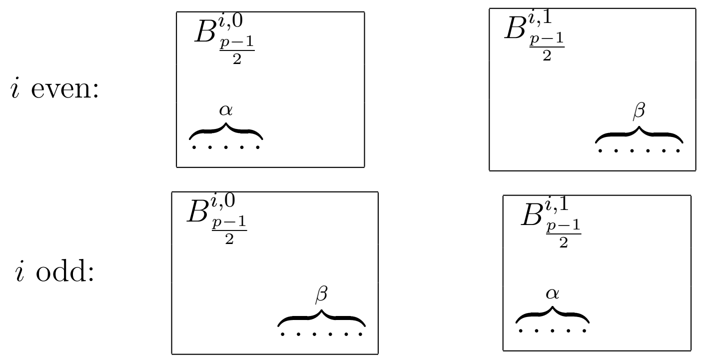


Figure 5: Intersections of $B^{i,k}_{\frac{p-1}{2}}$ with W.

entirety of Franklin patterns in S, and they specialize to the classical Franklin "V" patterns when $p = 2$. Therefore, we are now able to make the following definition:

Definition 3.1 *We say that a natural square S of order $n = p^3$ is a* **Franklin square of type** p *if it has the $p \times p$ property, the $1/p$-property for both rows and columns, and the numbers in every Franklin pattern in S add to the magic sum* $\dfrac{n(n^2-1)}{2}$.

The Franklin pattern requirement in Definition 3.1 applies to patterns arising from any partition $\alpha + \beta = p$ with $1 \leq \alpha, \beta < p$. One might reasonably weaken Definition 3.1 by only requiring the existence of a partition $\alpha + \beta$ of p such that all corresponding Franklin patterns have entries adding to the magic sum. Definition 3.1 and its weakened version both specialize to the definition of classical Franklin squares in the case $p = 2$.

4 Construction of Type-p Franklin Squares

Let p be prime and let R be a type-p most-perfect square of order p^3. Such squares exist; a linear construction is given in [4]. In this section we show that $S = \theta(R)$ is a pandiagonal type-p Franklin square, where θ is the involution introduced in Section 2. Proposition 2.6 says S is pandiagonal, has the $1/p$ row and column properties, and has the $p \times p$ property. It remains to show that the Franklin patterns of S (defined in Section 3) add to the magic sum. A similar verification for orders p^r with $r \geq 3$ is indicated in Section 5.

Lemma 4.1 *Let $m, n, p \in \mathbb{N}$ with $p \geq 2$, and consider a nonnegative integer array*

A of size $(mp+1)\times np$ *with*

$$A = \begin{array}{c|c|c|c|c} a & & b_1 & \cdots & b_{p-1} \\ \hline & & & \cdots & \\ & D & & \cdots & \\ & & & \cdots & \\ \hline c & & d_1 & \cdots & d_{p-1} \end{array}.$$

Here $a, b_i, c, d_i \in \mathbb{Z}$ *for* $1 \le i \le p-1$ *and* D *is an* $(mp-1)\times(n-1)p$ *array. If* A *possesses the* $p\times p$ *property then* $a + \sum_{i=1}^{p-1} b_i = c + \sum_{i=1}^{p-1} d_i$.

Proof: If $n = 1$ this follows immediately from the $p\times p$ property, so we assume $n \ge 2$. Rewrite A as

$$A = \begin{array}{c|c|c|c|c|c} a & & b_0 & b_1 & \cdots & b_{p-1} \\ \hline & & & & \cdots & \\ & D' & & & \cdots & \\ & & & & \cdots & \\ \hline c & & d_0 & d_1 & \cdots & d_{p-1} \end{array}$$

where $b_0, d_0 \in \mathbb{Z}$ and D' is an array of size $(mp-1)\times((n-1)p-1)$. By Lemma 2.3 we have $a + d_0 = c + b_0$. Also, because A has the $p\times p$ property, we have $b_0 + \cdots + b_{p-1} = d_0 + \cdots + d_{p-1}$. Therefore

$$a + d_0 = c + b_0 \Longrightarrow a + (\sum_{i=0}^{p-1} b_i - \sum_{i=1}^{p-1} d_i) = c + b_0 \Longrightarrow a + \sum_{i=1}^{p-1} b_i = c + \sum_{i=1}^{p-1} d_i.$$

□

If A as in the lemma has the $p\times p$ property, then the result of the lemma will continue to hold true if all other instances of p are replaced by a fixed multiple of p. Lemma 4.1 has a useful generalization:

Lemma 4.2 *Let* $m, n, k, p \in \mathbb{N}$ *with* $p \ge 2$ *and* $1 \le k < p$, *and consider a nonnegative integer array* A *of size* $(mp+1)\times np$ *with*

$$A = \begin{array}{ccc|c|ccc} a_1 & \cdots & a_k & & b_{k+1} & \cdots & b_p \\ \hline & & & & & \cdots & \\ & & & D & & \cdots & \\ & & & & & \cdots & \\ \hline c_1 & \cdots & c_k & & d_{k+1} & \cdots & d_p \end{array}.$$

Here all entries are integers and D *is an* $(mp-1)\times(n-1)p$ *array. If* A *possesses the* $p\times p$ *property then* $\sum_{i=1}^{k} a_i + \sum_{j=1}^{p-k} b_{k+j} = \sum_{i=1}^{k} c_i + \sum_{j=1}^{p-k} d_{k+j}$.

Proof: If $n = 1$ this follows immediately from the $p \times p$ property, so we assume $n \geq 2$. Let $b_1, \dots, b_k$ be the entries in A immediately preceding b_{k+1} in the same row and $b_{p+1}, \dots, b_{p+k}$ the entries immediately succeeding b_p in the same row. Similarly define $d_1, \dots, d_k$ and $d_{p+1}, \dots, d_{p+k}$. Applying Lemma 4.1 we have

$$a_j + (b_{j+1} + \cdots + b_{j+p-1}) = c_j + (d_{j+1} + \cdots + d_{j+p-1})$$

for $1 \leq j \leq k$. Adding gives

$$\sum_{j=1}^{k}[a_j + (b_{j+1} + \cdots + b_{j+p-1})] = \sum_{j=1}^{k}[c_j + (d_{j+1} + \cdots + d_{j+p-1})].$$

Upon rearrangement, one can see that a great deal of cancellation occurs in the previous equation. Note that by borrowing terms from the first summand and distributing them among the other summands, we obtain

$$\begin{aligned}\sum_{j=1}^{k}[a_j + (b_{j+1} + \cdots + b_{j+p-1})] &= [a_1 + (b_{k+1} + \cdots + b_p)] \\ &\quad + \sum_{j=2}^{k}[a_j + (b_j + \cdots + b_{j+p-1})] \\ &= \sum_{i=1}^{k} a_i + \sum_{j=1}^{p-k} b_{k+j} + \sum_{j=2}^{k}(b_j + \cdots + b_{j+p-1}).\end{aligned}$$

Likewise

$$\sum_{j=1}^{k}[c_j + (d_{j+1} + \cdots + d_{j+p-1})] = \sum_{i=1}^{k} c_i + \sum_{j=1}^{p-k} d_{k+j} + \sum_{j=2}^{k}(d_j + \cdots + d_{j+p-1}).$$

Finally, due to the $p \times p$ property, the sums $\sum_{j=2}^{k}(b_j + \cdots + b_{j+p-1})$ and $\sum_{j=2}^{k}(d_j + \cdots + d_{j+p-1})$ are equal (in fact they are equal term by term), so cancellation gives

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{p-k} b_{k+j} = \sum_{i=1}^{k} c_i + \sum_{j=1}^{p-k} d_{k+j},$$

as desired. □

Observe that the result of Lemma 4.2 still holds if the statement $1 \leq k \leq p$ is replaced by $1 \leq k \leq \ell p$ where $\ell \in \mathbb{Z}^+$.

Theorem 4.3 *Let p be prime and $n = p^3$. If R is a type-p most-perfect square of order n, then $\theta(R)$ is an order-n pandiagonal Franklin square of type p. Further, such squares R exist for every prime p.*

Proof: Type-p most-perfect squares of order $n = p^3$ exist due to [4]. Also, the square $\theta(R)$ has the $1/p$-property for rows and columns, is pandiagonal, and has the $p \times p$ property by Proposition 2.6. It remains to show that Franklin patterns in $\theta(R)$ add to the magic sum.

Let $p = \alpha + \beta$ with $1 \le \alpha, \beta < p$ and let W be a Franklin-up pattern in $\theta(R)$ corresponding to this partition of p. We establish the following notation concerning W:

- Let W_j^i denote $W \cap B_j^i$ and w_j^i denote the sum of the elements of W_j^i for $0 \le i, j \le p-1$, with $j \ne \frac{p-1}{2}$.
- W intersects B_j^i in two consecutive rows of B_j^i. For $0 \le i, j \le p-1$ with $j \ne \frac{p-1}{2}$, let $W_{j,t}^i$ denote the portion of W_j^i coming from the top-most of these two rows in B_j^i, and let $W_{j,b}^i$ denote the portion of W_j^i coming from the bottom-most of these two rows in B_j^i. Let $w_{j,t}^i$ denote the sum of the entries in $W_{j,t}^i$ and $w_{j,b}^i$ denote the sum of the entries in $W_{j,b}^i$. Note $W_j^i = W_{j,t}^i \cup W_{j,b}^i$ and $w_j^i = w_{j,t}^i + w_{j,b}^i$. The need for this distinction between "t" and "b" will be made clear later in the proof when we apply Lemma 4.2.
- If p is odd, let $W_{\frac{p-1}{2}}^{i,k}$ denote $B_{\frac{p-1}{2}}^{i,k} \cap W$, and let $w_{\frac{p-1}{2}}^{i,k}$ denote the sum of the elements of $W_{\frac{p-1}{2}}^{i,k}$.
- For $0 \le j < \frac{p-1}{2}$ we put $s_j = \displaystyle\sum_{i=0}^{p-1} (w_j^i + w_{p-1-j}^i)$.
- If p is odd, put $s_{\frac{p-1}{2}} = \displaystyle\sum_{i=0}^{p-1} \left(w_{\frac{p-1}{2}}^{i,0} + w_{\frac{p-1}{2}}^{i,1} \right)$. In the special case that $B_{\frac{p-1}{2}}^{i,0} = B_{\frac{p-1}{2}}^{i,1}$, the corresponding term in $s_{\frac{p-1}{2}}$ is just $w_{\frac{p-1}{2}}^{i,0}$, not $w_{\frac{p-1}{2}}^{i,0} + w_{\frac{p-1}{2}}^{i,1}$, as otherwise we would incur duplication.

Observe that the sum of the entries in W is $\displaystyle\sum_{0 \le j \le \frac{p-1}{2}} s_j$. We claim that $s_j = p^2(p^6 - 1)$ when $0 \le j < \frac{p-1}{2}$, and that $s_{\frac{p-1}{2}} = \dfrac{p^2(p^6-1)}{2}$ when p is odd. Assuming this claim, we have that the sum of the entries of W is

$$\sum_{0 \le j \le \frac{p-1}{2}} s_j = \frac{p-1}{2}[p^2(p^6-1)] + \frac{p^2(p^6-1)}{2} = \frac{p^3(p^6-1)}{2} = \frac{n(n^2-1)}{2}$$

when p is odd, and the sum is

$$\sum_{0 \le j \le \frac{p-1}{2}} s_j = s_0 = p^2(p^6-1) = \frac{p^3(p^6-1)}{2} = \frac{n(n^2-1)}{2}$$

when $p = 2$. In either case, the sum of the entries of W is the magic sum, as desired.

To finish, we need to verify the claims about the sums s_j. We first present an overview: If $S = \theta(R)$, then we can follow the entries in $W \subseteq S$, and hence the terms of the sums s_j, back to R by considering $\theta(S)$. Then we use the complementary property of R together with Lemma 4.2 to replace sums s_j with equivalent sums $\tilde{s}_j$ that have the claimed values.

And now on to details of the argument, which takes two cases: $0 \leq j < \frac{p-1}{2}$ and $j = \frac{p-1}{2}$. First suppose that $0 \leq j < \frac{p-1}{2}$. Observe that for $0 \leq i < p-1$, each entry of $W^i_{j,t} \cup W^i_{p-1-j,t}$ is p columns distant from its counterpart in $W^{i+1}_{j,t} \cup W^{i+1}_{p-1-j,t}$ in $S = \theta(R)$, with no repetition of columns. (Here "counterparts" lies in the same relative position within a block.) Further, we note that the columns of the subsquare frame array T for W coincide with the columns of the subsquare array $(S_{\ell,m})$ as in Equation 1. (This is not generally true for rows of T.) Also, for $0 \leq i \leq p-1$, $W^i_{j,t}$ lies wholly within band B_j, which in turn coincides with a natural band of p consecutive columns in the subsquare array $(S_{\ell,m})$. A similar statement is true for $W^i_{p-1-j,t}$. Therefore subsquares in $S_{\ell,m}$ containing a pair of counterparts in $W^i_{j,t} \cup W^i_{p-1-j,t}$ and $W^{i+1}_{j,t} \cup W^{i+1}_{p-1-j,t}$ must lie in consecutive columns in $S_{\ell,m}$. Taking all of this into account, upon applying Equation (2), we find that elements in $W^i_{j,t} \cup W^i_{p-1-j,t}$ are $p^2 = n/p$ columns distant from their counterparts in $W^{i+1}_{j,t} \cup W^{i+1}_{p-1-j,t}$ within $R = \theta(S)$, with no repetition of columns. (Another way to view this is that the squares containing these counterparts are p columns distant in the subsquare array $R_{\ell,m}$.) These same observations and conclusion are also true if $W^i_{j,t} \cup W^i_{p-1-j,t}$ is replaced with $W^i_{j,b} \cup W^i_{p-1-j,b}$.

We have established that as i varies from 0 to $p-1$, elements in $W^i_{j,t} \cup W^i_{p-j-1,t}$ are $p^2 = n/p$ columns apart from their counterparts in $W^{i+1}_{j,t} \cup W^{i+1}_{p-j-1,t}$ in R, and similarly when "t" is replaced by "b". If these same statements were also true with "rows" in place of "columns", then we could repeatedly apply the complementary property of R to obtain

$$\begin{aligned} s_j &= \sum_{i=0}^{p-1} w^i_j + w^i_{p-1-j} \\ &= \sum_{i=0}^{p-1} (w^i_{j,t} + w^i_{p-1-j,t}) + \sum_{i=0}^{p-1} (w^i_{j,b} + w^i_{p-1-j,b}) \\ &= p\left[\frac{p(p^6-1)}{2}\right] + p\left[\frac{p(p^6-1)}{2}\right] = p^2(p^6-1), \end{aligned}$$

as claimed. (Here the multiplications by p in the penultimate line are due to the fact that there are $a+b=p$ members of $W^i_{j,t} \cup W^i_{p-1-j,t}$, and similarly for $W^i_{j,b} \cup W^i_{p-1-j,b}$). Unfortunately, because the rows of the frame array $T = (T_{\ell,m})$ do *not* generally coincide with a natural band of p consecutive rows in $(S_{\ell,m})$, it is not always true that elements in $W^i_{j,t} \cup W^i_{p-j-1,t}$ are $p^2 = n/p$ rows apart from their counterparts in $W^{i+1}_{j,t} \cup W^{i+1}_{p-j-1,t}$ in R.

Lemma 4.2 can be used to rectify this problem. Elements in $W^{i+1}_{j,t} \cup W^{i+1}_{p-j-1,t}$ may not be $p^2 = n/p$ rows distant in R from elements in $W^i_{j,t} \cup W^i_{p-j-1,t}$, but this distance is some multiple of p due to our construction of W and to Equation (2). By moving vertically in R from $W_{j,t}^{i+1} \cup W_{p-j-1,t}^{i+1}$ by some appropriate multiple of p units (possibly zero), we encounter a set $\tilde{W}_{j,t}^{i+1} \cup \tilde{W}_{p-j-1,t}^{i+1}$ of p elements in R that is $n/p = p^2$ rows distant from $W_{j,t}^{i} \cup W_{p-j-1,t}^{i}$:

$$\begin{array}{ccc} W_{j,t}^{i+1} & \qquad & W_{p-1-j,t}^{i+1} \\ \cdots\cdot & & \cdots\cdots\cdot \\ \downarrow & & \downarrow \\ \cdots\cdot & & \cdots\cdots\cdot \\ \tilde{W}_{j,t}^{i+1} & & \tilde{W}_{p-1-j,t}^{i+1} \end{array}$$

Further, by applying Lemma 4.2, we have

$$w_{j,t}^{i+1} + w_{p-j-1,t}^{i+1} = \tilde{w}_{j,t}^{i+1} + \tilde{w}_{p-j-1,t}^{i+1},$$

where $\tilde{w}_{j,t}^{i+1}$ is the sum of the elements in $\tilde{W}_{j,t}^{i+1}$, and likewise for $\tilde{w}_{p-j-1,t}^{i+1}$. The vertical nature of this replacement has no effect on the relationship among columns: it is still true that an element in $W_{j,t}^{i} \cup W_{p-j-1,t}^{i}$ and its counterpart in $\tilde{W}_{j,t}^{i+1} \cup \tilde{W}_{p-j-1,t}^{i+1}$ are $n/p = p^2$ columns distant from one another. These statements are also true if "t" is replaced by "b". By making these replacements systematically and judiciously, so as to avoid repetition of rows, we may apply Lemma 4.2 together with the complementary property in R to obtain

$$\begin{aligned} s_j &= \sum_{i=0}^{p-1} w_j^i + w_{p-1-j}^i \\ &= \sum_{i=0}^{p-1} (w_{j,t}^i + w_{p-1-j,t}^i) + \sum_{i=0}^{p-1} (w_{j,b}^i + w_{p-1-j,b}^i) \\ &= \sum_{i=0}^{p-1} (\tilde{w}_{j,t}^i + \tilde{w}_{p-1-j,t}^i) + \sum_{i=0}^{p-1} (\tilde{w}_{j,b}^i + \tilde{w}_{p-1-j,b}^i) \\ &= p\left[\frac{p(p^6-1)}{2}\right] + p\left[\frac{p(p^6-1)}{2}\right] = p^2(p^6-1), \end{aligned} \tag{3}$$

thereby proving the first portion of our claim on the sums s_j.

Finally, we address the claimed value of $s_{\frac{p-1}{2}}$. Without loss of generality we assume that $B_{\frac{p-1}{2}}^{0,0} = B_{\frac{p-1}{2}}^{0,1}$. For each $1 \le i \le p-1$, we may use Lemma 4.2 to consider elements $\tilde{W}_{\frac{p-1}{2}}^{i,0} \cup \tilde{W}_{\frac{p-1}{2}}^{i,1}$ lying above $W_{\frac{p-1}{2}}^{i,0} \cup W_{\frac{p-1}{2}}^{i,1}$ and in the same row as $W_{\frac{p-1}{2}}^{0,0}$ as illustrated here:

$$\begin{array}{ccccc}
\overbrace{*\,*\,*}^{\tilde{W}^{2,0}_{\frac{p-1}{2}}} & \overbrace{*\,*\,*\,*}^{\tilde{W}^{1,0}_{\frac{p-1}{2}}} & \overbrace{*\,*\,*\,*\,*\,*\,*\,*}^{W^{0,0}_{\frac{p-1}{2}}} & \overbrace{*\,*\,*}^{\tilde{W}^{1,1}_{\frac{p-1}{2}}} & \overbrace{*\,*\,*\,*}^{\tilde{W}^{2,1}_{\frac{p-1}{2}}} \\
\uparrow & \uparrow & & \uparrow & \uparrow \\
 & *\,*\,*\,* & & *\,*\,* & \\
\uparrow & W^{1,0}_{\frac{p-1}{2}} & & W^{1,1}_{\frac{p-1}{2}} & \uparrow \\
\,\,* & & & & *\,*\,*\,* \\
W^{2,0}_{\frac{p-1}{2}} & & & & W^{2,1}_{\frac{p-1}{2}}
\end{array}$$

If we let $\tilde{w}^{i,0}_{\frac{p-1}{2}} + \tilde{w}^{i,1}_{\frac{p-1}{2}}$ be the corresponding sum of elements, we find by applying the $1/p$ row property of $\theta(R)$ (Proposition 2.2) that

$$\begin{aligned}
s_{\frac{p-1}{2}} &= w^{0,0}_{\frac{p-1}{2}} + \sum_{i=1}^{p-1}\left(w^{i,0}_{\frac{p-1}{2}} + w^{i,1}_{\frac{p-1}{2}}\right) \\
&= w^{0,0}_{\frac{p-1}{2}} + \sum_{i=1}^{p-1}\left(\tilde{w}^{i,0}_{\frac{p-1}{2}} + \tilde{w}^{i,1}_{\frac{p-1}{2}}\right) \\
&= \frac{1}{p}\left[\frac{n(n^2-1)}{2}\right] = \frac{1}{p}\left[\frac{p^3(p^6-1)}{2}\right] = \frac{p^2(p^6-1)}{2},
\end{aligned} \tag{4}$$

as claimed. The other Franklin pattern categories (right, down, and left) have similar verifications.

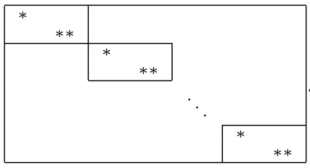
□

5 Type-p Franklin Squares of Order kp^3 with $k > 1$.

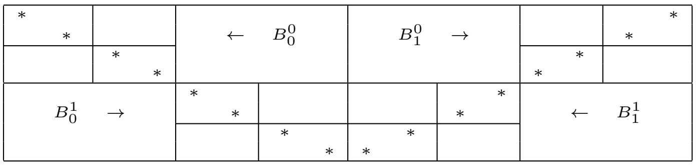
In this section we indicate how type-p Franklin squares of order kp^3 can be defined, and argue that these squares exist when $k = p^r$ for $r \geq 0$. This extends the results of Sections 3 and 4, where we addresed the special case $k = 1$. Terminology and ideas of Sections 3 and 4 will be used throughout.

The description in Section 1.4 characterizes type-p Franklin squares of order kp^3 except for the Franklin patterns. As in Section 3, we focus on describing Franklin-up patterns; the other varieties (right, down, and left) are obtained from Franklin-up locations by rotating the ambient square. Let S be a square of order $n = kp^3$, let $\alpha + \beta = p$ with $1 \leq \alpha, \beta < p$, and let W be a Franklin-up pattern in S. The frame for W consists of $\frac{n}{p} = kp^2$ consecutive rows of S. As in Section 3, we can partition this frame into a $p \times p^2$ array $(T_{i,j})$ where $T_{i,j}$ is an array of size $kp \times kp$. Therefore, each of the squares B^i_j and B_{p-1-j} should be of size $kp \times kp$, as should be $B^{k,l}_{\frac{p-1}{2}}$ in case p is odd. To determine W it is necessary to describe the intersection of these squares with W.

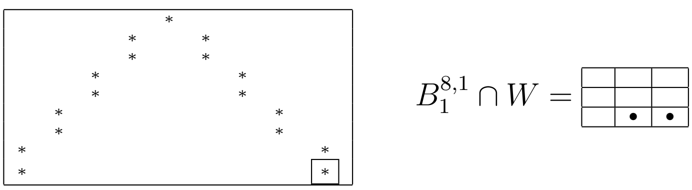
We first address $W \cap B_j^i$ with $0 \le j < \frac{p-1}{2}$. View B_j^i as a $k \times k$ array whose entries are $p \times p$ subarrays. If j is even, recall that as i increases from 0 to $p-1$, the squares B_j^i lie on a broken main diagonal of the array $(T_{i,j})$. In this case we declare that W intersects B_j^i in each of the $p \times p$ submatrices on the main block diagonal of B_j^i in the manner described in Section 3 (Figure 4). If j is odd, recall that the squares B_j^i lie on a broken off diagonal of the array $(T_{i,j})$. In this case we declare that W intersects B_j^i in each of the $p \times p$ submatrices occupying the off block diagonal of B_j^i in the manner of Section 3. Intersections of W with B_{p-1-j}^i are determined similarly. A figure illustrating $W \cap B_j^i$ with j even is shown below, where the smaller arrays along the main diagonal are of size $p \times p$.

$W \cap B_j^i =$  .

Also, here is a frame showing all blocks B_j^i in the classical case $p = 2$ and $n = 2 \cdot 2^3 = 16$:



It remains to address the intersection of the Franklin-up pattern W with the middle band $B_{\frac{p-1}{2}}$ in the case that p is odd. Unlike the other bands, we will continue to partition $B_{\frac{p-1}{2}}$ into $p \times p$ subsquares as we did in Section 3. (This is reasonable because we do not apply θ to this band in Theorem 5.1, and so we do not need a partition into squares of order $\frac{n}{p^2} = kp$.) Further, we define $B_{\frac{p-1}{2}}^{i,0}$ and $B_{\frac{p-1}{2}}^{i,1}$, as well as their intersections with W just as we did in Section 3, except that $0 \le i \le kp-1$ rather than $0 \le i \le p-1$ (Figure 5). We note that in the special case that $B_{\frac{p-1}{2}}^{i,0}$ and $B_{\frac{p-1}{2}}^{i,1}$ coincide, then the intersection with W is the entire bottom row of this square; this will happen when k is odd. Meanwhile, in the special case that $B_{\frac{p-1}{2}}^{i,0}$ and $B_{\frac{p-1}{2}}^{i,1}$ are adjacent (borders touching) then their intersection with W consists of the entire bottom row of both squares. This latter case, which happens when k is even, produces a row in the frame for W that intersects W in $2p$ locations rather than p locations. An illustration is given in the following figure, which shows the middle band B_1 in the case $n = kp^3 = 3 \cdot 3^3$. Each entry is a 3×3 array; the asterisks are the $B_1^{i,\ell}$'s. The boxed asterisk is $B_1^{8,1}$; its intersection with W is shown in the right portion of the figure (assuming $\alpha = 1$ and $\beta = 2$).



Theorem 5.1 *Let p be prime, $k \in \mathbb{Z}^+$, and $n = kp^3$. If R is a type-p most-perfect square of order n then $\theta(R)$ is an order-n pandiagonal type-p Franklin square. Further, such squares R exist when $k = p^r$ for any prime p and any $r \geq 0$.*

Proof: The proof, which shall be abridged, closely follows that for Theorem 4.3. Notation will be identical to that of Theorem 4.3, with the exception that w_j^i will be split into $2k$ summands rather than just two summands $w_{j,t}^i$ and $w_{j,b}^i$. This is due to the fact that W intersects B_j^i in $2k$ rows rather than 2 rows. (A similar adjustment is made for w_{p-1-j}^i.)

Let $S = \theta(R)$. Due to Proposition 2.6, to establish that S is a type-p Franklin square it remains to show that entries in Franklin patterns add to the magic sum. We verify this for Franklin-up patterns only, the other patterns have similar verifications. Following the proof of Theorem 4.3, and Equation (3) in particular, the use of Lemma 4.2 and the complementary property in R gives

$$s_j = \sum_{i=0}^{p-1} w_j^i + w_{p-1-j}^i = \underbrace{p\left[\frac{p(n^2-1)}{2}\right] + \cdots + p\left[\frac{p(n^2-1)}{2}\right]}_{2k \text{ times}} = \frac{n(n^2-1)}{p}$$

when $0 \leq j < \frac{p-1}{2}$. Likewise, in the case that p is odd, applying Lemma 4.2 together with the $1/p$-row property of S as in Equation (4) gives $s_{\frac{p-1}{2}} = \frac{n(n^2-1)}{2p}$. It follows that the sum of the entries in W is

$$\sum_{0 \leq j \leq \frac{p-1}{2}} s_j = \frac{n(n^2-1)}{2},$$

as desired.

Finally, the existence of type-p most-perfect squares of order p^s ($s \geq 3$) is guaranteed by [4]. □

6 Appendix

Appearing in Figure 6 is a larger version of the order-27, type-3 Franklin square given initially in Section 1.4.

0	691	401	432	151	509	621	259	212	567	286	239	27	718	347	459	97	536	405	124	563	594	313	185	54	664	374
457	140	495	646	248	198	25	680	387	52	707	333	484	86	522	592	275	225	619	302	171	79	653	360	430	113	549
635	261	196	14	693	385	446	153	493	473	99	520	581	288	223	41	720	331	68	666	358	419	126	547	608	315	169
16	698	378	448	158	486	637	266	189	583	293	216	43	725	324	475	104	513	421	131	540	610	320	162	70	671	351
437	144	511	626	252	214	5	684	403	32	711	349	464	90	538	572	279	241	599	306	187	59	657	376	410	117	565
639	250	203	18	682	392	450	142	500	477	88	527	585	277	230	45	709	338	72	655	365	423	115	554	612	304	176
23	675	394	455	135	502	644	243	205	590	270	232	50	702	340	482	81	529	428	108	556	617	297	178	77	648	367
441	160	491	630	268	194	9	700	383	36	727	329	468	106	518	576	295	221	603	322	167	63	673	356	414	133	545
628	257	207	7	689	396	439	149	504	466	95	531	574	284	234	34	716	342	61	662	369	412	122	558	601	311	180
21	676	395	453	136	503	642	244	206	588	271	233	48	703	341	480	82	530	426	109	557	615	298	179	75	649	368
442	161	489	631	269	192	10	701	381	37	728	327	469	107	516	577	296	219	604	323	165	64	674	354	415	134	543
629	255	208	8	687	397	440	147	505	467	93	532	575	282	235	35	714	343	62	660	370	413	120	559	602	309	181
1	692	399	433	152	507	622	260	210	568	287	237	28	719	345	460	98	534	406	125	561	595	314	183	55	665	372
458	138	496	647	246	199	26	678	388	53	705	334	485	84	523	593	273	226	620	300	172	80	651	361	431	111	550
633	262	197	12	694	386	444	154	494	471	100	521	579	289	224	39	721	332	66	667	359	417	127	548	606	316	170
17	696	379	449	156	487	638	264	190	584	291	217	44	723	325	476	102	514	422	129	541	611	318	163	71	669	352
435	145	512	624	253	215	3	685	404	30	712	350	462	91	539	570	280	242	597	307	188	57	658	377	408	118	566
640	251	201	19	683	390	451	143	498	478	89	525	586	278	228	46	710	336	73	656	363	424	116	552	613	305	174
15	697	380	447	157	488	636	265	191	582	292	218	42	724	326	474	103	515	420	130	542	609	319	164	69	670	353
436	146	510	625	254	213	4	686	402	31	713	348	463	92	537	571	281	240	598	308	186	58	659	375	409	119	564
641	249	202	20	681	391	452	141	499	479	87	526	587	276	229	47	708	337	74	654	364	425	114	553	614	303	175
22	677	393	454	137	501	643	245	204	589	272	231	49	704	339	481	83	528	427	110	555	616	299	177	76	650	366
443	159	490	632	267	193	11	699	382	38	726	328	470	105	517	578	294	220	605	321	166	65	672	355	416	132	544
627	256	209	6	688	398	438	148	506	465	94	533	573	283	236	33	715	344	60	661	371	411	121	560	600	310	182
2	690	400	434	150	508	623	258	211	569	285	238	29	717	346	461	96	535	407	123	562	596	312	184	56	663	373
456	139	497	645	247	200	24	679	389	51	706	335	483	85	524	591	274	227	618	301	173	78	652	362	429	112	551
634	263	195	13	695	384	445	155	492	472	101	519	580	290	222	40	722	330	67	668	357	418	128	546	607	317	168

Figure 6:

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