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Dominator coloring of Mycielskian graphs

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Abstract

A dominator coloring of a graph G is a proper vertex coloring of G such that each vertex of G is adjacent to all the vertices of at least one color class or else lies alone in its color class. In this paper we have characterized the results on dominator coloring of Mycielskian graphs and iterated Mycielskian graphs.

1 Introduction

Let G be a simple graph, where V is the vertex set, E is the edge set, n is the order of G and m is the size of G. For graph theoretic terminology we refer to [4] and for colorings and domination in graphs we refer to [5, 14, 15].

In order to build a graph having small clique number and high chromatic number, Mycielski [20] introduced a Mycielskian graph $\mu(G)$ defined as follows: Let $V(\mu(G)) = V \cup V^{'} \cup \{u\}$ with $v_i v_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, with $v_i v_j^{'} \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, with $v_i^{'} u \in E(\mu(G)), 1 \leq i \leq n$, and with no other edges in $\mu(G)$, where $v_i \in V(G)$ and $v_i^{'} \in V^{'}$. For recent results on Mycielskian graph we refer to [2, 3, 7–9, 13, 16–18, 20].

A *dominator coloring* (DC) of a graph G is a proper vertex coloring of G such that each vertex dominates some color class or else lies alone in its color class. A *dominator chromatic number* $\chi_d(G)$ is the minimum cardinality among all DC of G. The idea of DC was presented by Gera et al. [10] and further studied by [1,6,11,12,19]. It has been shown in [1] that for every graph G, $\chi_d(G)+1 \leq \chi_d(\mu(G)) \leq \chi_d(G)+2$. In this context we have characterized the results attaining the bounds. Also we have proved that $\chi_d(G)+2k-1 \leq \chi_d(\mu^k(G)) \leq \chi_d(G)+2k$ and characterized the results attaining the bounds, where $\mu^k(G)$ is the iterated Mycielskian of G.

The *open neighborhood* and *closed neighborhood* of $v \in V$ are the sets $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$ respectively. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a DC of a graph G, where each V_i is a color class. A DC using $\chi_d(G)$ colors is called

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as χ_d-coloring of G. A color class $V_i, 1 \leq i \leq k$, is called a *spare color class* with respect to $\mathcal{C}$ if each vertex $v \in V(G)$ dominates some color class $V_j, j \neq i$, of $\mathcal{C}$ in G. A vertex v is a *solitary vertex* if $\{v\} \in \mathcal{C}$ of G and $N(v)$ do not contain any color class.

2 Mycielskian graph

In [1], Arumugam et al. proved that dominator chromatic number of Mycielskian graph $\mu(G)$ is either $\chi_d(G) + 1$ or $\chi_d(G) + 2$. We now characterize the results attaining the bounds.

Theorem 2.1. *Given a graph G, $\chi_d(\mu(G)) = \chi_d(G) + 1$ if and only if for some χ_d-coloring $\mathcal{C}$ of G:*

(i) each vertex v dominates some color class V_i with $v \notin V_i$;

(ii) a vertex v is a solitary vertex and $\mathcal{C}$ contains a spare color class V_i which does not contain any vertex of $N(v)$.

Proof. Suppose condition (i) holds. Let us consider a coloring of $\mu(G)$ such that each twin vertex $v_i^{'}$ is assigned the color of v_i and the root vertex u is assigned a new color. Clearly each vertex $v \in V(G)$ dominates some color class as in $\mathcal{C}$ of G. The twin vertices $v_i^{'}$ and the root vertex u dominate the color class $\{u\} \in \mathcal{C}_1$. Thus $\chi_d(\mu(G)) \leq \chi_d(G) + 1$ and hence equality holds.

Suppose condition (ii) holds and let v_1 be a solitary vertex. Let us consider a coloring $\mathcal{C}_1 = (\mathcal{C} - V_i) \cup \{\{V_i \cup \{v_1^{'}\}\}, \{u\}\}$ of $\mu(G)$ such that each twin vertex $v_i^{'}, i \neq 1$, is assigned the color of v_i and the twin vertex $v_1^{'}$ is assigned the color i. Further the root vertex u is assigned a new color. Clearly each vertex $v \in V(G)$ dominates some color class as in $\mathcal{C}$ of G. The twin vertices $v_i^{'}$ and the root vertex u dominate the color class $\{u\} \in \mathcal{C}_1$. Thus $\chi_d(\mu(G)) \leq \chi_d(G) + 1$ and hence equality holds.

Conversely, suppose $\chi_d(\mu(G)) = \chi_d(G) + 1$. Let $\mathcal{C} = \{V_1, V_2, \ldots, V_k\}$ be a χ_d-coloring of $\mu(G)$, where $k = \chi_d(G) + 1$ and let $u \in V_1$, then

Case 1. $V_1 = \{u\}$.

Assuming that each vertex $v \in V_i$ such that $|V_i| \geq 2, i \neq 1$, we consider a restricted coloring $\mathcal{C}_1$ to G in such a way that, for each color class $V_i \subset V^{'}$, we select an arbitrary vertex $v^{'} \in V_i$ and recolor its twin $v \in V$ with color $i, i \neq 1$, remaining vertices are colored as in $\mathcal{C}$ of $\mu(G)$. The restricted coloring $\mathcal{C}_1$ to G is a DC with $\chi_d(\mu(G)) - 1$ colors. Thus condition (i) holds.

Suppose $\{v_i\} \in \mathcal{C}$ and $\{v_i^{'}\} \in \mathcal{C}$ of $\mu(G)$. Then consider a restricted coloring $\mathcal{C}_1$ to G in such a way that, for each color class $V_i \subset V^{'}$, we randomly choose a vertex $v^{'} \in V_i$ and recolor its twin $v \in V$ with color $i, i \neq 1$, remaining vertices are colored as in $\mathcal{C}$ of $\mu(G)$. The restricted coloring $\mathcal{C}_1$ to G is a DC with $\chi_d(\mu(G)) - 2$ colors, which is a contradiction. Thus either $\{v_i\} \notin \mathcal{C}$ or $\{v_i^{'}\} \notin \mathcal{C}$ of $\mu(G)$. Let $\{v_i^{'}\} \notin \mathcal{C}$

of $\mu(G)$. Then the vertex $v_i^{'} \in V_i$ such that $|V_i| \geq 2$. Suppose the vertex $v_i^{'} \in V^{'}$ dominate the color classes $V_1 \in \mathcal{C}$ and $V_j \in \mathcal{C}, j \neq \{1, i\}$. Then consider a restricted coloring $\mathcal{C}_1$ to G in such a way that, for each color class $V_k \subset V^{'}$, we randomly choose a vertex $v^{'} \in V_k$ and recolor its twin v with color $k, k \neq 1$, remaining vertices are colored as in $\mathcal{C}$ of $\mu(G)$. Thus $\mathcal{C}_1$ is a DC of G with $\chi_d(\mu(G)) - 1$ colors. Now we prove that $V_i \in \mathcal{C}_1$ is a spare color class. If any vertex $x \in V(G)$ dominate the color class $V_i \in \mathcal{C}$ of $\mu(G)$, then in the restricted coloring $\mathcal{C}_1$ to G the vertex x continues to dominate the color class V_i and the color class $\{v_i\} \in \mathcal{C}_1$. Hence V_i is a spare color class and V_i does not contain any vertex of $N(v_i)$, since v_i and $v_i^{'}$ are twin vertices. Hence condition (i) holds. Suppose $v_i^{'} \in V^{'}$ does not dominate any color class $V_j, j \neq 1$. Then condition (ii) holds.

Case 2. $V_1 \neq \{u\}$.

Clearly no vertex $v \in V(G)$ dominates the color class $V_1 \in \mathcal{C}$ of $\mu(G)$. In this case the root vertex u dominates some color class, say $V_k \subseteq V^{'}$. Suppose any vertex $v \in V(G)$ dominate the color class V_k or v is a solitary vertex. Then its twin vertex $v^{'}$ cannot dominate the color class V_k or $\{v\}$. In this case $v^{'}$ dominate the color class V_1. Now we consider a restricted coloring $\mathcal{C}_1$ to G such that for each color class $V_i \subset V^{'}, i \neq k$, we randomly choose a vertex $v^{'} \in V_i$ and recolor its twin v with color $i, i \neq k$, remaining vertices are colored as in $\mathcal{C}$ of $\mu(G)$. The restricted coloring $\mathcal{C}_1$ to G is a DC with $\chi_d(\mu(G)) - 1$ colors. Thus condition (i) holds. $\square$

3 Iterated Mycielskian graphs

In this section, we give bounds on dominator coloring of iterated Mycielskian of graphs and characterize the results attaining the bounds.

Definition 3.1. Iteratively applying the Mycielskian operator k-times for a graph G, we get iterated Mycielskian $\mu^k(G)$ of G. That is, $\mu^k(G) = \underbrace{\mu(\mu(\mu(\ldots\mu}_{k\text{-times}}(G)))).$

Lemma 3.2. *Suppose some χ_d-coloring of G has no spare color class and v is a solitary vertex. Then we have a χ_d-coloring $\mathcal{C}_i$ of $\mu^i(G)$ such that $\mathcal{C}_i$ has a solitary vertex.*

Proof. Suppose condition holds. Then by Theorem 2.1, we have a χ_d-coloring $\mathcal{C}_1$ of $\mu(G)$ such that $\chi_d(\mu(G)) = \chi_d(G) + 2$ in which the vertex v continues to be a solitary vertex. Again based on Theorem 2.1, we have $\chi_d(\mu^2(G)) = \chi_d(\mu(G)) + 2$ in which the vertex v continues to be a solitary vertex. Applying Theorem 2.1 iteratively, we have a χ_d-coloring $\mathcal{C}_i$ of $\mu^i(G)$ such that $\mathcal{C}_i$ has a solitary vertex. $\square$

Lemma 3.3. *Let $\mathcal{C}$ and $\mathcal{C}_1$ be a χ_d-coloring of G and $\mu(G)$ respectively. Then either $\mathcal{C}$ or $\mathcal{C}_1$ has a solitary vertex.*

Proof. Suppose a χ_d-coloring $\mathcal{C}$ of G has no solitary vertex. Then by Theorem 2.1, we have a χ_d-coloring $\mathcal{C}_1$ of $\mu(G)$ in which the root vertex u is a solitary vertex. This completes the proof. □

Theorem 3.4. *For a graph G, $\chi_d(G)+2k-1 \leq \chi_d(\mu^k(G)) \leq \chi_d(G)+2k$. Further $\chi_d(\mu^k(G)) = \chi_d(G)+2k$ if and only if some χ_d-coloring $\mathcal{C}$ of G contains no spare color class and $\{v\} \in \mathcal{C}$ is a solitary vertex.*

Proof. We know that

$$\chi_d(\mu(G)) \geq \chi_d(G)+1.$$

Then by Lemma 3.3 and Theorem 2.1, we have $\chi_d(\mu^2(G)) \geq \chi_d(G)+3$. Again by Lemma 3.3 and Theorem 2.1, we have $\chi_d(\mu^3(G)) \geq \chi_d(G)+5$. Continuing this process, we obtain $\chi_d(\mu^k(G)) \geq \chi_d(G)+2k-1$.

Next we claim that $\chi_d(\mu^k(G)) \leq \chi_d(G)+2k$. Consider the following two cases, 3 and 4.

Case 3. *Let $\mathcal{C}$ be a χ_d-coloring of G having no spare color class and $\{v\} \in \mathcal{C}$ be a solitary vertex.*

By Lemma 3.2, we have a χ_d-coloring $\mathcal{C}_i$ of the iterated Mycielskian graph $\mu^i(G)$, $1 \leq i \leq k$, such that $\mathcal{C}_i$ contains a vertex v which is a solitary vertex. Then by Theorem 2.1, we have $\chi_d(\mu^i(G)) = \chi_d(\mu^{i-1}(G))+2$, where $\mu^0(G)=G$. This implies that

$$\begin{aligned}
\chi_d(\mu^k(G)) &= \chi_d(\mu^{k-1}(G))+2 \\
&= \chi_d(\mu^{k-2}(G))+4 \\
&\vdots \\
&= \chi_d(\mu^{k-k}(G))+2k \\
&= \chi_d(\mu^{0}(G))+2k \\
&= \chi_d(G)+2k.
\end{aligned}$$

Case 4. *Every χ_d-coloring $\mathcal{C}$ of G satisfies one of the two conditions of Theorem 2.1.*

Then by Lemma 3.3, we have a χ_d-coloring $\mathcal{C}_1$ of $\mu(G)$ containing a solitary vertex v. Let $\mu(G)=G_1$. Then by Lemma 3.2, we have a χ_d-coloring $\mathcal{C}_i$ of $\mu^i(G), 1 \leq i \leq k-1$, containing a solitary vertex v. From Theorem 2.1 we have $\chi_d(\mu(G)) = \chi_d(G)+1$ and $\chi_d(\mu^i(G)) = \chi_d(\mu^{i-1}(G))+2, 2 \leq i \leq k$, which implies that

$$\begin{aligned}
\chi_d(\mu^k(G)) &= \chi_d(\mu^{k-1}(G))+2 \\
&= \chi_d(\mu^{k-2}(G))+4 \\
&\vdots \\
&= \chi_d(\mu^{k-(k-1)}(G))+2(k-1) \\
&= \chi_d(\mu(G))+2k-2 \\
&= \chi_d(G)+2k-1 \qquad (\text{since } \chi_d(\mu(G)) = \chi_d(G)+1).
\end{aligned}$$

Now we prove the "if and only if" case of the theorem. Let $\chi_d(\mu^k(G)) = \chi_d(G) + 2k$. Suppose some χ_d-coloring $\mathcal{C}$ of G satisfies one of the two conditions of Theorem 2.1. Then it follows from Case 4 that $\chi_d(\mu^k(G)) < \chi_d(G) + 2k$, which is a contradiction.

Conversely, let $\mathcal{C}$ be a χ_d-coloring of G having no spare color class and $\{v\}$ is a solitary vertex. Then by Case 3, we have $\chi_d(\mu^k(G)) = \chi_d(G) + 2k$. Thus it is proved. $\square$

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