AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **74(1)** (2019), Pages 74–85

On the genus of the essential graph of commutative rings

K. Selvakumar M. Subajini

Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli 627 012, Tamil Nadu
India
selva_158@yahoo.co.in subashini.m05@gmail.com

M. J. Nikmehr

Faculty of Mathematics
K.N. Toosi University of Technology, Tehran
Iran
nikmehr@kntu.ac.ir

Abstract

Let R be a commutative ring with identity and let $Z(R)$ be the set of zero-divisors of R. The *essential graph* of R is defined as the graph $EG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$ such that two distinct vertices x and y are adjacent if and only if $\text{ann}(xy)$ is an essential ideal. In this paper, we classify all finite commutative rings with identity for which the genus and crosscap of $EG(R)$ are at most one.

1 Introduction

The study linking commutative ring theory with graph theory was started with the concept of the zero-divisor graph of a commutative ring. Let R be a commutative ring and $Z(R)^*$ be the set of all non-zero zero-divisors of R. The *zero-divisor graph* of R, denoted $\Gamma(R)$, is the simple graph with $Z(R)^*$ as the vertex set such that two distinct vertices x and y are joined by an edge if and only if $xy = 0$. This definition was introduced by Beck, Anderson and Livingston in [1, 5] and later was studied extensively in [2, 6, 9, 12, 17, 18, 19]. For $a \in R$, let $\text{ann}(a) = \{d \in R : da = 0\}$ be the annihilator of a in R. In 2014, Badawi [3] introduced the *annihilator graph* $AG(R)$ as the simple graph with vertex set $Z(R)^*$ such that two distinct vertices x and y are adjacent if and only if $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. One can see that the zero-divisor graph $\Gamma(R)$ is a subgraph of the annihilator graph $AG(R)$. In view of this, Nikmehr et al. [11] have introduced and investigated a graph called the essential

ISSN: 2202-3518

graph of a commutative ring. A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with any non-zero ideal of R. The *essential graph* of R is defined as the graph $EG(R)$ with the vertex set $Z(R)^* = Z(R) \smallsetminus \{0\}$ such that distinct vertices x and y are adjacent if and only if $\text{ann}(xy)$ is an essential ideal. The authors in [11] discussed some basic properties of $EG(R)$ and studied the affinity between essential graph and zero-divisor graph. One can see that the zero-divisor graph $\Gamma(R)$ is a subgraph of the essential graph $EG(R)$.

The main objective of topological graph theory is to embed a graph into a surface. There are many studies [2, 6, 9, 12, 14, 15, 16, 18, 19] concerning orientable and non-orientable embeddings of the zero-divisor graph and other graphs. In this paper, we classify all finite commutative rings with identity for which the genus and crosscap of $EG(R)$ are at most one.

Let S_g and $\bar{S}_k$ denote the sphere with g handles and k crosscaps respectively, where g and k are non-negative integers, that is S_g and $\bar{S}_k$ are the oriented and non-oriented with g handles and k crosscaps. The *genus* $\gamma(G)$ of a simple graph G is the minimum g such that G can be embedded in S_g. Similarly, *crosscap number* $\overline{\gamma}(G)$ is the minimum k such that G can be embedded in $\bar{S}_k$. When considering orientability, the surfaces S_g and the sphere are orientable $\bar{S}_k$ is not orientable. A graph G is planar if $\gamma(G) = 0$. A graph G such that $\gamma(G) = 1$ is called a toroidal graph and $\overline{\gamma}(G) = 1$ is called a projective graph. It is easy to see that $\gamma(H) \leq \gamma(G)$ and $\overline{\gamma}(H) \leq \overline{\gamma}(G)$ for all subgraphs H of G. One of the most remarkable theorems in topological graph theory, known as Euler's formula, states that if G is a finite connected graph with n vertices, e edges and of genus g, then $n - e + f = 2 - 2g$, where f is the number of faces obtained when G is cellularly embedded in S_g.

Note that the zero divisor graph $\Gamma(R)$ is a subgraph of $EG(R)$. In [11] it has been shown that for any reduced ring R, $EG(R)$ is identical to $\Gamma(R)$. Using this result, one can establish that for any reduced ring, $EG(R)$ is complete if and only if $\Gamma(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. If $G = K_{1,n}$ where $n \geq 1$, then G is a star graph. A *split graph* is a simple graph in which the vertices can be partitioned into a clique and an independent set. A graph G is said to be *unicyclic* if it contains a unique cycle. An undirected graph is an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. An edge $e = uv$ of G is said to be contracted if it is deleted and its ends are identified and is denoted by $[u, v]$.

Throughout this paper, we assume that R is a finite commutative ring with identity, $Z(R)$ its set of zero-divisors and $\mathrm{Nil}(R)$ its set of nilpotent elements, $R^{\times}$ its group of units, $\mathbb{F}_q$ denote the field with q elements, and $R^* = R - \{0\}$. For every ideal I of R, we denote the *annihilator* of I by $\mathrm{ann}(I)$. The following results are useful in the subsequent sections.

Theorem 1.1. [1, Theorem 2.10] *Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with char $R = p$ or p^2, and $|\Gamma(R)| = p^n - 1$, where p is prime and $n \geq 1$.*

Theorem 1.2. [1, Theorem 2.13] *Let R be a finite commutative ring with $|\Gamma(R)| \geq 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a finite field.*

Theorem 1.3. [11, Theorem 2.2] *Let R be a reduced ring. Then $EG(R) = \Gamma(R)$.*

Theorem 1.4. [11, Lemma 3.1] *Let R be a non reduced commutative ring. Then the following statements hold.*
(i) For every $x \in Nil(R)^*$, x is adjacent to all other vertices.
(ii) $EG(R)[Nil(R)^*]$ is a (induced) complete subgraph of $EG(R)$.

In view of Theorem 1.4, if R is a local ring, then $EG(R)$ is complete.

$\lvert Z(R)^*\rvert$	Local Ring R	$EG(R)$
1	$\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$	K_1
2	$\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$	K_2
3	$\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$	K_3
	$\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$	K_3
4	$\mathbb{Z}_{25}$, $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$	K_4
6	$\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}$	K_6
7	$\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^4,x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3-2,x^4\rangle}$	K_7
	$\frac{\mathbb{Z}_4[x]}{\langle x^4,x^3+x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x]}{\langle x^3,x^2-2x\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle}$	K_7
	$\frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,xy,x^2-2,y^2-2,y^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}$	K_7
	$\frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2,xy\rangle}$	K_7
	$\frac{\mathbb{Z}_4[x]}{\langle x^3,2x\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_8[x]}{\langle x^2,2x\rangle}$, $\frac{\mathbb{F}_8[x]}{\langle x^2\rangle}$	K_7
	$\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}$, $\frac{\mathbb{Z}_4[x,y]}{\langle 2x,2y,x^2,y^2,xy\rangle}$, $\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}$	K_7

Table 1.1

2 Basic Properties of an Essential Graph

In this section, we study some fundamental properties of the essential graph. Especially we identify when the essential graph is isomorphic to some well-known graphs.

Remark 2.1. Note that $\Gamma(R)$ is a subgraph of $EG(R)$. Then by Theorems 1.1 and 1.4, R is a local ring or $\mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if $EG(R)$ is complete. We list in Table 1.1, some small commutative rings R for which $EG(R)$ is complete.

Remark 2.2. Let R be a reduced ring. Then $EG(R)$ is a subgraph of $AG(R)$.

Theorem 2.3. *Let R be a finite commutative ring with identity but not a field. Then $EG(R)$ is a tree if and only if R is isomorphic to one of the following rings. $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$, or $\mathbb{Z}_2 \times F$, where F is a finite field.*

Proof. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. Suppose $EG(R)$ is a tree. Suppose $n \geq 3$. Then $(1,0,\ldots,0)-(0,1,0,\ldots,0)-(0,0,1,0,\ldots,0)$ $-$ $(1,0,\ldots,0)$ is a cycle in $EG(R)$, a contradiction. Hence $n \leq 2$.

Suppose $R \cong R_1 \times R_2$. If R_1 is local with $\mathfrak{m}_1 \neq \{0\}$, then there exists $x_1 \in \mathfrak{m}_1^*$ such that $\mathrm{ann}(x_1) = \mathfrak{m}_1$. Let $x = (0,1)$, $y = (x_1,0)$ $z = (x_1,1)$ and $w = (1,0) \in Z(R)^*$. Then $x - y - z - w - x$ is a cycle in $EG(R)$, a contradiction. Hence R_1 and R_2 are fields and so $EG(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $EG(R)$ is tree, $|R_1| = 2$ or $|R_2| = 2$ and so $R \cong \mathbb{Z}_2 \times F$, where F is a field.

Suppose $R \cong R_1$. Since R is not a field, $Z(R) \neq 0$ and so $EG(R)$ is complete. Since $EG(R)$ is a tree, we have $|Z(R)^*| \leq 2$. Hence $R \cong \mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, $\mathbb{Z}_9$, or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$.

Converse follows from Table 1.1 and Theorem 1.2. □

Theorem 2.4. *Let R be a finite commutative ring with identity but not a field. Then $EG(R)$ is unicyclic if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $EG(R)$ is unicyclic. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. Suppose $n \geq 4$. Let $x_1 = (1,0,0,\ldots,0)$, $x_2 = (0,1,0,\ldots,0)$, $x_3 = (0,0,1,0,\ldots,0)$, $x_4 = (0,0,0,1,0,\ldots,0)$, $y_1 = (1,1,0,0,\ldots,0) \in Z(R)^*$. Then $x_1 - x_2 - x_3 - x_1$ as well as $x_3 - y_1 - x_4 - x_3$ are two distinct cycles in $EG(R)$, a contradiction. Hence $n \leq 3$.

Case 1. Suppose $n = 3$. Suppose $|R_1| \geq 3$. Then $(1,0,0)-(0,1,0)-(0,0,1)-(1,0,0)$ and $(a,0,0)-(0,1,0)-(0,0,1)-(a,0,0)$ are cycles in $EG(R)$ for some $1 \neq a \in R_1^*$, a contradiction. Hence $|R_i| = 2$ for all i and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Suppose $n = 2$. If $\mathfrak{m}_1 \neq \{0\}$, then there exists $x \in \mathfrak{m}_1^*$ such that $\mathrm{ann}(x) = \mathfrak{m}_1$. Then $(1,0)-(x,0)-(0,1)-(1,0)$ and $(u,0)-(x,0)-(0,1)-(u,0)$ are cycles in $EG(R)$ for some $1 \neq u \in R_1^{\times}$, a contradiction. Hence R_1 and R_2 are fields and so $EG(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $EG(R)$ is unicyclic, $R_1 \cong \mathbb{Z}_3$ and $R_2 \cong \mathbb{Z}_3$.

Case 3. Suppose $n = 1$. Now R is a local ring but not a field. Then $EG(R)$ is complete. Since $EG(R)$ is unicyclic, $|Z(R)^*| = 3$ and by Table 1.1, R is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2, xy, y^2 \rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle}$. □

Theorem 2.5. *Let G be a connected graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$, C_4, C_5.*

Theorem 2.6. *Let R be a finite commutative non-local ring with identity and $|Z(R)^*| \geq 2$. Then $EG(R)$ is a split graph if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a field or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Assume that $EG(R)$ is a split graph. By the assumption on R, $R \cong R_1 \times \cdots \times R_n$ where each R_i is local and $n \geq 2$. If $n \geq 4$, then $(1,1,0,\ldots,0)-(0,0,1,1,0,\ldots,0)$ and $(1,0,1,0\ldots,0)-(0,1,0,1,0,\ldots,0)$ induce $2K_2$ in $EG(R)$ and by Theorem 2.5, $EG(R)$ is not split, a contradiction. Hence $n \leq 3$.

Case 1. Suppose that $n=3$. If $|R_1| \geq 3$, then $(1,0,0)-(0,1,1)-(u,0,0)-(0,1,0)-(1,0,0)$ is a cycle of length 4 in $EG(R)$ for some $1 \neq u \in R_1^\times$, a contradiction. Hence $|R_i| = 2$ for all i and hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Suppose $n = 2$. If $\mathfrak{m}_1 \neq \{0\}$, then there exists $x \in \mathfrak{m}_1^*$ such that $\operatorname{ann}(x) = \mathfrak{m}_1$. Consider $\Omega = \{x_1, x_2, x_3, x_4\}$ where $x_1 = (1,0)$, $x_2 = (0,1)$, $x_3 = (v,0)$, $x_4 = (x,1)$, $1 \neq v \in R_1^\times$. Clearly $x_1x_2 = x_2x_3 = 0$. Also $\operatorname{ann}(x_1x_4) = \mathfrak{m}_1 \times R_2 = \operatorname{ann}(x_3x_4)$, which is essential. Hence $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle of length 4 in $EG(R)$, a contradiction. Thus R_1 and R_2 are fields and so $EG(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $EG(R)$ is split, $|R_1|-1 = 1$ or $|R_2|-1 = 1$ and so $R \cong \mathbb{Z}_2 \times F$ where F is a field. □

Theorem 2.7. *Let R be a finite commutative ring with identity. Then $EG(R)$ is outerplanar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3, x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_2 \times F$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, where F is a field or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Since R is finite, $R \cong R_1 \times \cdots \times R_n$, where each R_i is a local ring. Assume that $EG(R)$ is outerplanar. Suppose $n \geq 4$. Consider $\Omega = \{x_1, x_2, x_3, x_4\}$ where $x_1 = (1,0,0,\ldots,0)$, $x_2 = (0,1,0,\ldots,0)$, $x_3 = (0,0,1,0,\ldots,0)$, $x_4 = (0,0,0,1,0\ldots,0)$. Then the subgraph induced by Ω in $EG(R)$ is isomorphic to K_4, a contradiction. Hence $n \leq 3$.

Case 1. Assume that $n = 3$. Suppose $|R_1| \geq 3$. Consider $\Omega' = \{x_1, x_2, x_3, x_4, x_5\}$ where $x_1 = (1,0,0)$, $x_2 = (a,0,0)$, $x_3 = (0,1,0)$, $x_4 = (0,0,1)$, $x_5 = (0,1,1)$, $1 \neq a \in R_1^*$. Then the subgraph induced by Ω' in $EG(R)$ contains $K_{2,3}$ as a subgraph, a contradiction. Therefore $|R_i| = 2$ for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Assume that $n = 2$. If $\mathfrak{m}_1 \neq \{0\}$, then there exists $x \in \mathfrak{m}_1^*$ such that $\operatorname{ann}(x) = \mathfrak{m}_1$ for some $x \in R_1$. Consider the set $\Omega'' = \{y_1, y_2, y_3, y_4, y_5\}$ where $y_1 = (1,0)$, $y_2 = (x,0)$, $y_3 = (u,0)$, $y_4 = (x,1)$, $y_5 = (0,1)$, $1 \neq u \in R_1^\times$. Then the subgraph induced by Ω'' in $EG(R)$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence R_1 and R_2 are fields and so $EG(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $EG(R)$ is outerplanar, R is isomorphic to $\mathbb{Z}_2 \times F$ where F is a finite field or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Case 3. Suppose that $n = 1$. Since R is local, $EG(R)$ is complete. Since $EG(R)$ is outerplanar, $1 \leq |Z(R)^*| \leq 3$ and hence R is isomorphic to one of the following rings:

$$\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle},\ \mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle},\ \frac{\mathbb{F}_4[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle},\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}.$$

□

3 Planarity of $EG(R)$

In this section, we characterize all finite commutative rings R with identity whose essential graph $EG(R)$ is planar. The followings results are useful in this section.

Theorem 3.1. ([8], Kuratowski) *A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 3.2. [17, Theorem 3.5.1] *Let $(R,\mathfrak{m})$ be a finite local ring and $\Gamma(R)$ be the zero-divisor graph of R. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2\rangle},\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle x^2\rangle},\ \mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle},\ \frac{\mathbb{F}_4[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle},\ \mathbb{Z}_{25},\ \frac{\mathbb{Z}_5[x]}{\langle x^2\rangle},$$
$$\mathbb{Z}_{16},\ \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2-2,x^4\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3-2,x^4\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3+x^2-2,x^4\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^3,xy,y^2-x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2x\rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2-4,2x\rangle},$$
$$\frac{\mathbb{Z}_4[x,y]}{\langle x^3,x^2-2,xy,y^2-2,y^3\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle x^2,y^2,xy-2\rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^2,y^2\rangle},\ \mathbb{Z}_{27},\ \frac{\mathbb{Z}_3[x]}{\langle x^3\rangle},\ \frac{\mathbb{Z}_9[x]}{\langle x^2-3,x^3\rangle},\ \frac{\mathbb{Z}_9[x]}{\langle x^2+3,x^3\rangle}.$$

Theorem 3.3. [12, Theorem 3.7] *Let $R = F_1 \times \cdots \times F_n$ be a finite ring, where each F_i is a field and $n \geq 2$. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times F$, $\mathbb{Z}_3 \times F$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ where F is a finite field.*

Theorem 3.4. *Let $(R,\mathfrak{m})$ be a finite commutative local ring with identity. Then $EG(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$, $\mathbb{Z}_9$, $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle}$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{\langle x^3\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^3,x^2-2\rangle}$, $\frac{\mathbb{Z}_2[x,y]}{\langle x^2,xy,y^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle}$, $\frac{\mathbb{F}_4[x]}{\langle x^2\rangle}$, $\frac{\mathbb{Z}_4[x]}{\langle x^2+x+1\rangle}$, $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[x]}{\langle x^2\rangle}$.*

Proof. Since $EG(R)$ is complete, the proof follows from Theorem 3.1 and Table 1.1. □

Theorem 3.5. *Let $R = F_1 \times \cdots \times F_n$ be a finite ring, where each F_i is a field and $n \geq 2$. Then $EG(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times F$, $\mathbb{Z}_3 \times F$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ where F is a finite field.*

Proof. Since R is reduced and by Theorem 1.3, $\Gamma(R) = EG(R)$. Now the proof follows from Theorem 3.3. □

Note that if $R \cong R_1 \times \cdots \times R_n$ is a commutative ring with identity where each $(R_i,\mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and $n \geq 2$, then $K_{5,5}$ is a subgraph of $EG(R)$ and hence $\gamma(EG(R)) \geq 3$. Hence if $\gamma(EG(R)) = 0$ or 1, then one of the component R_i must be a field.

Theorem 3.6. *Let $R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring but not field, F_j is a field and $m, n \geq 1$. Then $EG(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_2$.*

Proof. Suppose $EG(R)$ is planar. Note that $|\mathfrak{m}_i^*| \geq 1$ and $|R_i^*| \geq 3$ for all i, $1 \leq i \leq n$. Suppose $n + m \geq 3$. Consider $\Omega = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, where $x_1 = (a_1, 0, \ldots, 0), x_2 = (a_2, 0, \ldots, 0), x_3 = (a_3, 0, \ldots, 0)$, $y_1 = (0, 1, 0, \ldots, 0)$, $y_2 = (0, 0, 1, 0, \ldots, 0)$, $y_3 = (0, 1, 1, 0 \ldots, 0) \in Z(R)^*$, where $a_i \in R_1^*$. Then the subgraph induced by Ω of $EG(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $n + m = 2$ and so $R \cong R_1 \times F_1$.

Suppose $|\mathfrak{m}_1^*| \geq 2$. Then $|R_1^\times| \geq 3$. Note that $|F_1| \geq 2$. Since R_1 is local, $\text{ann}(x) = \mathfrak{m}_1$ for some $x \in \mathfrak{m}_1^*$. Consider $\Omega' = \{a_1, a_2, a_3, b_1, b_2, b_3\}$, where $a_1 = (u_1, 0)$, $a_2 = (u_2, 0)$, $a_3 = (u_3, 0)$, $b_1 = (0, 1)$, $b_2 = (x, 0)$, $b_3 = (x, 1) \in Z(R)^*$, where u_1, u_2 and u_3 are distinct units in R_1. Then $a_i b_1 = 0$ in R for $i = 1, 2, 3$. Clearly $\text{ann}(a_i b_j) = \mathfrak{m}_1 \times F_1$, which is essential and so a_i is adjacent to b_j in $EG(R)$ for $i = 1, 2, 3$ and $j = 2, 3$. From this, we get $K_{3,3}$ is a subgraph of $\langle \Omega' \rangle$ in $EG(R)$, a contradiction. Hence $|\mathfrak{m}_1^*| = 1$ and so $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Suppose $|F_1| \geq 3$. Let $a \in \mathfrak{m}_1^*$ such that $a^2 = 0$. Consider $\Omega'' = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, where $x_1 = (1, 0), x_2 = (a, 0), x_3 = (u, 0), y_1 = (0, 1), y_2 = (0, v)$, $y_3 = (a, 1) \in Z(R)^*$, where $1 \neq u \in R_1^\times$ and $1 \neq v \in F_1^*$. Then the subgraph induced by Ω'' in $EG(R)$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $|F_1| = 2$ and so $F_1 \cong \mathbb{Z}_2$. □

4 Genus of $EG(R)$

In this section, we characterize all finite commutative rings R with identity whose essential graph $EG(R)$ is toroidal. The following results are useful in this section.

Lemma 4.1. [20] $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *if $n \geq 3$. In particular, $\gamma(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 4.2. [20] $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if $m, n \geq 2$. In particular, $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,3}) = 2$ if $m = 7, 8, 9, 10$.*

Theorem 4.3. [17, Theorem 3.5.2] *Let $(R, \mathfrak{m})$ be a finite local ring and $\Gamma(R)$ be the zero-divisor graph of R. Then $\gamma(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_{49},\ \frac{\mathbb{Z}_7[x]}{\langle x^2 \rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3, 2x \rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle x^3, x^2-2, xy, y^2 \rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2, 2x \rangle},\ \frac{\mathbb{F}_8[x]}{\langle x^2 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3+x+1 \rangle},\ \frac{\mathbb{Z}_4[x,y]}{\langle 2x, 2y, x^2, xy, y^2 \rangle},$$
$$\frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z \rangle^2},\ \mathbb{Z}_{32},\ \frac{\mathbb{Z}_2[x]}{\langle x^5 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^3-2, x^5 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^4-2, x^5 \rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2-2, x^5 \rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2-2x+2, x^5 \rangle},\ \frac{\mathbb{Z}_8[x]}{\langle x^2+2x-2, x^5 \rangle}.$$

Theorem 4.4. [19, Theorem 3.1] *Let $R = F_1 \times \cdots F_n$ where each F_i is a finite field. Then $\gamma(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings:*

$\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem 4.5. *Let $(R, \mathfrak{m})$ be a finite commutative local ring with identity. Then $\gamma(EG(R)) = 1$ if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_{16}, \frac{\mathbb{Z}_2[x]}{\langle x^4\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^4, x^2-2\rangle}, \frac{\mathbb{Z}_2[x]}{\langle x^3-2, x^4\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^4, x^3+x^2-2\rangle}, \frac{\mathbb{Z}_2[x]}{\langle x^3, x^2-2x\rangle}, \frac{\mathbb{Z}_2[x,y]}{\langle x^3, xy, y^2-x^2\rangle}, \frac{\mathbb{Z}_8[x]}{\langle x^2-4, 2x\rangle},$$
$$\frac{\mathbb{Z}_4[x,y]}{\langle x^3, xy, x^2-2, y^2-2, y^3\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^2\rangle}, \frac{\mathbb{Z}_4[x,y]}{\langle x^2, y^2, xy-2\rangle}, \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2\rangle}, \frac{\mathbb{Z}_2[x,y]}{\langle x^2, y^2, xy\rangle}, \frac{\mathbb{Z}_4[x]}{\langle x^3, 2x\rangle}, \frac{\mathbb{Z}_4[x,y]}{\langle x^3, x^2-2, xy, y^2\rangle}, \frac{\mathbb{Z}_8[x]}{\langle x^2, 2x\rangle}, \frac{\mathbb{F}_8[x]}{\langle x^2\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle x^3+x+1\rangle}, \frac{\mathbb{Z}_4[x,y]}{\langle 2x, 2y, x^2, y^2, xy\rangle}, \frac{\mathbb{Z}_2[x,y,z]}{\langle x,y,z\rangle^2}, \mathbb{Z}_{49} \text{ or } \frac{\mathbb{Z}_7[x]}{\langle x^2\rangle}.$$

Proof. Since $EG(R)$ is complete, by Lemma 4.1, $5 \leq |Z(R)^*| \leq 7$. Now the proof follows from Table 1.1. □

Theorem 4.6. *Let $R = F_1 \times \cdots \times F_n$ be a finite ring, where each F_i is a field and $n \geq 2$. Then $\gamma(EG(R)) = 1$ if and only if R is isomorphic to one of the following rings:* $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since R is a reduced ring, by Theorem 1.3, $EG(R) = \Gamma(R)$. Then the proof completes by Theorem 4.4. □

Theorem 4.7. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$, F_j is a field and $m, n \geq 1$. Then $\gamma(EG(R)) = 1$ if and only if R is isomorphic to* $\mathbb{Z}_4 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{F}_4$.

Proof. By Fig. 4.2 and Fig. 4.3, $\gamma(EG(R)) = 1$ where R is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{F}_4$, or $\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle} \times \mathbb{F}_4$.

Assume $\gamma(EG(R)) = 1$. Suppose $n + m \geq 3$. Since R_1 is local, there exists $a \in \mathfrak{m}_1^*$ with $\mathrm{ann}(a) = \mathfrak{m}_1$. Consider $\Omega = \{x_1, \ldots, x_{10}\}$, where $x_1 = (1, 0, \ldots, 0), x_2 = (a, 0, \ldots, 0), x_3 = (u, 0, \ldots, 0)$, $x_4 = (0, 1, 0, \ldots, 0), x_5 = (0, 0, 1, 0, \ldots, 0)$, $x_6 = (0, 1, 1, 0, \ldots, 0)$, $x_7 = (a, 1, 0, \ldots, 0), x_8 = (a, 0, 1, 0 \ldots, 0), x_9 = (a, 1, 1, 0, \ldots, 0)$, $x_{10} = (1, 0, 1, 0, \ldots, 0)$, $1 \neq u \in R_1^\times$. Consider $G_1 = \langle\Omega\rangle$. Then G_1 is a subgraph of $EG(R)$ and G_1 contains G' in Fig. 4.1 as a subgraph. By Fig. 4.1, $K_{3,6}$ is a subgraph of G' and hence by Theorem 4.2, $\gamma(G') \geq 1$.

Assume that $\gamma(G') = 1$. Fix an embedding of $K_{3,6}$ on the surface of torus. By Euler's formula, there are 9 faces in the embedding of $K_{3,6}$, say $\{F'_1, \ldots, F'_9\}$. Let f_i be the length of the face F'_i. Note that $\sum_{i=1}^{9} f_i = 2e = 36$ and $f_i \geq 4$ for every i. Thus $f_i = 4$ for every i. Let $S = \{x_7, x_8, [x_4, x_{10}]\} \subset V(G')$. Further, the subgraph H of G' induced by the vertices in S is K_3, $E(H) \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangles as faces, one cannot have an embedding of H and $K_{3,6}$ together in the torus. This implies that $\gamma(G') \geq 2$. Since G' is a subgraph of $EG(R)$, $\gamma(EG(R)) \geq 2$. Hence $n + m = 2$, $R \cong R_1 \times F_1$ and by Theorem 3.6, $R \ncong Z_4 \times \mathbb{Z}_2$ and $\frac{\mathbb{Z}_3[x]}{\langle x^2\rangle} \times \mathbb{Z}_2$.

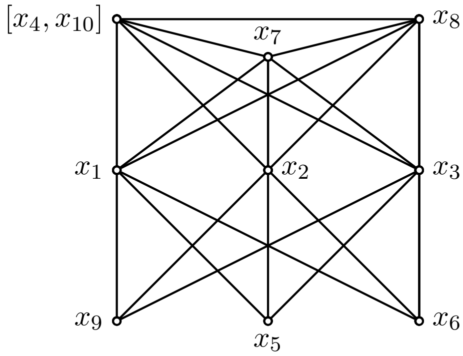

Fig. 4.1 Graph G'

Now we claim that if $|\mathfrak{m}_1^*| \geq 2$, then $\gamma(EG(R)) \geq 2$. Suppose $|\mathfrak{m}_1^*| = 2$. Then $R_1 \cong \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle}$ and hence $|R_1^{\times}| = 6$. Let $a, b \in \mathfrak{m}_1^*$ such that $ab = 0$ and $\mathrm{ann}(a) = \mathfrak{m}_1$. Consider $\Omega' = \{x_1, \ldots, x_7, y_1, y_2, y_3\} \subseteq Z(R)^*$ where $x_1 = (u_1, 0)$, $x_2 = (u_2, 0)$, $x_3 = (u_3, 0)$, $x_4 = (u_4, 0)$, $x_5 = (u_5, 0)$, $x_6 = (u_6, 0)$, $x_7 = (b, 0)$, $y_1 = (0, 1)$, $y_2 = (a, 0)$, $y_3 = (a, 1)$. Then the subgraph induced by Ω' of $EG(R)$ contains $K_{3,7}$ as a subgraph and by Lemma 4.2, $\gamma(EG(R)) \geq 2$, a contradiction.

Suppose $|\mathfrak{m}_1^*| \geq 3$. Then $|R_1^{\times}| \geq 4$. Let $x, y, z \in \mathfrak{m}_1^*$ such that $xy = yz = 0$ and $\mathrm{ann}(y) = \mathfrak{m}_1$ and let $\{v_1, \ldots, v_4\} \subseteq R_1^{\times}$. Consider $\Omega'' = \{a_1, \ldots, a_5, b_1, \ldots, b_4\} \subseteq Z(R)^*$, where $a_1 = (v_1, 0)$, $a_2 = (v_2, 0)$, $a_3 = (v_3, 0)$, $a_4 = (v_4, 0)$, $a_5 = (x, 0)$, $b_1 = (0, 1)$, $b_2 = (y, 0)$, $b_3 = (y, 1)$, $b_4 = (x, 0)$, $b_5 = (x, 1)$. Then $a_i b_1 = 0$ for $1 \leq i \leq 5$ and $a_5 b_2 = a_5 b_3 = 0$. Since $b_4 \in \mathrm{Nil}(R)^*$, by Theorem 1.4, $\mathrm{ann}(a_i b_4) = (y) \times F_1$ is essential for $1 \leq i \leq 5$. Also $\mathrm{ann}(a_i b_2) = \mathfrak{m}_1 \times F_1 = \mathrm{ann}(a_i b_3)$ for $1 \leq i \leq 4$. Then the subgraph induced by Ω'' of $EG(R)$ contains $K_{4,5}$ as a subgraph and by Lemma 4.2, $\gamma(EG(R)) \geq 2$, a contradiction.

Hence we conclude that $|\mathfrak{m}_1^*| = 1$ and so $R_1 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. By Theorem 3.6, $F_1 \ncong \mathbb{Z}_2$ and so $|F_1| \geq 3$. Suppose $|F_1| \geq 5$. Let $a \in \mathfrak{m}_1^*$ with $a^2 = 0$. Consider $S = \{x_1, \ldots, x_{11}\} \subset Z(R)^*$, where $x_1 = (u_1, 0)$, $x_2 = (u_2, 0)$, $x_3 = (a, 0)$, $x_4 = (0, v_1)$, $x_5 = (0, v_2)$, $x_6 = (0, v_3)$, $x_7 = (0, v_4)$, $x_8 = (a, v_1)$, $x_9 = (a, v_2)$, $x_{10} = (a, v_3)$, $x_{11} = (a, v_4)$ and $u_1, u_2 \in R_1^{\times}$, $v_j \in F_1^*$. Then the subgraph induced by S of $EG(R)$ contains $K_{3,8}$ as a subgraph and by Lemma 4.2, $\gamma(EG(R)) \geq 2$, a contradiction. Hence F_1 is isomorphic to either $\mathbb{Z}_3$ or $\mathbb{F}_4$.

□

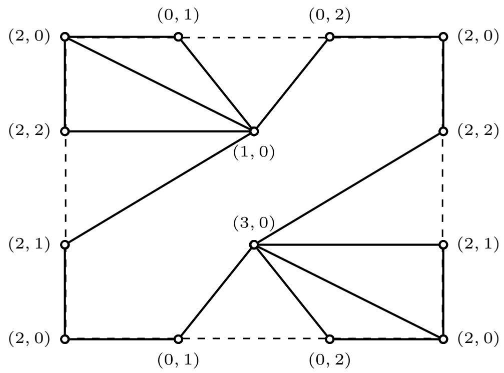

Fig. 4.2 : $EG(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong EG\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3\right)$ in S_1

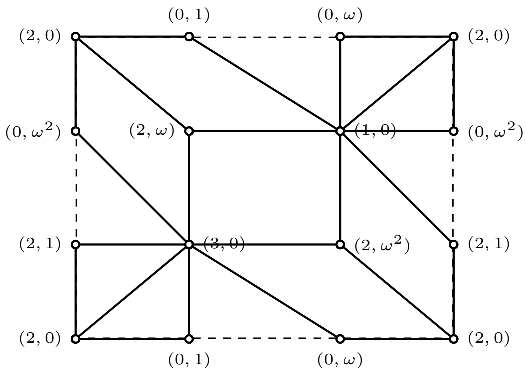

Fig. 4.3 : $EG(\mathbb{Z}_4 \times \mathbb{F}_4) \cong EG\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{F}_4\right)$ in S_1

5 Crosscap of $EG(R)$

In this section, we characterize all finite commutative rings R with identity whose essential graph $EG(R)$ is projective. The following result is very useful for further reference in this section.

Theorem 5.1. *[7] Let m, n be integers and for a real number x, $\lceil x \rceil$ is the least integer that is greater than or equal to x. Then*
(i) $\overline{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geq 3 \text{ and } n \neq 7; \\ 3 & \text{if } n = 7 \end{cases}$
(ii) $\overline{\gamma}(K_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$, *where* $m, n \geq 2$.

Theorem 5.2. *[9] Let $R = F_1 \times \cdots \times F_n$, where each F_i is finite field. Then $\overline{\gamma}(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 5.3. *Let R be a finite local ring with identity. Then $\overline{\gamma}(EG(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_7[x]}{\langle x^2 \rangle}$.*

Proof. Since $EG(R)$ is complete, proof follows from Theorem 5.1. □

Theorem 5.4. *Let $R = F_1 \times \cdots \times F_n$, where each F_i is finite field. Then $\overline{\gamma}(EG(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. The proof follows from Theorems 1.3 and 5.2. □

By slight modifications in the proof of Theorem 3.6 with Lemma 5.1 and Fig. 5.1, one can prove the following theorem.

Theorem 5.5. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$, where each $(R_i, \mathfrak{m}_i)$ is a local ring and F_j is finite field and $n, m \geq 1$. Then $\overline{\gamma}(EG(R)) = 1$ if and only if R is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3$.*

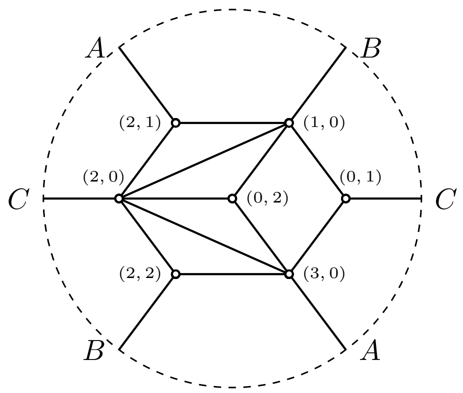

Fig. 5.1 Embedding of $EG(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong EG\left(\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times \mathbb{Z}_3\right)$ in $\bar{S}_1$

Acknowledgments

The authors would like to thank the referees for careful reading of the manuscript and helpful comments. The work reported here is supported by the INSPIRE programme (IF 140700) of the Department of Science and Technology, Government of India, for the third author.

References

[1] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434–447.

[2] S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph, *J. Algebra* **270** (2003), 169–180.

[3] A. Badawi, On the annihilator graph of a commutative ring, *Comm. Algebra* **42** (2014), 108–121.

[4] J. Battle, F. Harary, Y. Kodama and J. W. T. Youngs, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962), 565–568.

[5] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208–226.

[6] R. Belshoff, and J. Chapman, Planar zero-divisor graphs, *J. Algebra* **316** (1) (2007), 471–480.

[7] A. Bouchet, Orientable and non-orientable genus of the complete bipartite graph, *J. Combin. Theory, Ser. B* **24** (1) (1978), 24–33.

[8] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Monterey: Wadsworth and Brooks/Cole, 1986.

[9] H. J. Chiang-Hsieh, Classification of rings with projective zero-divisor graphs, *J. Algebra* **307** (2008), 2789–2802.

[10] W. Massey, *Algebraic Topology: An Introduction*, New York: Harcourt, Brace & World, Inc, 1967.

[11] M. J. Nikmehr, R. Nikandish and M. Bakhtyiaria, On the Essential Graph of a Commutative Ring, *J. Algebra Appl.* **16** (7) (2017), 1750132 (14 pp.).

[12] N. O. Smith, Planar zero-divisor graphs, *Int. J. Comm. Rings* **2** (4) (2002), 177–188.

[13] K. Selvakumar and V. Ramanathan, Classification of nonlocal rings with genus one 3-zero-divisor hypergraphs, *Comm. Algebra* **45** (1) (2017), 275–284.

[14] K. Selvakumar and M. Subajini, Commutative rings with genus two annihilator graphs, *Comm. Algebra* **46** (1) (2018), 28–37.

[15] T. Tamizh Chelvam, K. Selvakumar and V. Ramanathan, On the planarity of the k-zero-divisor hypergraphs, *AKCE Int. J. Graphs Combin.* **12** (2) (2015), 169–176.

[16] T. Tamizh Chelvam and K. Selvakumar, On the genus of the annihilator graph of a commutative ring, *Algebra* & *Discrete Math.* **24** (2) (2017), 191–208.

[17] H. J. Wang, Zero-divisor graphs of genus one, *J. Algebra* **304** (2) (2006), 666–678.

[18] H. J. Wang and N. O. Smith, Commutative rings with toroidal zero-divisor graphs, *Houston J. Math.* **36** (1) (2010), 1–31.

[19] C. Wickham, Classification of rings with genus one zero-divisor graphs, *Comm. Algebra* **36** (2008), 325–345.

[20] A. T. White, *Graphs, Groups and Surfaces*, Amsterdam: North-Holland, 1973.

(Received 11 Mar 2018; revised 28 Dec 2018)