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(g, f)-Chromatic spanning trees and forests

Kazuhiro Suzuki*

Department of Information Science
Kochi University
Japan
kazuhiro@tutetuti.jp

Abstract

A rainbow graph is an edge-colored graph whose edges have distinct colors, that is, where each color appears on at most one edge. Akbari and Alipour (2007), and Suzuki (2006), independently presented a necessary and sufficient condition for an edge-colored graph to have a rainbow spanning tree. In this paper, we define a (g, f)-chromatic graph as an edge-colored graph where each color c appears on at least $g(c)$ edges and at most $f(c)$ edges. We also present a necessary and sufficient condition for an edge-colored graph to have a (g, f)-chromatic spanning tree. Using this criterion, we can show that an edge-colored complete graph K_n has a spanning tree with a color probability distribution "similar" to that of K_n. Finally, we conjecture that an edge-colored complete graph K_{2n} ($n \geq 3$) can be partitioned into n edge-disjoint spanning trees such that each has a color probability distribution "similar" to that of K_{2n}. This conjecture is a generalization of the conjecture by Brualdi and Hollingsworth (1996).

1 Introduction

We consider finite undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. An *edge-coloring* of a graph G is a mapping $color : E(G) \to \mathbb{C}$, where $\mathbb{C}$ is a set of colors. Then, the triple $(G, \mathbb{C}, color)$ is called an *edge-colored graph*. We often abbreviate an edge-colored graph $(G, \mathbb{C}, color)$ as G. Note that an edge colored graph is not necessarily proper: some edges colored with the same color may have a common end vertex.

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1.1 Rainbow spanning trees

An edge-colored graph G is said to be *rainbow*[1] if no two edges of G have the same color, that is, $color(e_i) \neq color(e_j)$ for any two distinct edges e_i and e_j of G. As far as I know, there are three topics about rainbow graphs: the Anti-Ramsey problem introduced by Erdős et al. [5], rainbow connection problems introduced by Chartrand et al. [4], and rainbow subgraph problems (see the surveys [8] [13] [11]). This paper focuses on rainbow subgraph problems.

We denote by $\omega(G)$ the number of components of a graph G. Given an edge-colored graph G and a color set R, we define $E_R(G) = \{e \in E(G) \mid color(e) \in R\}$. For simplicity, we denote the graph $(V(G), E(G) \setminus E_R(G))$ by $G - E_R(G)$, and also denote $E_{\{c\}}(G)$ by $E_c(G)$ for a color c.

Akbari and Alipour [1], and Suzuki [15], independently presented the following necessary and sufficient condition for an edge-colored graph to have a rainbow spanning tree.

Theorem 1.1 (Akbari and Alipour [1], Suzuki [15]). *An edge-colored graph G has a rainbow spanning tree if and only if*

$$\omega(G - E_R(G)) \leq |R| + 1 \qquad \textit{for any } R \subseteq \mathbb{C}.$$

Suzuki [15] proved the following theorem by using Theorem 1.1.

Theorem 1.2 (Suzuki [15]). *An edge-colored complete graph K_n has a rainbow spanning tree if $|E_c(K_n)| \leq n/2$ for any color $c \in \mathbb{C}$.*

The complete graph K_n has $(n-1)n/2$ edges, and thus the condition of Theorem 1.2 is equivalent to

$$\frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \leq 1 \qquad \text{for any color } c \in \mathbb{C}.$$

We can regard $|E_c(K_n)|/|E(K_n)|$ as the probability of a color c appearing in K_n. The term "rainbow" means that each color appears on one or zero edges. Thus, we can interpret Theorem 1.2 as saying that if each color probability is at most $1/(n-1)$ in K_n then K_n has a spanning tree T such that each color probability is $1/(n-1)$ or 0 in T.

1.2 f-Chromatic spanning trees and forests

The term "rainbow" means that each color appears on at most one edge. Suzuki [16] generalized "one" to a mapping f from a given color set $\mathbb{C}$ to the set $\mathbb{Z}_{\geq 0}$ of non-negative integers, and defined f*-chromatic* graphs as follows.

[1] A rainbow graph is also said to be *heterochromatic*, *multicolored*, *totally multicolored*, *polychromatic*, or *colorful*, and so on.

Definition 1.3 (Suzuki [16]). *Let G be an edge-colored graph. Let f be a mapping from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Then G is said to be f-chromatic if $|E_c(G)| \leq f(c)$ for any color $c \in \mathbb{C}$.*

Suzuki [16] presented the following necessary and sufficient condition for an edge-colored graph to have an f-chromatic spanning forest with exactly m components.

Theorem 1.4 (Suzuki [16]). *Let G be an edge-colored graph of order n. Let f be a mapping from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Let m be a positive integer such that $n \geq m$. Then G has an f-chromatic spanning forest with exactly m components if and only if*

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} f(c) \qquad \text{for any } R \subseteq \mathbb{C}.$$

Suzuki [16] proved the following Theorem by using Theorem 1.4.

Theorem 1.5 (Suzuki [16]). *Let G be an edge-colored graph of order n. Let f be a mapping from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Let m be a positive integer such that $n \geq m$. If $|E(G)| > \binom{n-m}{2}$ and*

$$\frac{|E_c(G)|}{|E(G)|}(n - m) \leq f(c) \qquad \text{for any color } c \in \mathbb{C},$$

then G has an f-chromatic spanning forest with exactly m components.

A rainbow graph is an f-chromatic graph with $f(c) = 1$ for every color c. Thus, these two theorems include Theorems 1.1 and 1.2. In this paper we will further generalize these theorems and study color probability distributions of edge-colored complete graphs and their spanning trees.

2 Main results

We begin with the definition of a (g, f)-*chromatic* graph.

Definition 2.1. *Let G be an edge-colored graph. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Then G is said to be (g, f)-chromatic if $g(c) \leq |E_c(G)| \leq f(c)$ for any color $c \in \mathbb{C}$.*

Fig. 1 shows a (g, f)-chromatic spanning tree of an edge-colored graph. For the color set $\mathbb{C} = \{1, 2, 3, 4, 5, 6, 7\}$, mappings g and f are given as follows:

$$g(1) = 1, g(2) = 1, g(3) = 2, g(4) = 0, g(5) = 0, g(6) = 1, g(7) = 0,$$
$$f(1) = 3, f(2) = 2, f(3) = 3, f(4) = 0, f(5) = 0, f(6) = 1, f(7) = 2.$$

The left edge-colored graph has the right (g, f)-chromatic spanning tree, where each color c appears on at least $g(c)$ edges and at most $f(c)$ edges.

We will see more examples. First, we suppose that g and f are given as follows:

$$g(1) = 3, g(2) = 1, g(3) = 3, g(4) = 0, g(5) = 0, g(6) = 1, g(7) = 2,$$
$$f(1) = 3, f(2) = 2, f(3) = 3, f(4) = 0, f(5) = 0, f(6) = 1, f(7) = 2.$$

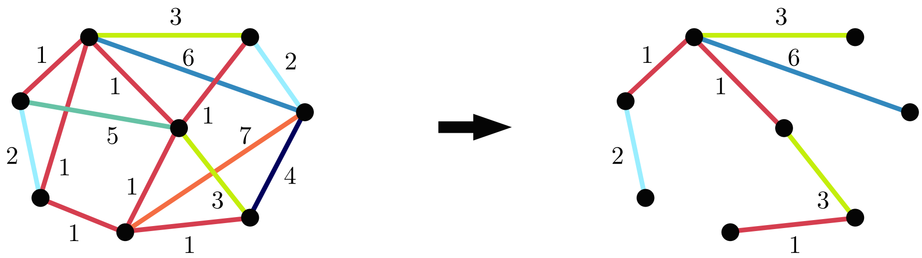

Fig. 1: A (g, f)-chromatic spanning tree of an edge-colored graph.

Then, the left edge-colored graph in Fig. 1 has no (g, f)-chromatic spanning trees, because $g(1) + g(2) + \cdots + g(7)$ exceeds 7, the size of a spanning tree of the graph.

Next, in Fig. 2, we suppose that g and f are given as follows:

$$g(1) = 0, g(2) = 2, g(3) = 2, g(4) = 0, g(5) = 0, g(6) = 1, g(7) = 0,$$
$$f(1) = 3, f(2) = 2, f(3) = 3, f(4) = 0, f(5) = 0, f(6) = 1, f(7) = 2.$$

Then, in the left edge-colored graph, any subgraph having $g(2)$, $g(3)$, and $g(6)$ edges colored with 2, 3, and 6, respectively, contains the right subgraph, which has a cycle. Thus, the left graph has no (g, f)-chromatic spanning trees.

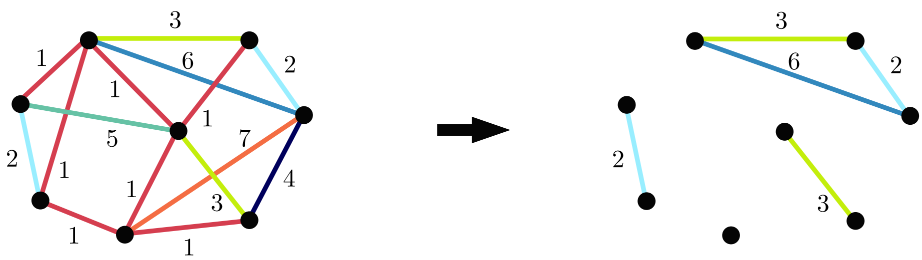

Fig. 2: The mapping g forces us to use a cycle.

The following is the main theorem, which gives a necessary and sufficient condition for an edge-colored graph to have a (g, f)-chromatic spanning tree as a corollary.

Theorem 2.2. *Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$ such that $g(c) \leq f(c)$ for any $c \in \mathbb{C}$. Let m be a positive integer such that $n \geq m + \sum_{c \in \mathbb{C}} g(c)$. Then G has a (g, f)-chromatic spanning forest with exactly m components if and only if*

$$\omega(G - E_R(G)) \leq \min\{\ m + \sum_{c \in R} f(c),\quad n - \sum_{c \in \mathbb{C} \setminus R} g(c)\ \} \qquad \text{for any } R \subseteq \mathbb{C}.$$

This theorem is proved in Section 3.2. Note that the size of a spanning forest with exactly m components of G is $n-m$. If G has a (g,f)-chromatic spanning forest with exactly m components, then the size of the forest is at least $\sum_{c\in\mathbb{C}} g(c)$. Thus, the condition $n \geq m + \sum_{c\in R} g(c)$ is necessary.

We see the above last example again. Let G be the left graph in Fig. 2. G has no (g,f)-chromatic spanning trees. Thus, by Theorem 2.2,

$$\omega(G - E_R(G)) > \min\{\ 1 + \sum_{c\in R} f(c),\quad 8 - \sum_{c\in\mathbb{C}\setminus R} g(c)\ \} \qquad \text{for some } R \subseteq \mathbb{C}.$$

Actually, for $R = \{1,4,5,7\}$, $G - E_R(G)$ is the right graph in Fig. 2 and we have

$$\omega(G - E_R(G)) = 4,\quad 1 + \sum_{c\in R} f(c) = 6,\quad 8 - \sum_{c\in\mathbb{C}\setminus R} g(c) = 3.$$

We can prove the following theorem by using Theorem 2.2.

Theorem 2.3. *Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Let m be a positive integer such that $n \geq m$. If $|E(G)| > \binom{n-m}{2}$ and*

$$g(c) \leq \frac{|E_c(G)|}{|E(G)|}(n-m) \leq f(c) \qquad \textit{for any color } c \in \mathbb{C},$$

then G has a (g,f)-chromatic spanning forest with exactly m components.

This theorem is proved in Section 3.3. Note that an f-chromatic graph is a (g,f)-chromatic graph with $g(c) = 0$ for any color c, and $\omega(G - E_R(G)) \leq n$ for any $R \subseteq \mathbb{C}$ since the number of components of any subgraph of a graph of order n is at most n. Thus, Theorem 2.2 and 2.3 include Theorem 1.4 and 1.5.

Let K_n be an edge-colored complete graph of order n, and set

$$g(c) = \left\lfloor \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rfloor \quad \text{and} \quad f(c) = \left\lceil \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rceil \quad \text{for any color } c \in \mathbb{C}.$$

Then, by Theorem 2.3, K_n has a (g,f)-chromatic spanning tree T. By the definition 2.1, $g(c) \leq |E_c(T)| \leq f(c)$ for any color $c \in \mathbb{C}$. Thus, the following theorem holds.

Theorem 2.4. *Any edge-colored complete graph K_n has a spanning tree T such that*

$$\left\lfloor \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rfloor \leq |E_c(T)| \leq \left\lceil \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rceil \quad \textit{for any color } c \in \mathbb{C}.$$

We call $|E_c(G)|/|E(G)|$ the *color probability* of a color c in an edge-colored graph G. The *color probability distribution* of G is the sequence of the color probabilities. Since $|E(T)| = n-1$, Theorem 2.4 implies that an edge-colored complete graph K_n has a spanning tree T such that $|E_c(T)|/|E(T)|$ is about $|E_c(K_n)|/|E(K_n)|$. Then the color probability distribution of T is said to be *similar* to that of K_n.

From Theorem 2.4, we can get the following theorem, proved in Section 3.4.

Theorem 2.5. *An edge-colored complete graph K_n has a spanning tree with the same color probability distribution as that of K_n if and only if $|E_c(K_n)|$ is an integral multiple of $n/2$ for any color $c \in \mathbb{C}$.*

In Section 4, we will give a conjecture for a spanning tree decomposition of an edge-colored complete graph.

3 Proofs

In this section, we prove Theorem 2.2, Theorem 2.3, and Theorem 2.5. In order to prove Theorem 2.2, we first state and prove two lemmas.

3.1 Lemmas

Lemma 3.1. *Let G be an edge-colored graph of order n. Let g be a mapping from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Then G has a (g,g)-chromatic forest if and only if*

$$\omega(G - E_R(G)) \leq n - \sum_{c \in \mathbb{C}\setminus R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Note that this lemma requires the forest neither to be a spanning forest nor to have a fixed number of components.

Proof. Let G be an edge-colored graph of order n. Let g be a mapping from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$.

First, we prove the necessity. Suppose that G has a (g,g)-chromatic forest F. By Definition 2.1, $|E_c(F)| = g(c)$ for any color c. For any $R \subseteq \mathbb{C}$, the graph $(V(G), E_{\mathbb{C}\setminus R}(F))$ is a spanning forest of $G - E_R(G)$. Thus,

$$\begin{aligned}\omega(G - E_R(G)) &\leq \omega((V(G), E_{\mathbb{C}\setminus R}(F))) \\ &= |V(G)| - |E_{\mathbb{C}\setminus R}(F)| \\ &= |V(G)| - \sum_{c \in \mathbb{C}\setminus R} |E_c(F)| \\ &= n - \sum_{c \in \mathbb{C}\setminus R} g(c).\end{aligned}$$

Next, we prove the sufficiency. Suppose that

$$\omega(G - E_R(G)) \leq n - \sum_{c \in \mathbb{C}\setminus R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Set $m = n - \sum_{c \in \mathbb{C}} g(c)$. Then,

$$n - m = \sum_{c \in \mathbb{C}} g(c) = \sum_{c \in R} g(c) + \sum_{c \in \mathbb{C}\setminus R} g(c) \quad \text{for any } R \subseteq \mathbb{C},$$

that is,

$$n - \sum_{c \in \mathbb{C} \setminus R} g(c) = m + \sum_{c \in R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Thus, we have

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Hence, by Theorem 1.4, G has a g-chromatic spanning forest F with exactly m components. By Definition 1.3, $|E_c(F)| \leq g(c)$ for any color $c \in \mathbb{C}$. On the other hand, we have

$$\sum_{c \in \mathbb{C}} |E_c(F)| = |E(F)| = n - m = n - (n - \sum_{c \in \mathbb{C}} g(c)) = \sum_{c \in \mathbb{C}} g(c).$$

Thus $|E_c(F)| = g(c)$ for any color $c \in \mathbb{C}$. Therefore, by Definition 2.1, F is a (g, g)-chromatic forest of G. □

Lemma 3.2. *Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$ such that $g(c) \leq f(c)$ for any $c \in \mathbb{C}$. Let m be a positive integer. Then G has a (g, f)-chromatic spanning forest with exactly m components if and only if G has both an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components, and a (g, g)-chromatic forest.*

Note that the f-chromatic spanning forest and the (g, g)-chromatic forest may be different in this Lemma.

Proof. Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$ such that $g(c) \leq f(c)$ for any $c \in \mathbb{C}$. Let m be a positive integer.

First, we prove the necessity. Suppose that G has a (g, f)-chromatic spanning forest F with exactly m components. By Definition 2.1, $g(c) \leq |E_c(F)|$ for any color $c \in \mathbb{C}$. Thus, $\sum_{c \in \mathbb{C}} g(c) \leq \sum_{c \in \mathbb{C}} |E_c(F)| = |E(F)|$. Hence, F is an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components of G. Since F is a (g, f)-chromatic forest, F contains some (g, g)-chromatic forest, which is also a (g, g)-chromatic forest in G.

Next, we prove the sufficiency. Suppose that G has both an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components, and a (g, g)-chromatic forest F_g. Let F_f be an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components of G such that it has the maximum number of edges of F_g.

We will prove that F_f is the desired (g, f)-chromatic spanning forest with exactly m components of G by contradiction.

Suppose that F_f is not a (g, f)-chromatic spanning forest with exactly m components of G. Then, since F_f is f-chromatic but not (g, f)-chromatic, we may assume that for some color, say color 1, $|E_1(F_f)| \leq g(1) - 1$.

Since F_g is (g, g)-chromatic, $|E_1(F_g)| = g(1)$. Thus, $|E_1(F_f)| < |E_1(F_g)|$. Hence, $E_1(F_g) \setminus E_1(F_f) \neq \emptyset$. Let e be an edge in $E_1(F_g) \setminus E_1(F_f)$. Adding the edge e to

F_f, we consider the resulting graph $(V(F_f), E(F_f) \cup \{e\})$ denoted by F_f^+. Since F_f is f-chromatic and $e \notin E_1(F_f)$, we have

$$|E_c(F_f^+)| = \begin{cases} |E_c(F_f)| + 1 \leq g(c) \leq f(c) & \text{if } c = 1, \\ |E_c(F_f)| \leq f(c) & \text{if } c \neq 1. \end{cases}$$

Thus, F_f^+ is also an f-chromatic spanning subgraph of G.

If the edge e connects two distinct components of F_f in F_f^+, then F_f^+ is an f-chromatic spanning forest with exactly $m-1$ components of G. Since F_g is (g, g)-chromatic, $|E(F_g)| = \sum_{c \in \mathbb{C}} g(c)$. Since $|E(F_f)| \geq \sum_{c \in \mathbb{C}} g(c)$, we have

$$|E(F_f^+)| = |E(F_f)| + 1 \geq \sum_{c \in \mathbb{C}} g(c) + 1 = |E(F_g)| + 1 > |E(F_g)|.$$

Thus, $E(F_f^+) \setminus E(F_g) \neq \emptyset$. Let e' be an edge in $E(F_f^+) \setminus E(F_g)$. Then, we have

$$\omega(F_f^+ - e') = \omega(F_f^+) + 1 = m,$$
$$|E(F_f^+ - e')| = |E(F_f^+)| - 1 = |E(F_f)| \geq \sum_{c \in \mathbb{C}} g(c),$$

where $F_f^+ - e'$ denotes the graph $(V(F_f^+), E(F_f^+) \setminus \{e'\})$. Hence, since F_f^+ is an f-chromatic spanning forest of G, $F_f^+ - e'$ is an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components of G. Recall that $e \in E(F_g)$ and $e' \notin E(F_g)$. Then, $F_f^+ - e'$, namely, $(V(F_f), (E(F_f) \cup \{e\}) \setminus \{e'\})$ has more edges of F_g than F_f, which is a contradiction to the maximality of F_f.

Therefore, we may assume that the both endpoints of e are contained in one component of F_f. Then, $\omega(F_f^+) = \omega(F_f) = m$ and F_f^+ has exactly one cycle C, which contains e. Since F_g has no cycles, C has some edge $e' \notin E(F_g)$. Then, $F_f^+ - e'$ is a forest and

$$\omega(F_f^+ - e') = \omega(F_f^+) = m,$$
$$|E(F_f^+ - e')| = |E(F_f^+)| - 1 = |E(F_f)| \geq \sum_{c \in \mathbb{C}} g(c).$$

Thus, since F_f^+ is an f-chromatic spanning subgraph of G, $F_f^+ - e'$ is an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components of G. Recall that $e \in E(F_g)$ and $e' \notin E(F_g)$. Then, $F_f^+ - e'$, namely, $(V(F_f), (E(F_f) \cup \{e\}) \setminus \{e'\})$ has more edges of F_g than F_f, which is a contradiction to the maximality of F_f.

Consequently, F_f is the desired (g, f)-chromatic spanning forest with exactly m components of G.

□

3.2 Proof of Theorem 2.2

Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$ such that $g(c) \leq f(c)$ for any $c \in \mathbb{C}$. Let m be a positive integer such that $n \geq m + \sum_{c \in \mathbb{C}} g(c)$.

First, we prove the necessity. Suppose that G has a (g,f)-chromatic spanning forest F with exactly m components. Since F is a (g,f)-chromatic forest, F contains some (g,g)-chromatic forest. Thus, by Lemma 3.1, we have

$$\omega(G - E_R(G)) \leq n - \sum_{c \in \mathbb{C} \setminus R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

On the other hand, since F is a (g,f)-chromatic spanning forest with exactly m components of G, F is an f-chromatic spanning forest with exactly m components of G. Thus, by Theorem 1.4, we have

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} f(c) \qquad \text{for any } R \subseteq \mathbb{C}.$$

Therefore,

$$\omega(G - E_R(G)) \leq \min\{\ m + \sum_{c \in R} f(c),\ \ n - \sum_{c \in \mathbb{C} \setminus R} g(c)\ \} \quad \text{for any } R \subseteq \mathbb{C}.$$

Next, we prove the sufficiency. Suppose that

$$\omega(G - E_R(G)) \leq \min\{\ m + \sum_{c \in R} f(c),\ \ n - \sum_{c \in \mathbb{C} \setminus R} g(c)\ \} \quad \text{for any } R \subseteq \mathbb{C}. \tag{1}$$

By (1), we have

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} f(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Thus, by Theorem 1.4, G has an f-chromatic spanning forest F with exactly m components of G. By our assumption that $n \geq m + \sum_{c \in \mathbb{C}} g(c)$, we have

$$|E(F)| = n - m \geq \sum_{c \in \mathbb{C}} g(c).$$

Thus, F is an f-chromatic spanning forest of size at least $\sum_{c \in \mathbb{C}} g(c)$ with exactly m components.

On the other hand, by (1), we have

$$\omega(G - E_R(G)) \leq n - \sum_{c \in \mathbb{C} \setminus R} g(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Thus, by Lemma 3.1, G has a (g,g)-chromatic forest.

Therefore, by Lemma 3.2, G has a (g,f)-chromatic spanning forest with exactly m components.

3.3 Proof of Theorem 2.3

In order to prove Theorem 2.3, we will use the following Lemma.

Lemma 3.3 (Suzuki [16]).

$$|E(G)| \leq \binom{|V(G)| - \omega(G) + 1}{2} \quad \textit{for any graph } G.$$

Let G be an edge-colored graph of order n. Let g and f be mappings from $\mathbb{C}$ to $\mathbb{Z}_{\geq 0}$. Let m be a positive integer such that $n \geq m$. Suppose that $|E(G)| > \binom{n-m}{2}$ and

$$g(c) \leq \frac{|E_c(G)|}{|E(G)|}(n-m) \leq f(c) \qquad \text{for any color } c \in \mathbb{C}. \tag{2}$$

Then, since $\sum_{c\in\mathbb{C}} |E_c(G)| = |E(G)|$, we have

$$\sum_{c\in\mathbb{C}} g(c) \leq \sum_{c\in\mathbb{C}} \frac{|E_c(G)|}{|E(G)|}(n-m) = n-m, \text{ that is, } n \geq m + \sum_{c\in\mathbb{C}} g(c). \tag{3}$$

We will prove that G has a (g,f)-chromatic spanning forest with exactly m components by contradiction.

Suppose that G has no (g,f)-chromatic spanning forests with exactly m components. By (3) and our assumption, we can apply Theorem 2.2 to G and we have

$$\omega(G - E_R(G)) > \min\{\ m + \sum_{c\in R} f(c), \quad n - \sum_{c\in\mathbb{C}\setminus R} g(c)\ \} \quad \text{for some } R \subseteq \mathbb{C}.$$

That is, $\omega(G-E_R(G)) \geq m+\sum_{c\in R} f(c)+1$ or $\omega(G-E_R(G)) \geq n-\sum_{c\in\mathbb{C}\setminus R} g(c)+1$ for some $R \subseteq \mathbb{C}$. We denote $G - E_R(G)$ by G'.

Claim 1.

$$\omega(G') \geq m+1 \text{ and } \omega(G') \geq n+1-\frac{|E(G')|}{|E(G)|}(n-m).$$

Proof. First, we suppose that $\omega(G') \geq m + \sum_{c\in R} f(c) + 1$ for some $R \subseteq \mathbb{C}$. Since $f(c) \geq 0$ for any color c, $\omega(G') \geq m + \sum_{c\in R} f(c) + 1 \geq m+1$.

By our assumption (2),

$$\begin{aligned}\sum_{c\in R} f(c) &\geq \sum_{c\in R} \frac{|E_c(G)|}{|E(G)|}(n-m) = \frac{n-m}{|E(G)|}\sum_{c\in R}|E_c(G)| = \frac{n-m}{|E(G)|}|E_R(G)| \\ &= \frac{n-m}{|E(G)|}(|E(G)| - |E(G')|) = n-m-\frac{|E(G')|}{|E(G)|}(n-m).\end{aligned}$$

Thus, we have

$$\begin{aligned}\omega(G') &\geq m + \sum_{c \in R} f(c) + 1 \\ &\geq m + n - m - \frac{|E(G')|}{|E(G)|}(n-m) + 1 = n + 1 - \frac{|E(G')|}{|E(G)|}(n-m).\end{aligned}$$

Next, we suppose that $\omega(G') \geq n - \sum_{c \in \mathbb{C} \setminus R} g(c) + 1$ for some $R \subseteq \mathbb{C}$. By (3), $\sum_{c \in \mathbb{C}} g(c) \leq n - m$. Thus, we have

$$\omega(G') \geq n - \sum_{c \in \mathbb{C} \setminus R} g(c) + 1 \geq n - \sum_{c \in \mathbb{C}} g(c) + 1 \geq n - (n-m) + 1 = m + 1.$$

By our assumption (2),

$$\sum_{c \in \mathbb{C} \setminus R} g(c) \leq \sum_{c \in \mathbb{C} \setminus R} \frac{|E_c(G)|}{|E(G)|}(n-m) = \frac{|E_{\mathbb{C} \setminus R}(G)|}{|E(G)|}(n-m) = \frac{|E(G')|}{|E(G)|}(n-m).$$

Thus, we have

$$\begin{aligned}\omega(G') &\geq n - \sum_{c \in \mathbb{C} \setminus R} g(c) + 1 \\ &\geq n - \frac{|E(G')|}{|E(G)|}(n-m) + 1 = n + 1 - \frac{|E(G')|}{|E(G)|}(n-m).\end{aligned}$$

□

By Claim 1,

$$n - \omega(G') + 1 \leq \frac{|E(G')|}{|E(G)|}(n-m).$$

Since $n \geq \omega(G')$, $n - \omega(G') + 1 \geq 1$, that is, $n - \omega(G') + 1 \neq 0$. Thus,

$$|E(G)| \leq \frac{n-m}{n - \omega(G') + 1}|E(G')|.$$

Since $|V(G')| = |V(G)| = n$, by Lemma 3.3,

$$\begin{aligned}|E(G)| &\leq \frac{n-m}{n - \omega(G') + 1}\binom{|V(G')| - \omega(G') + 1}{2} \\ &\leq \frac{n-m}{n - \omega(G') + 1} \times \frac{(n - \omega(G') + 1)(n - \omega(G'))}{2} \\ &= \frac{(n-m)(n - \omega(G'))}{2}.\end{aligned}$$

By Claim 1, $\omega(G') \geq m + 1$. Thus,

$$|E(G)| \leq \frac{(n-m)(n-(m+1))}{2} = \binom{n-m}{2},$$

which contradicts our assumption that $|E(G)| > \binom{n-m}{2}$.

Therefore G has a (g, f)-chromatic spanning forest with exactly m components.

3.4 Proof of Theorem 2.5

If an edge-colored complete graph K_n has a spanning tree T with the same color probability distribution as that of K_n, that is,

$$\frac{|E_c(K_n)|}{|E(K_n)|} = \frac{|E_c(T)|}{|E(T)|} \quad \text{for any color } c \in \mathbb{C},$$

Then

$$|E_c(K_n)| = \frac{|E_c(T)||E(K_n)|}{|E(T)|} = \frac{|E_c(T)|n(n-1)/2}{n-1} = \frac{|E_c(T)|n}{2} \quad \text{for any color } c \in \mathbb{C}.$$

Thus, since $|E_c(T)|$ is an integer, $|E_c(K_n)|$ is an integral multiple of $n/2$.

Next, let K_n be an edge-colored complete graph of order n. For any color $c \in \mathbb{C}$, we suppose that $|E_c(K_n)| = k_c \times n/2$ for some $k_c \in \mathbb{Z}_{\geq 0}$. By Theorem 2.4, K_n has a spanning tree T such that

$$\left\lfloor \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rfloor \leq |E_c(T)| \leq \left\lceil \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rceil \quad \text{for any color } c \in \mathbb{C}.$$

Since $|E(K_n)| = n(n-1)/2$ and $|E_c(K_n)| = k_c \times n/2$ ($k_c \in \mathbb{Z}_{\geq 0}$), we have

$$k_c = \lfloor k_c \rfloor = \left\lfloor \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rfloor \leq |E_c(T)| \leq \left\lceil \frac{|E_c(K_n)|}{|E(K_n)|}(n-1) \right\rceil = \lceil k_c \rceil = k_c.$$

Thus, $|E_c(T)| = k_c$. Then,

$$\frac{|E_c(K_n)|}{|E(K_n)|} = \frac{k_c \times n/2}{n(n-1)/2} = \frac{k_c}{n-1} = \frac{|E_c(T)|}{|E(T)|} \quad \text{for any color } c \in \mathbb{C}.$$

Therefore, the color probability distribution of T is the same as that of K_n.

4 Spanning tree decomposition conjectures

In 1996, Brualdi and Hollingsworth [2] presented the following conjecture.

Conjecture 4.1 (Brualdi and Hollingsworth [2])**.** *An edge-colored complete graph K_{2n} ($n \geq 3$) having the property*
(*) *for every color c, the set of edges colored with c induces its perfect matching can be partitioned into n edge-disjoint rainbow spanning trees.*

Brualdi and Hollingsworth [2] proved that an edge-colored complete graph K_{2n} ($n \geq 3$) having the property (*) has two edge-disjoint rainbow spanning trees. Krussel, Marshall, and Verrall [12] proved that the graph has three edge-disjoint rainbow spanning trees. Fu and Lo [6] proved that if $n \geq 14$ then the graph has three edge-disjoint isomorphic rainbow spanning trees. Horn [9] proved, using the probabilistic method, that there exist positive constants ϵ, n_0 such that every edge-colored complete graph K_{2n} ($2n \geq n_0$) having the property (*) has at least ϵn edge-disjoint rainbow spanning trees. Fu, Lo, Perry, and Rodger [7] proved, using a constructive method, that an edge-colored complete graph K_{2n} having the property (*) has $\lfloor \sqrt{6n+9}/3 \rfloor$ edge-disjoint rainbow spanning trees. Pokrovskiy and Sudakov [14] proved that every properly edge-colored complete graph K_n with exactly $n-1$ colors has $n/9 - 6$ edge-disjoint rainbow spanning trees.

Kaneko, Kano, and Suzuki [10] proved that a properly edge-colored complete graph K_n ($n \geq 5$) has two edge-disjoint rainbow spanning trees. Pokrovskiy and Sudakov [14] proved that every properly edge-colored complete graph K_n has $10^{-6}n$ edge-disjoint isomorphic rainbow spanning trees.

Akbari and Alipour [1] proved that an edge-colored complete graph K_n ($n \geq 5$) has two edge-disjoint rainbow spanning trees if $|E_c(K_n)| \leq n/2$ for any color $c \in \mathbb{C}$. Carraher, Hartke, and Horn [3] proved that an edge-colored complete graph K_n ($n \geq 1000000$) has at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees if $|E_c(K_n)| \leq n/2$ for any color $c \in \mathbb{C}$.

Based on these previous results, we conjecture the following as a generalization of Conjecture 4.1.

Conjecture 4.2. *An edge-colored complete graph K_{2n} ($n \geq 3$) can be partitioned into n edge-disjoint spanning trees $T_1, T_2, \ldots, T_n$ such that each has a color probability distribution is similar to that of K_{2n}, that is, each T_i satisfies that*

$$\left\lfloor \frac{|E_c(K_{2n})|}{|E(K_{2n})|}(2n-1) \right\rfloor \leq |E_c(T_i)| \leq \left\lceil \frac{|E_c(K_{2n})|}{|E(K_{2n})|}(2n-1) \right\rceil \quad \textit{for any color } c \in \mathbb{C}.$$

Since $|E(K_{2n})| = 2n(2n-1)/2$ for the complete graph K_{2n}, we have

$$\frac{|E_c(K_{2n})|}{|E(K_{2n})|}(2n-1) = \frac{|E_c(K_{2n})|}{n}.$$

Thus, this conjecture implies that $E_c(K_{2n})$ can be partitioned into n almost equal parts. If K_{2n} has the property (*) then $|E_c(K_{2n})| = n$ for any color c. Hence, if Conjecture 4.2 holds then K_{2n} ($n \geq 3$) having the property (*) can be partitioned into n edge-disjoint spanning trees $T_1, T_2, \ldots, T_n$ such that each T_i satisfies $|E_c(T_i)| = 1$ for any color c, that is, it can be partitioned into n edge-disjoint rainbow spanning trees. Therefore, Conjecture 4.2 is a generalization of Conjecture 4.1.

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References

[1] S. Akbari and A. Alipour, Multicolored trees in complete graphs, *J. Graph Theory* **54** (2007), 221–232.

[2] R. A. Brualdi and S. Hollingsworth, Multicolored trees in complete graphs, *J. Combin. Theory Ser. B* **68** (1996), 310–313.

[3] J. M. Carraher, S. G. Hartke and P. Horn, Edge-disjoint rainbow spanning trees in complete graphs, *European J. Combin.* **57** (2016), 71–84.

[4] G. Chartrand, G. L. Johns, K. A. McKeon and P. Zhang, Rainbow connection in graphs, *Math. Bohem.* **133** (2008), 85–98.

[5] P. Erdős, M. Simonovits and V. T. Sós, Anti-Ramsey theorems, *Infinite and finite sets Vol. II, Colloq. Math. Soc. János Bolyai* **10** (1975), 633–643.

[6] H. L. Fu and Y. H. Lo, Multicolored isomorphic spanning trees in complete graphs, *Ars Combin.* **122** (2015), 423–430.

[7] H. L. Fu, Y. H. Lo, K. E. Perry and C. A. Rodger, On the number of rainbow spanning trees in edge-colored complete graphs, *Discrete Math.* **341** (2018), 2343–2352.

[8] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* **26** (2010), 1–30.

[9] P. Horn, Rainbow spanning trees in complete graphs colored by one-factorizations, *J. Graph Theory* **87** (2018), 333–346.

[10] A. Kaneko, M. Kano and K. Suzuki, Two edge-disjoint heterochromatic spanning trees in colored complete graphs, *Matimyás Mat.* **29** (2006), 49–51.

[11] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey, *Graphs Combin.* **24** (2008), 237–263.

[12] J. Krussel, S. Marshall and H. Verrall, Spanning trees orthogonal to one-factorizations of K_{2n}, *Ars Combin.* **57** (2000), 77–82.

[13] X. Li, Y. Shi and Y. Sun, Rainbow connections of graphs: a survey, *Graphs Combin.* **29** (2013), 1–38.

[14] A. Pokrovskiy and B. Sudakov, Linearly many rainbow trees in properly edge-coloured complete graphs, *J. Combin. Theory Ser. B* **132** (2018), 134–156.

[15] K. Suzuki, A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph, *Graphs Combin.* **22** (2006), 261–269.

[16] K. Suzuki, A generalization of heterochromatic graphs and f-chromatic spanning forests, *Graphs Combin.* **29** (2013), 715–727.

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