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The damage number of a graph

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Abstract

We introduce a natural parameter in the game of Cops and Robbers, the *damage number* of a graph, which is the minimum number of distinct vertices the robbers can visit without capture. Minimizing robbers' access to vertices becomes a natural and wise cops' decision when dealing with networks with vulnerable vertices or in networks with less than cop number cops, where capture is not possible. Clearly, in cop-win graphs the damage number is bounded by the capture time; however, we show that for some graph classes the damage number is approximately half of the capture time. We also prove that in almost all graphs the damage number is less than $\frac{n}{2}$, where n is the order of a graph, confirm this result for graphs of small order and find graphs whose damage numbers exceed $\frac{n}{2}$. We also find or bound the damage number for a variety of classes of graphs and study the damage density of a graph.

1 Introduction

The game of Cops and Robbers is a pursuit-evasion game played on a reflexive graph $G = (V, E)$; i.e., a graph that has a loop at every vertex. There are two opposing sides namely a set of $k > 0$ cops and a set of robbers; however, most games, including the study of this paper, involve only one robber. The two sides play in rounds where each round consists of a move from each side with the cops going first followed by the robber's move. Before the game starts, or at round 0, the cops begin the game

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by each choosing a vertex to occupy, and then the robber chooses a vertex; a vertex may be occupied by multiple players any time during the game. The two sides move alternately by sliding along an edge or along a loop (i.e. pass) from vertex to vertex. This is a perfect information game and the cops win if a cop *captures* the robber by moving onto the vertex that is occupied by the robber after a finite number of moves; else the robber wins. Graphs for which one cop suffices to win are called *cop-win* and the graphs for which $k > 1$ cops can guarantee a win are called k*-cop-win.* Cop-win and k-cop-win graphs have been completely characterized in [8, 9] and [5], respectively.

A multitude of variations of the game of Cops and Robbers have been introduced in the past years; burning number, firefighters and seepage are only a few examples [3, 4, 6]. In this article, we introduce a variation of the game where the robber damages the vertices he visits. The minimum number of vertices of G that can be damaged by the robber is called the *damage number* of G. We assume that the damage is permanent, but the damaged vertices are part of the graph and may be used by the players in their subsequent moves. The robber is unable to damage any more vertices if he is captured by the cops. The damage is done when the robber completes a move; that is if the robber moves from u to v, then u is damaged. The damage number is so broad that it can be studied for the games with single or multiple cops and robbers and also for k-cop-win and robber-win graphs. Note that here we assume that the players play *optimally*, in the sense that the robber strategy is to visit as many distinct vertices as possible in order to inflict the most damage and the cops' strategy is to minimize the damage; that is to minimize the number of distinct vertices the robber visits.

The motivation for this article lies in considering situations where the nature of the damage done by an intruder is severe or costly, or situations where not enough cops are available to guarantee capture of a robber. In these scenarios containing the damage becomes the highest priority as chasing down the robber until capture is not wise or may not be possible.

We follow the standard graph theory notations as in [10]. In Section 2 we formally define the damage number of a graph, present bounds in terms of known graph parameters and show that in almost all graphs the damage number is less than $\frac{n}{2}$, where n is the order of a graph. In Section 3, we study the damage number of all graphs with order at most eight; in these graphs the damage number is at most $\frac{n}{2}$. We show that the Pappus graph has a damage number exceeding $\frac{n}{2}$. For the graph class H_n as defined in [7], a class of cop-win graphs with maximum capture time, the damage number is shown to be $\lfloor \frac{n-3}{2} \rfloor - 1$. We also present bounds on the damage number of classes of strongly regular graphs. In particular, we find a lower bound for the damage number of strongly regular graphs, compute the damage number of the Petersen graph, and bound the damage number of Paley graphs as a function of n. In Section 5, we define the damage density of a graph and conclude by proposing future research directions.

2 Definitions, Motivation and Early Results

In this section we will formally define the damage number of a graph and present some bounds in terms of known graph parameters. We compare the damage number and capture time as defined in [1] and we show that with extreme probability, the robber can damage less than half of the vertices. Although this version of the game and our definitions are stated for any number of cops, our paper studies the game with one cop and one robber, represented as $\mathcal{C}$ and $\mathcal{R}$.

Definition 2.1 We say a vertex u is *damaged* by the robber if the robber occupies u in round $i \geq 0$ and moves to a neighbouring vertex in the next round.

We can imagine the damage is inflicted in the time period between moving onto u in round i and moving from it in round $i+1$. Note that the robber may move from u to u, as our graphs are reflexive.

Definition 2.2 The *damage number of* G, denoted $\text{dmg}(G)$, is the minimum number of unique vertices that can be damaged by the robber.

In the following example we use P_6 to illustrate how the vertices are damaged by the robber at each time step after $\mathcal{R}$ visits them and how to compute the damage number.

Example 2.3 Three rounds of the game on the path P_6 are shown in Figure 1. Since $\mathcal{R}$ cannot inflict any more damage, $\text{dmg}(P_6) = 2$.

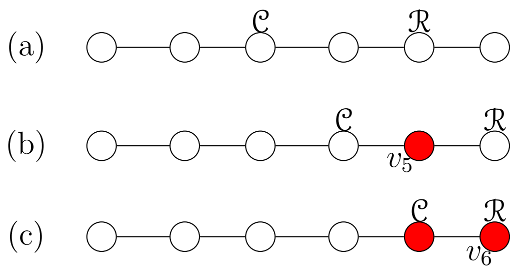

Figure 1: (a) Round 0: cop and robber initial positions; (b) Round 1: $\mathcal{R}$ moves and v_5 is damaged (c) Round 2: $\mathcal{R}$ passes and v_6 is damaged

As noted in the introduction, both sides are assumed to play optimally. In the path graph P_6, had the robber chosen any other vertex, for example v_6, then the capture time might have been the same but the number of damaged vertices would have been less than 2. It is easy to see that the optimal cop strategy is to start on a central vertex of the path.

An *undamaged* vertex is one that has not been damaged. The cop is *protecting* a vertex or a number of vertices if they are in the closed neighbourhood of the cop's position. We note that a protected vertex could be an undamaged vertex or a damaged vertex. When we say the *cop passes in all the rounds*, we mean he passes unless capture becomes possible, in which case he captures the robber.

Example 2.4 If G is a complete multi-partite graph, then $\text{dmg}(G) \leq 1$. If G has a universal vertex, then $\text{dmg}(G) = 0$. Else, $\mathcal{C}$ passes in all the rounds and $\mathcal{R}$ starts on a vertex in the same partition as the cop.

The following theorem gives the damage number for an n-cycle, which is not cop-win when $n \geq 4$.

Theorem 2.5 *If* $n \geq 4$, $dmg(C_n) = \lfloor \frac{n-1}{2} \rfloor$.

Proof. We show that $\mathcal{C}$ has a strategy to prevent damage to at least $\lceil (n+1)/2 \rceil$ vertices and $\mathcal{R}$ can damage at least $\lfloor (n-1)/2 \rfloor$ vertices. Let the vertices of C_n be $v_1, v_2, \ldots, v_n$ in clockwise order. The cop starts on v_1, and the robber starts on v_i for some $3 \leq i \leq n-1$. In the first round, $\mathcal{C}$ passes and in the following rounds, he passes whenever $\mathcal{R}$ passes. Otherwise, he moves the opposite direction of the $\mathcal{R}$'s move unless he is protecting a neighbour of $\mathcal{R}$, in which case he passes. This ensures that $\mathcal{C}$ can prevent damage to $\lceil (n+1)/2 \rceil$ vertices.

Now we show that $\mathcal{R}$ can damage $\lfloor (n-1)/2 \rfloor$ vertices. If $\mathcal{C}$ starts on v_1, then without loss of generality, $\mathcal{R}$ starts on v_3. Regardless of the subsequent moves $\mathcal{C}$ makes, $\mathcal{R}$ can continue to move clockwise, guaranteeing that vertices v_3 through $v_{\lceil \frac{n+1}{2} \rceil}$ are damaged. If n is even, we are done. If n is odd and $v_{\lceil \frac{n}{2} \rceil + 1}$ can be damaged, then the robber has achieved damaging $\lfloor (n-1)/2 \rfloor$, else the cop must be on $v_{\lceil \frac{n}{2} \rceil + 2}$, which guarantees that $\mathcal{R}$ can move counterclockwise to damage v_2. In each case, $\mathcal{R}$ damages at least $\lfloor (n-1)/2 \rfloor$ vertices. $\square$

In a cop-win graph, a vertex may be visited multiple times by the robber before he is captured and the robber cannot inflict any more damage after capture. Therefore, we have the following lemma where $\text{capt}(G)$ is the capture time of G.

Lemma 2.6 *If G is a cop-win graph, then* $dmg(G) \leq capt(G) - 1$.

This bound is tight for some families of graphs, such as graphs with a universal vertex and trees. However, as is illustrated in Example 2.7, the bound given in Lemma 2.6 is not tight for all cop-win graphs. In fact, the damage number can be significantly less than the capture time.

Example 2.7 Figure 2 shows a graph with $\text{capt}(G) = n - 4 = 4$; see [7]. However, if the cop begins at the vertex labelled $\mathcal{C}$ and passes in all the rounds, then the robber can occupy one of the two vertices not adjacent to $\mathcal{C}$ to avoid capture; therefore, $\text{dmg}(G) = 1$.

By Theorem 1 of [7] and Lemma 2.6, we have the following corollary.

Corollary 2.8 *For a cop-win graph G of order* $n \geq 7$, $dmg(G) \leq n - 5$.

For any graph G, we have the following bound.

Theorem 2.9 *If G is a graph on n vertices, then* $\lfloor \frac{rad(G)}{2} \rfloor \leq dmg(G) \leq n - \Delta - 1$ *where $rad(G)$ is the radius of the graph.*

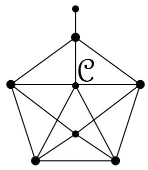

Figure 2: A graph G with $\text{capt}(G) = |V(G)| - 4$ whose damage number is 1

Proof. If $\mathcal{C}$ starts on a vertex v of degree Δ, then he can protect $N[v]$ by passing in all the rounds to stay on v; thus $\text{dmg}(G) \leq n - (\Delta + 1)$.

Now consider placing $\mathcal{C}$ on a central vertex w and place $\mathcal{R}$ on a vertex $rad(G)$ away. Every path between $\mathcal{R}$ and $\mathcal{C}$ is at least $rad(G)$ in length. If at the end of each round the distance between $\mathcal{C}$ and $\mathcal{R}$ decreases, as the cop and robber move toward each other along a path then at least $\lfloor \frac{rad(G)}{2} \rfloor$ vertices are damaged, as they meet in the middle of a path of length $rad(G)$. □

Perhaps surprisingly, many families of graphs on n vertices have a damage number of less than half the order of the graph.

Theorem 2.10 *For a graph of order n, with extreme probability a robber can damage less than $(1 + o(1))\frac{n}{2}$ vertices.*

Proof. It is known that if $p \in (0, 1)$, then random graph $G \in G(n, p)$ has a vertex of degree $(1+o(1))pn$. Assuming $p = 1/2$, then G has a vertex of degree $(1+o(1))\frac{n}{2}$, implying the cop can save at least $(1 + o(1))\frac{n}{2}$ vertices. □

3 Damage Number in Graphs of Small Orders

In this section, we investigate the damage number of small graphs and verify that the robber can damage at most half of the vertices when $n \leq 8$ and, more importantly, we find an example showing that the robber can damage more than half the vertices in a graph.

3.1 Graphs of Small Orders

We show that in graphs of order at most 8, the robber can damage at most half of the vertices.

Theorem 3.1 *If $n \leq 8$, then $dmg(G) \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. If G is chordal, it is known that $\text{capt}(G) \leq \lfloor \frac{n}{2} \rfloor$ (see [1]) and therefore, by Theorem 2.5 and Lemma 2.6, we only need to show $\text{dmg}(G) \leq \frac{n}{2}$ when G is not chordal and is not a cycle. Such a graph G has a vertex v with $\deg_G(v) \geq 3$. If the cop starts at v and passes in all the rounds, then he can protect at least four vertices and $\text{dmg}(G) \leq n - 4 \leq \frac{n}{2}$ when $n \leq 8$. □

It is easy to verify the information in Table 1 for $n = 1, 2, \ldots, 7$ and Theorem 3.2 confirms the case $n = 8$.

n	Maximum dmg(G)	Example of a graph with maximum damage number
1, 2, 3	0	Any
4	1	C_4
5	2	C_5
6	2	C_6
7	3	C_7
8	4	The Möbius ladder M_8

Table 1: Maximum damage number of all graphs with $1 \leq n \leq 8$

Theorem 3.2 *For the Möbius ladder M_8, dmg$(M_8) = 4$.*

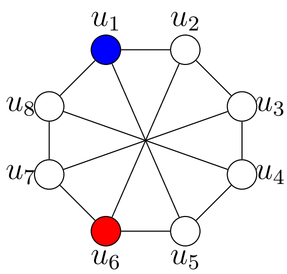

Figure 3: The Möbius ladder M_8

Proof. The Möbius ladder M_8 is shown in Figure 3. Observe that M_8 does not have a 3-cycle, every pair of vertices have at most two common neighbours, and each vertex u of M_8 has a non-neighbour v such that $|N(u) \cap N(v)| = 2$. We prove the theorem in two steps.

Step 1: The robber can damage three vertices in the first three consecutive moves. Without loss of generality, assume that $\mathcal{C}$ starts at u_1. Then the robber starts on u_6, the non-neighbour of $\mathcal{C}$ that has two common neighbours with cop's position. We now show that regardless of the cop's moves in the next two rounds, $\mathcal{R}$ can move onto two more consecutive undamaged vertices. We discuss three cases based on cop's move in round 1.

1. If the cop passes: First, $\mathcal{R}$ moves to u_7. Then he moves to u_8 if $\mathcal{C}$ moves to u_2 or u_5. Likewise, $\mathcal{R}$ moves to u_3 if $\mathcal{C}$ passes or moves to u_8.
2. If the cop moves to u_5 (or u_2 similarly): First, the robber moves to u_2. If the cop passes or moves back to u_1, $\mathcal{R}$ moves to u_3. If cop moves to u_4 or u_6, $\mathcal{R}$ moves to u_1.
3. If the cop moves to u_8: First, the robber moves to u_2. If the cop passes or moves back to u_1, $\mathcal{R}$ moves to u_3. If cop moves to u_4 or u_7, $\mathcal{R}$ moves to u_1.

Next we show that $\mathcal{R}$ can move onto another undamaged vertex without capture.

Step 2: The robber can damage a fourth vertex. Now that we know the robber can move onto three consecutive undamaged vertices, let his path be uvw with w being his current position as shown in Figure 4 (a). Due to the absence of 3-cycles, w has two undamaged neighbours and it is now the cop's turn to move. If the cop cannot move onto a vertex to protect the two undamaged neighbours of w, then $\mathcal{R}$ in his next move can enter an undamaged vertex and we are done. If $\mathcal{C}$ moves to a vertex, say v_1, that is adjacent to both undamaged neighbours of $\mathcal{R}$, say v_2 and v_3, then the robber back-tracks to v (note that $\mathcal{C}$'s position v_1 is not adjacent to v, otherwise w and v_1 would have three common neighbours). Let v_4 be the undamaged neighbour of v. For v_5, the last vertex of M_8, we have $uv_5 \in E(M_8)$; otherwise, it would be impossible to complete the adjacencies into a cubic graph isomorphic to M_8. Also, $v_1v_5 \notin E(M_8)$, else any collection of possible edges will create a bipartite graph whereas M_8 has a 5-cycle. Let's go back in time where $\mathcal{R}$ was on v and $\mathcal{C}$ moves to protect v_4.

Now we have the following two cases.

1. If $v_1v_4 \in E(M_8)$, then $v_4v_5 \in E(M_8)$ and the rest of the edges are as shown in Figure 4 (b). The cop either passes to stay on v_1 or moves to v_4 to protect v_4. Then, $\mathcal{R}$ moves back to u which has two undamaged neighbours v_5 and v_3. Either way, the cop does not have a move from his current position to protect both v_5 and v_3 at the same time.

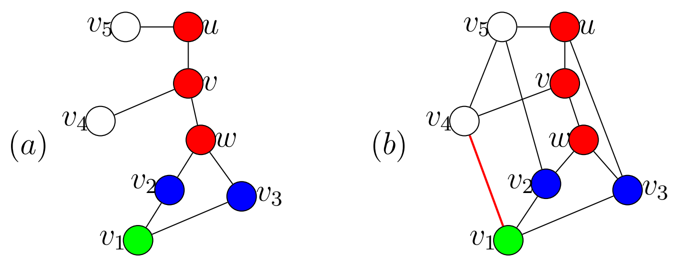

Figure 4: (a) A subgraph of M_8 showing damaged vertices u, v, w and some of the edges; (b) M_8 showing all the vertices and edges assuming $v_1v_4 \in E(M_8)$

2. If $v_1u \in E(M_8)$, then $v_2v_5, v_3v_4 \in E(M_8)$ or $v_2v_4, v_3v_5 \in E(M_8)$. Firstly, if $v_2v_5, v_3v_4 \in E(M_8)$, then the rest of the edges are as in Figure 5 (b) and for the cop to protect v_4, he has to move to v_3. Then $\mathcal{R}$ moves to u which has two undamaged neighbours v_1 and v_5. It turns out that the cop cannot move to a vertex that can protect both v_1 and v_5. Secondly, if $v_2v_4, v_3v_5 \in E(M_8)$, then the rest of the edges are as in Figure 5 (c) and for the cop to protect v_4, he must move to v_2. Again, $\mathcal{R}$ moves to u which has two undamaged neighbours v_5 and v_1. In cop's turn from v_2, he does not have a move that puts him in a position to protect both of the robber's undamaged neighbours.

In any case, from u, the robber safely moves onto a fourth undamaged vertex. □

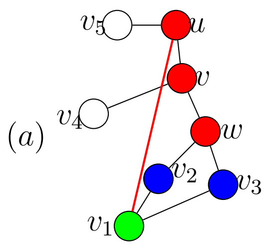

Figure 5: (a) A subgraph of M_8 showing damaged vertices u, v, w and some of the edges assuming $v_1u \in E(M_8)$; (b) M_8 assuming $v_1u \in E(M_8)$ and $v_2v_5, v_3v_4 \in E(M_8)$; (c) M_8 assuming $v_1u \in E(M_8)$ and $v_2v_4, v_3v_5 \in E(M_8)$

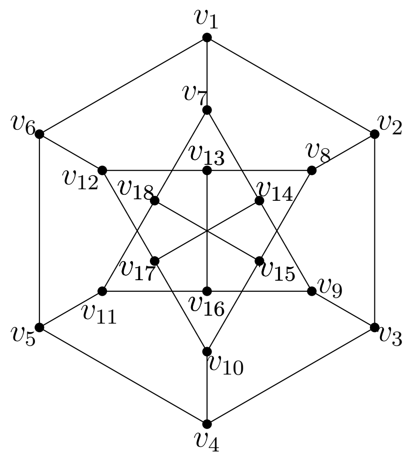

Figure 6: Pappus graph

3.2 Damage Number Exceeding $\frac{n}{2}$

In this subsection we show that the damage number can exceed half the order of the graph. The only known such graph is the Pappus graph (Figure 6), a symmetric, 3-regular, bipartite graph on 18 vertices for which we will show that $\operatorname{dmg}(G) \geq 10$.

Theorem 3.3 *The damage number of the Pappus graph is at least* 10*.*

Proof. Let G be the Pappus graph. As G is bipartite, we will consider the scenarios where $\mathcal{C}$ forces $\mathcal{R}$ to complete an even length cycle, and when he does not.

First suppose $\mathcal{R}$ does not complete a cycle by revisiting a damaged vertex. Since G is 3-regular, the robber always has a choice of two neighbouring vertices to move to and as the Pappus graph is bipartite and non-adjacent vertices have at most one common neighbour, the cop cannot protect both the undamaged neighbours. Since the robber is not forced to complete a cycle by revisiting a damaged vertex, $\mathcal{R}$ can damage at least 10 vertices.

We now consider if a cycle is completed. Clearly, if $\mathcal{C}$ forces $\mathcal{R}$ to complete a 10-cycle or larger cycle, we are done. Now suppose that $\mathcal{C}$ forces $\mathcal{R}$ to complete a

smaller cycle. Since G is a bipartite graph of girth 6, we need only consider if $\mathcal{R}$ is forced to complete a 6-cycle or an 8-cycle.
Case 1: The 6-cycle

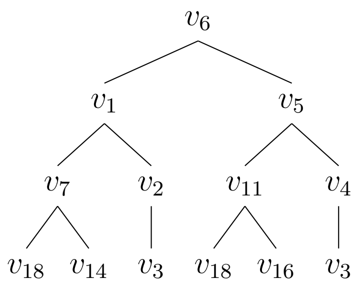

Figure 7: Possible moves for $\mathcal{R}$ when v_6 can be damaged.

In this case the robber will need to damage at least four more vertices. Every 6-cycle can be mapped via graph automorphisms to $C = v_{12}v_{13}v_8v_{15}v_{10}v_{17}$, so we need only consider if the robber is on this cycle.

Assume that $\mathcal{C}$ forces $\mathcal{R}$ to complete C. Once the cycle is completed, by rotation starting on v_{12} is the same as starting on v_8 or v_{10} and starting on v_{13} is the same as starting on v_{15} or v_{17}, thus we need only consider if $\mathcal{R}$ is on v_{12} or v_{13}. There is an automorphism that maps v_{13} onto v_{12}. Assume that $\mathcal{R}$ is on v_{13} and then moves to v_{12} to complete a cycle.

We will look at two cases.

1) First assume that $\mathcal{R}$ can damage v_6. Figure 7 shows the undamaged vertices that $\mathcal{R}$ moves onto from v_6. Clearly if $\mathcal{R}$ moves onto v_{11} or v_7, we are done.

If $\mathcal{R}$ must move to v_2 or v_4, we have several scenarios to consider. We will suppose that $\mathcal{R}$ moves onto v_2 (moving onto v_4 is a similar argument). Since $\mathcal{C}$ was protecting v_7 they are on v_7, v_{18} or v_{14}. For each scenario we will show that $\mathcal{R}$ can damage the required fourth vertex.

If $\mathcal{C}$ is on v_7 or v_{18} that is far enough away that $\mathcal{R}$ can damage v_3 and we are done.

If $\mathcal{C}$ is on v_{14} then his next move must be to v_9 to protect v_3, thus the robber moves to v_8. If $\mathcal{C}$ stays on v_9 or moves to v_3, the robber can damage v_{18}. If instead $\mathcal{C}$ moves to v_{16} then $\mathcal{R}$ will move to v_{15} and v_{10}, forcing $\mathcal{C}$ to v_{11} and v_5, then $\mathcal{R}$ can damage v_{14} in subsequent moves. Lastly, if from v_9 the cop moves back to v_{14}, the robber will move to v_{15}, forcing $\mathcal{C}$ to v_7, then $\mathcal{R}$ can damage v_4 in subsequent moves.

Thus, if v_6 is damaged the damage of the Pappus graph is at least 10.

2) Assume that $\mathcal{R}$ cannot damage v_6. In this case $\mathcal{R}$ completes a cycle by moving to v_{12}. This means that in the previous round $\mathcal{R}$ was unable to move from v_{13} to v_{16}, thus $\mathcal{C}$ was on v_{11} and when $\mathcal{R}$ moved to v_{12}, $\mathcal{C}$ then moved to v_5 to protect v_6 from damage.

In the next two rounds $\mathcal{R}$ can move to v_{14}. Figure 8 shows the undamaged vertices the robber can move to from v_{14}. Clearly if the robber moves onto v_1 or v_3 we are done.

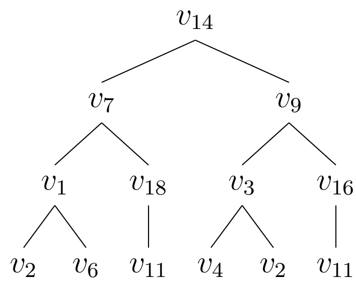

Figure 8: Possible moves for R when v_{14} is damaged.

Suppose that $\mathcal{R}$ must move to v_{16} (a similar argument holds for v_{18}). This means that $\mathcal{C}$ must be on v_2, v_3 or v_4 in order to protect v_3. For each scenario we will show that $\mathcal{R}$ can damage the required fourth vertex.

If $\mathcal{C}$ is on v_2 or v_3 then $\mathcal{R}$ can move to v_{11} and we are done.

Assume $\mathcal{C}$ is on v_4 and $\mathcal{R}$ moves to v_{16}. Then $\mathcal{C}$ must move to v_5, and $\mathcal{R}$ chooses to move to v_{13}.

If $\mathcal{C}$ stays on v_5 then $\mathcal{R}$ can damage v_2. If $\mathcal{C}$ moves back to v_4 then $\mathcal{R}$ moves to v_8, which forces $\mathcal{C}$ to v_3, thus $\mathcal{R}$ can damage v_{18} in two moves. If $\mathcal{C}$ moves to v_6 then $\mathcal{R}$ can move to v_{15} in two moves, forcing $\mathcal{C}$ onto v_1 then v_7, which allows $\mathcal{R}$ to damage v_4 in subsequent rounds.

Thus if v_6 is not damaged the damage of the Pappus graph is at least 10.

Case 2: The 8-cycle

In this scenario, the robber will need to damage at least two undamaged vertices to ensure a damage number of at least 10. All 8-cycles can be mapped by an automorphism onto $C = v_2v_8v_{13}v_{12}v_{17}v_{10}v_4v_3$, thus we need only consider if $\mathcal{R}$ is forced to complete C.

We should note if $\mathcal{R}$ moves onto any undamaged vertex other than v_{15} then $\mathcal{R}$ can damage at least one more vertex, as each of those undamaged vertices have at least two undamaged neighbours. We have three scenarios to consider, $\mathcal{R}$ is on v_{12}, v_{13}, v_8. We will show that in each scenario $\mathcal{R}$ can move onto an undamaged vertex that is not v_{15}.

Suppose $\mathcal{R}$ is on vertex v_{12}. By an automorphism this is the same as $\mathcal{R}$ being on v_3. If the robber can damage v_6 we're done, otherwise regardless of the cops next two moves, the robber can move onto one of v_{14} or v_{16}.

Now suppose $\mathcal{R}$ is on v_{13}. By symmetry a similar argument will hold if the robber is on v_{17} and by an automorphism this is the same as $\mathcal{R}$ being on v_2 or v_4. If v_{16} is not protected then we are done. Otherwise $\mathcal{R}$ can move to v_{12} and we are in the previous case.

Lastly, assume $\mathcal{R}$ is on v_8. By symmetry a similar argument will hold if the robber is on v_{10}. The cop can not protect both v_{13} and v_2, so the robber can move onto one of those vertices and we are in the previous case.

Hence, regardless of whether the robber completes a cycle or not, $\text{dmg}(G) \geq 10$. □

4 Classes of Cop-win Graphs and Strongly Regular Graphs

In this section, we look at the damage number of a few classes of graphs that are known to be cop-win or to be robber-win. In Example 2.7, we saw a graph whose damage number is much smaller than the capture time. Next, we look more closely at the class of cop-win graphs that the graph belongs to; i.e., a class of graphs with maximum capture time.

4.1 Cop-win Graphs with Maximum Capture Time

Consider the family of graphs H_n as shown in Figure 9 and described in [7]. This family has the capture time of $n-4$; i.e., it achieves the upper bound for the capture time of a graph on n vertices. Although the capture time for this family is extreme, the damage number is not.

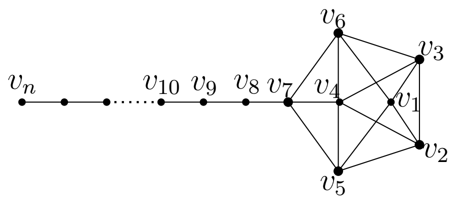

Figure 9: A graph H_n with $\text{capt}(H_n) = |V(H_n)| - 4$

Theorem 4.1 *If $n \geq 7$, then $dmg(H_n) = \lfloor \frac{n-3}{2} \rfloor - 1$.*

Proof. If $n = 7, 8$, then $\mathcal{C}$ begins on v_4 and the damage number is 1. Let $n \geq 9$. Suppose that $\mathcal{C}$ and $\mathcal{R}$ initial positions are of distance two on the path $v_7 v_8 \dots v_n$. When the robber comes off the path and onto vertex v_7, if the cop is right behind, the robber can only damage four vertices in H_7. Assume that $\mathcal{C}$ starts on vertex $v_{5+(\lfloor \frac{n-3}{2} \rfloor - 1)}$. If $\mathcal{R}$ starts anywhere on the path to the left of $\mathcal{C}$, he will damage at most $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices. Suppose $\mathcal{R}$ starts to the right of $\mathcal{C}$. If they are of distance two, then $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices will be damaged by the robber.

Should the cop start on a vertex to the right of vertex $v_{5+(\lfloor \frac{n-3}{2} \rfloor - 1)}$, at least $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices will be damaged, as the path to the left of $\mathcal{C}$ will be at least $\lfloor \frac{n-3}{2} \rfloor$ in length. If the cop starts to the left of vertex $v_{5+(\lfloor \frac{n-3}{2} \rfloor - 1)}$, then the robber potentially damages at least one more vertex in the path leading up to H_7, thus the damage number is $\lfloor \frac{n-3}{2} \rfloor - 1$. □

The robber's strategy to obtain the capture time of $n-4$ has him visiting all but five vertices. To contrast, the damage number for this graph is approximately half of the capture time.

We now turn our attention to a family of graphs that are not cop-win.

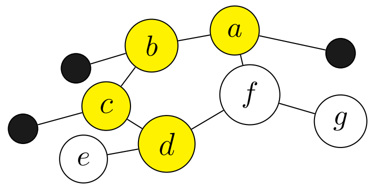

Figure 10: Robber's path on Petersen graph damaging four consecutive vertices

4.2 Strongly Regular Graphs

A regular cop-win graph is a complete graph, so we may assume that $G \neq K_n$. A k-regular graph G on n vertices in which each pair of adjacent vertices has exactly λ common neighbours and each pair of non-adjacent vertices has exactly μ common neighbours is called strongly regular, denoted $\mathrm{SRG}(n, k, \lambda, \mu)$. It is natural to first compute the damage number of Petersen graph, $\mathrm{SRG}(10, 3, 0, 1)$. If the cop starts on some vertex v and passes on all his moves, then he can protect four vertices. In addition, the subgraph induced by $V(P) \setminus N[v]$ is connected and therefore the robber can damage all the remaining six vertices. So this, possibly, is not the best cop strategy and actually as we see in the following theorem, the damage number of Petersen graph is 5.

Theorem 4.2 *For the Petersen graph P, $dmg(P) = 5$.*

Proof. The girth of the Petersen graph is five. We note that the cop can move to protect any vertex of the Petersen graph. This is because P is $\mathrm{SRG}(10, 3, 0, 1)$ and thus every vertex is either adjacent to $\mathcal{C}$'s position (already protected) or has a common neighbour with it (cop can move onto that common neighbor). A subgraph of P illustrating the proof is shown in Figure 10.

To prove our claim, we show that (a) regardless of the cop's movements, $\mathcal{R}$ can visit five consecutive undamaged vertices at least, and that (b) the cop can protect five vertices by preventing $\mathcal{R}$ from damaging a sixth vertex.

(a) $\mathcal{R}$ can visit five consecutive undamaged vertices: Since girth of the graph is five and each vertex has three neighbours, $\mathcal{R}$ can visit four consecutive undamaged vertices say a, b, c and d in the first four moves. This is because, as long as the robber has two undamaged neighbours, he can move onto one of them since the cop can be adjacent or on at most one of the undamaged neighbours, else the girth would be ≤ 4. Assume that $\mathcal{R}$ is now on d and it is his move. By the structure of the graph, the second neighbour of d is an undamaged neighbour of a, say f, and its third neighbour is some undamaged vertex e. The robber can move to either of the vertices and damage a fifth vertex. At this point however, the cop can force the robber to move to f by protecting e. So, the robber's fifth vertex to visit is f.

(b) $\mathcal{C}$ can prevent $\mathcal{R}$ from damaging a sixth vertex: At this point all the remaining vertices have either one or two undamaged neighbours, all damaged vertices have one undamaged neighbour and the robber's current position, f, has only one undamaged

neighbour, g. It is now cop's turn and the cop can prevent the robber from moving onto g. In turns out that $\mathcal{C}$ can prevent the robber from moving onto an undamaged vertex in all the following moves by protecting $\mathcal{R}$'s undamaged neighbour.

We note that as long as the robber is on a vertex with two undamaged neighbours, he can move onto an undamaged vertex, whereas if he has only one undamaged neighbour, the cop is able to protect that undamaged neighbour after his move. That is why the cop forced the robber to move into f in round 4 and also why revisiting a damaged vertex does not help the robber at any stage. □

In general, for SRG(n, k, λ, μ) we can say:

Theorem 4.3 *If G is a SRG(n, k, λ, μ), then dmg$(G) \geq \min\{k - \lambda, k - \mu + 1\}$.*

Proof. Assume that we are in a state of the game where $\mathcal{R}$ has done all the damage and cannot inflict any more damage. Let v_r and v_c be the cop and robber current positions and assume that it is robber's turn. If $v_r v_c \in E(G)$, then v_r has $k - \lambda - 1$ neighbours that the robber can move onto without getting caught. Since no more vertices can be damaged, it turns out that all those $k - \lambda - 1$ vertices were already damaged. These plus the damaged v_r make a total of at least $k - \lambda$ damaged vertices.

Now assume that $v_r v_c \notin E(G)$. By a similar argument, the $k - \mu$ neighbours of the robber that are not adjacent to the cop must have been already damaged. These damaged vertices and the damaged v_r make a total of at least $k - \mu + 1$ vertices. Therefore, $\mathrm{dmg}(G) \geq \min\{k - \lambda, k - \mu + 1\}$. □

Theorem 4.4 *For a Paley graph $\mathcal{P}_n$ with $n > 9$, $\frac{n+3}{4} + 1 \leq$ dmg$(\mathcal{P}_n) \leq \frac{n-1}{2}$.*

Proof. Paley graph $\mathcal{P}_n$ is SRG$(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$. If $\mathcal{C}$ passes in all the rounds to occupy one vertex, then he can protect $\frac{n-1}{2}+1$ vertices and therefore, $\mathrm{dmg}(\mathcal{P}_n) \leq \frac{n-1}{2}$.

Now we establish the lower bound. By Theorem 4.3, damage number is at least $\frac{n+3}{4}$. Now we show that the robber can damage at least one more than $\frac{n+3}{4}$ vertices. Let $X = \{v_{r_1}, v_{r_2}, \ldots, v_{r_{\frac{n+3}{4}}}\}$ be the set of first $\frac{n+3}{4}$ vertices damaged by the robber and assume that $\mathcal{R}$ is at $v_{r_{\frac{n+3}{4}}}$ and that it is the robber's turn. Note that $\mathcal{P}_n[X]$ cannot be a clique as the maximum clique in a Paley graph of order n has $\sqrt{n}$ vertices. We have the following two cases:

Case 1: $v_{r_{\frac{n+3}{4}}}$ is not adjacent to all the vertices in X. So there is an undamaged vertex not adjacent to $\mathcal{C}$ that the robber can move onto.

Case 2: $v_{r_{\frac{n+3}{4}}}$ is adjacent to every vertex in X. Since $\mathcal{P}_n[X]$ is not a clique, there is some vertex in X, say v_i, which is adjacent to no more than $\frac{n+3}{4} - 2$ vertices of X. If all common neighbours of $\mathcal{C}$ and $\mathcal{R}$ are undamaged vertices, then $\mathcal{C}$ is not adjacent to any vertex in X and in particular $\mathcal{C}$ is not adjacent to v_i. Therefore, $\mathcal{R}$ safely moves to v_i. In the next round, $\mathcal{C}$ can not become adjacent to all the undamaged vertices adjacent to v_i (because v_i has at least $\frac{n+3}{4}$ undamaged neighbours and v_i have at most $\frac{n-1}{4}$ common neighbours with a vertex), and therefore, $\mathcal{R}$ can move from v_i to one more undamaged vertex.

Therefore, the damage number is at least $\frac{n+3}{4} + 1$. □

If $\mathcal{C}$ passes in all the rounds to occupy one vertex, the subgraph induced by the set of his non-neighbours, B, is a $(\frac{n-1}{4})$–regular graph and $|B| = \frac{n-1}{2}$. Therefore, $\mathcal{P}_n[B]$ is Hamiltonian and the robber can cause damage to all the vertices in B. Given that $\text{dmg}(\mathcal{P}_9) = 3$ (this is easily seen considering that $\mathcal{P}_9 = C_3 \square C_3$, where $\square$ denotes the Cartesian product), we propose the following conjecture.

Conjecture 4.5 *For a Paley graph $\mathcal{P}_n$, dmg$(\mathcal{P}_n) = \frac{n-1}{2}$ for $n > 9$.*

5 Density of the Damage of a Graph

Thus far we have been looking at the number of vertices that can be damaged by the robber. Another quantity that we can study is the proportion of the graph that is damaged by the robber. This is the *damage density* of the graph and is computed as

$$D_{dmg}(G) = \text{dmg}(G)/|V(G)|.$$

Similar such ratios have been investigated for cop-number and capture time [1, 2]. For example, for the path P_{2n}, $D_{dmg}(P_{2n}) = \frac{2n-2}{4n}$, so as n gets arbitrarily large, the damage density approaches 1/2. In fact, using a family of trees, we can show the following:

Theorem 5.1 *Given any $r \in [0, 1/2]$ and $\epsilon < 5/4 - \sqrt{16r^2 - 8r + 25}/4$ we can find a graph G such that $D_{dmg}(G)$ is within ϵ of r.*

Proof. We define the graph $P_{2\ell,y}$, $\ell \geq 1$, to be $P_{2\ell}$ with y leaves incident to vertex ℓ. Let $r \in (0, 1/2)$ and $0 < \epsilon < 5/4 - \sqrt{16r^2 - 8r + 25}/4$. Without loss of generality we can assume that $0 < r - \epsilon < r + \epsilon < 1/2$.

It is the case that $\text{dmg}(P_{2\ell,y}) = \text{dmg}(P_{2\ell}) = \frac{2\ell-2}{2} = \ell - 1$. Our claim is that for a given r and ϵ there exist ℓ and y such that $D_{dmg}(P_{2\ell,y})$ is within ϵ of r. That is, we want to find ℓ and y so that

$$r - \epsilon < \frac{\ell - 1}{2\ell + y} < r + \epsilon. \tag{1}$$

From Equation 1 we find the following bounds on y:

$$\frac{(\ell - 1) - (r + \epsilon)(2\ell)}{r + \epsilon} < y < \frac{(\ell - 1) - (r - \epsilon)(2\ell)}{r - \epsilon}.$$

If this interval has a length of at least 1 and has values all greater than 0, then such a $P_{2\ell,y}$ exists. We will first find the values of ℓ for which the width of the interval is at least 1.

By basic algebra we can show that

$$\frac{(\ell - 1) - (r - \epsilon)(2\ell)}{r - \epsilon} - \frac{(\ell - 1) - (r + \epsilon)(2\ell)}{r + \epsilon} \geq 1$$

when

$$\ell \geq \frac{(r-\epsilon)(r+\epsilon)}{2\epsilon} + 1.$$

So given r and ϵ, if we pick ℓ as above, then our interval will have width at least 1, so it will contain an integer. Let

$$f(\ell, r, \epsilon) = \frac{(\ell - 1) - (r+\epsilon)(2\ell)}{r+\epsilon}.$$

To ensure the interval contains a positive integer, we want $f(\ell, r, \epsilon) \geq 0$, so that the lower bound of our interval is at least 0. When $r - \epsilon < 1/2$, which is always true, $f(\ell+1, r, \epsilon) > f(\ell, r, \epsilon)$, so if $f(\ell, r, \epsilon) \geq 0$ for $\ell = [(r-\epsilon)(r+\epsilon)/2\epsilon] + 1$, it is true for larger values of ℓ as well.

When $\ell = \frac{(r-\epsilon)(r+\epsilon)}{2\epsilon} + 1$, $f(\ell, r, \epsilon) = \frac{2\epsilon^2 - 2r^2 - 5\epsilon + r}{2\epsilon}$ and this is at least 0 for $\epsilon < 5/4 - \sqrt{16r^2 - 8r + 25}/4$, thus we can find positive integers ℓ and y such that for $r \in [0, 1/2]$ we can find a graph $P_{2\ell,y}$ so that $D_{dmg}(P_{2\ell,y})$ is arbitrarily close to r. □

6 Future Work

We end this article with open problems and future research directions.

Problem 6.1 For cop-win graphs, clearly $0 \leq \dfrac{\text{dmg}(G)}{\text{capt}(G)} < 1$, but what is the closure of the values this ratio can take on? For any $r \in [0, 1]$, can we find a graph G such that $\dfrac{\text{dmg}(G)}{\text{capt}(G)}$ is arbitrarily close to r?

Problem 6.2 Find families of graphs whose damage number is extreme, that is $\text{dmg}(G) = n - \Delta - 1$.

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