

Characterisation and classification of signatures of spanning trees of the n -cube

HOWIDA A. AL FRAN DAVID J. W. SIMPSON
CHRISTOPHER P. TUFFLEY*

*School of Fundamental Sciences, Massey University
Private Bag 11222, Palmerston North 4442
New Zealand*

Abstract

The signature of a spanning tree T of the n -dimensional cube Q_n is the n -tuple $\text{sig}(T) = (a_1, a_2, \dots, a_n)$ such that a_i is the number of edges of T in the i th direction. We characterise the n -tuples that can occur as the signature of a spanning tree, and classify a signature \mathcal{S} as *reducible* or *irreducible* according to whether or not there is a proper nonempty subset R of $[n]$ such that restricting \mathcal{S} to the indices in R gives a signature of $Q_{|R|}$. If so, we say moreover that \mathcal{S} and T *reduce over* R .

We show that reducibility places strict structural constraints on T . In particular, if T reduces over a set of size r then T decomposes as a sum of 2^r spanning trees of Q_{n-r} , together with a spanning tree of a certain contraction of Q_n with underlying simple graph Q_r . Moreover, this decomposition is realised by an isomorphism of *edge slide graphs*, where the edge slide graph of Q_n is the graph $\mathcal{E}(Q_n)$ on the spanning trees of Q_n , with an edge between two trees if and only if they are related by an *edge slide*. An edge slide is an operation on spanning trees of the n -cube given by “sliding” an edge of a spanning tree across a 2-dimensional face of the cube to get a second spanning tree.

The signature of a spanning tree is invariant under edge slides, so the subgraph $\mathcal{E}(\mathcal{S})$ of $\mathcal{E}(Q_n)$ induced by the trees with signature \mathcal{S} is a union of one or more connected components of $\mathcal{E}(Q_n)$. Reducible signatures may be further divided into *strictly reducible* and *quasi-irreducible* signatures, and as an application of our results we show that $\mathcal{E}(\mathcal{S})$ is disconnected if \mathcal{S} is strictly reducible. We conjecture that the converse is also true. If true, this would imply that the connected components of $\mathcal{E}(Q_n)$ can be characterised in terms of signatures of spanning trees of subcubes.

* Corresponding author: c.tuffley@massey.ac.nz

1 Introduction

The n -cube is the graph Q_n whose vertices are the subsets of the set $[n] = \{1, 2, \dots, n\}$, with an edge between X and Y if they differ by the addition or removal of a single element. The element added or removed is the *direction* of the edge. Given a spanning tree T of Q_n , we may then define the *signature* of T to be the n -tuple

$$\text{sig}(T) = (a_1, a_2, \dots, a_n),$$

where a_i is the number of edges of T in direction i . The signature of T carries exactly the same information as the *direction monomial* $q^{\text{dir}(T)}$ appearing in Martin and Reiner’s weighted count [10] of the spanning trees of Q_n . With respect to certain weights q_1, \dots, q_n and x_1, \dots, x_n they show that

$$\sum_{T \in \text{Tree}(Q_n)} q^{\text{dir}(T)} x^{\text{dd}(T)} = q_1 \cdots q_n \prod_{\substack{S \subseteq [n] \\ |S| \geq 2}} \sum_{i \in S} q_i (x_i^{-1} + x_i),$$

where

$$q^{\text{dir}(T)} = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}.$$

Thus, the signature and direction monomial completely determine each other. (The second factor $x^{\text{dd}(T)}$ appearing here is the *decoupled degree monomial* of T . It plays no role in this paper, so we refer the interested reader to Martin and Reiner [10] for the definition, and Tuffley [12, Sec. 2.2] for an alternate formulation in terms of a canonical orientation of the edges of T .)

The goal of this paper is to study the signatures of spanning trees of Q_n , and to understand what $\text{sig}(T)$ tells us about the structure of T . We begin by using Hall’s Theorem to characterise the n -tuples that can occur as the signature of a spanning tree of Q_n . We then classify T and $\mathcal{S} = \text{sig}(T)$ as *reducible* or *irreducible* according to whether or not there is a proper nonempty subset R of $[n]$ such that restricting \mathcal{S} to the indices in R gives a signature of $Q_{|R|}$. We say that such a set R is a *reducing set* for \mathcal{S} , and that T and \mathcal{S} *reduce over* R . Each signature \mathcal{S} has an *unsaturated part* $\text{unsat}(\mathcal{S})$, and we further classify reducible signatures as *strictly reducible* or *quasi-irreducible* according to whether or not $\text{unsat}(\mathcal{S})$ is reducible or irreducible.

We show that reducibility places strict structural constraints on T . In particular, if T reduces over R then T decomposes as a sum of a spanning tree T^X of $Q_{[n]-R}$ for each $X \subseteq R$, together with a spanning tree T_R of the multigraph Q_n/\bar{R} obtained by contracting every edge of Q_n in directions belonging to $\bar{R} = [n] - R$. Moreover, this decomposition may be realised as an isomorphism of *edge slide graphs*.

An *edge slide* is an operation on spanning trees of Q_n , in which an edge of a spanning tree T is “slid” across a 2-dimensional face of Q_n to get a second spanning tree T' . The *edge slide graph* of Q_n is the graph $\mathcal{E}(Q_n)$ with vertices the spanning trees of Q_n , and an edge between two trees if they are related by an edge slide. Edge slides were introduced by the third author [12] as a means to combinatorially count the spanning trees of Q_3 , and thereby answer the first nontrivial case of a

question implicitly raised by Stanley. The number of spanning trees of Q_n is known by Kirchhoff's Matrix Tree Theorem to be

$$|\text{Tree}(Q_n)| = 2^{2^n - n - 1} \prod_{k=1}^n k^{\binom{n}{k}}$$

(see for example Stanley [11]), and Stanley implicitly asked for a combinatorial proof of this fact. Tuffley's method to count the spanning trees of Q_3 using edge slides does not readily extend to higher dimensions, but the edge slide graph may nevertheless carry insight into the structure of the spanning trees of Q_n . Stanley's question has since been answered in full by Bernardi [2].

In particular, a natural question of interest is to determine the connected components of $\mathcal{E}(Q_n)$. The signature is easily seen to be constant on connected components, and consequently the subgraph $\mathcal{E}(\mathcal{S})$ induced by the spanning trees with signature \mathcal{S} is a union of connected components of $\mathcal{E}(Q_n)$. We say that a signature \mathcal{S} is *connected* if $\mathcal{E}(\mathcal{S})$ is connected, and *disconnected* otherwise. We conclude the paper by using our results to show that all strictly reducible signatures are disconnected, and conjecture that \mathcal{S} is connected if and only if \mathcal{S} is irreducible or quasi-irreducible. If true, this would imply that the connected components of $\mathcal{E}(Q_n)$ can be characterised in terms of signatures of spanning trees of subcubes. We show that it suffices to consider the irreducible case only.

1.1 Organisation

The paper is organised as follows. Section 2 sets out the bulk of the definitions and notation needed for the paper, with the introduction of some further definitions not needed until Section 6 postponed until then. We characterise signatures of spanning trees of Q_n in Section 3, and classify them in Section 4. In Sections 5 and 6 we study the structural consequences of reducibility, considering first upright trees in Section 5 and then arbitrary reducible trees in Section 6. We then use our results from Section 6 to prove that strictly reducible signatures are disconnected in Section 7, and conclude with a discussion in Section 8.

2 Definitions and notation

This section sets out some definitions and notation used throughout the paper. Some additional definitions not needed until Section 6 are set out in a second definitions section there.

2.1 General notation

Given a graph G we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. We write $\text{Tree}(G)$ for the set of spanning trees of G .

Given a set S , we denote the power set of S by $\mathcal{P}(S)$. For $1 \leq k \leq |S|$ we write

$$\mathcal{P}_{\geq k}(S) = \{X \subseteq S : |X| \geq k\}.$$

For $n \in \mathbb{N}$ we let $[n] = \{1, 2, \dots, n\}$, and also write $\mathcal{P}_{\geq k}^n$ for $\mathcal{P}_{\geq k}([n])$. For example, $\mathcal{P}_{\geq 2}^3 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

2.2 The n -cube

Definition 2.1. We regard the n -dimensional cube or n -cube as the graph Q_n with vertex set the power set of $[n]$, and an edge between vertices X and Y if and only if they differ by adding or removing exactly one element. The **direction** of the edge $e = \{X, Y\}$ is the unique element i such that $X \oplus Y = \{i\}$, where \oplus denotes symmetric difference.

For any $S \subseteq [n]$, we define Q_S to be the induced subgraph of Q_n with vertices the subsets of S . Observe that Q_S is an $|S|$ -cube.

2.3 The signature of a spanning tree of Q_n

Definition 2.2. Given a spanning tree T of Q_n , the **signature** of T is the n -tuple

$$\text{sig}(T) = (a_1, a_2, \dots, a_n),$$

where for each i the entry a_i is the number of edges of T in direction i . We will say that $\mathcal{S} = (a_1, \dots, a_n)$ is a *signature of Q_n* if there is a spanning tree T of Q_n such that $\text{sig}(T) = \mathcal{S}$, and we let

$$\text{Sig}(Q_n) = \{\text{sig}(T) : T \in \text{Tree}(Q_n)\}.$$

Figure 1 shows a pair of spanning trees of Q_3 with signature $(2, 2, 3)$. We note that the signature of T carries exactly the same information as the *direction monomial* $q^{\text{dir}(T)}$ of Martin and Reiner [10], because

$$q^{\text{dir}(T)} = q_1^{a_1} q_2^{a_2} \dots q_n^{a_n} \quad \Leftrightarrow \quad \text{sig}(T) = (a_1, a_2, \dots, a_n).$$

The entries of $\text{sig}(T)$ satisfy $1 \leq a_i \leq 2^{n-1}$, because Q_n has 2^{n-1} edges in direction i and deleting them disconnects Q_n , and

$$\sum_{i=1}^n a_i = |E(T)| = 2^n - 1.$$

These conditions are not sufficient conditions for an n -tuple (a_1, a_2, \dots, a_n) to be a signature of Q_n . We find necessary and sufficient conditions in Section 3.

If $\mathcal{S} = (a_1, \dots, a_n)$ is a signature of Q_n then so is any permutation of \mathcal{S} , because any permutation of $[n]$ induces an automorphism of Q_n . It follows that \mathcal{S} is a signature if and only if the n -tuple \mathcal{S}' obtained by permuting \mathcal{S} to nondecreasing order is a signature. Accordingly we make the following definition:

Definition 2.3. A signature (a_1, a_2, \dots, a_n) of Q_n is **ordered** if $a_1 \leq a_2 \leq \dots \leq a_n$.

We will characterise signatures by characterising ordered signatures.

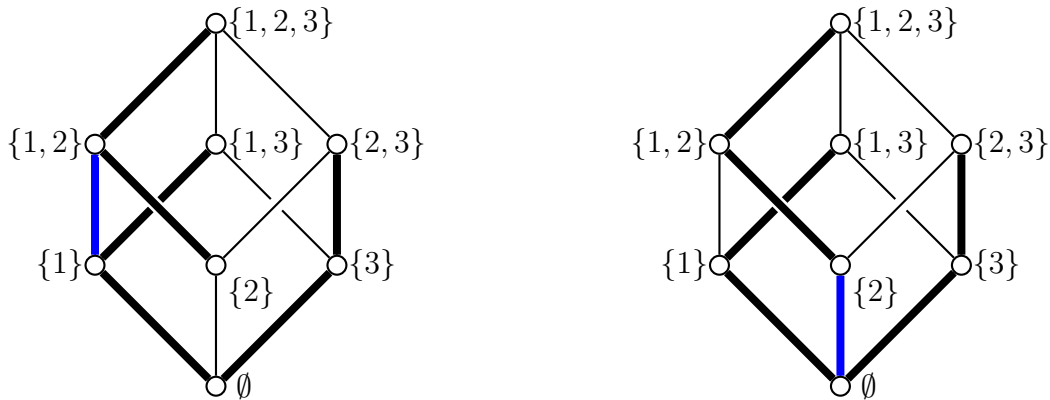


Figure 1: A pair of spanning trees of Q_3 with signature $(2, 2, 3)$. The two trees are related by an edge slide in direction 1 (Section 2.4): the tree on the right is obtained from the tree on the left by deleting the edge $\{\{1\}, \{1, 2\}\}$, and replacing it with the edge $\{\emptyset, \{2\}\}$. The tree on the right is upright (Section 2.5), with associated section defined by $\psi_T([3]) = 3$, $\psi_T(\{1, 2\}) = 1$, $\psi_T(\{1, 3\}) = 3$, $\psi_T(\{2, 3\}) = 2$, and $\psi_T(\{i\}) = i$ for $1 \leq i \leq 3$.

Example 2.4 (Signatures in low dimensions). The 1-cube Q_1 has a unique spanning tree, with signature (1) . The 2-cube has a total of four spanning trees: two with each of the signatures $(1, 2)$ and $(2, 1)$. The 3-cube Q_3 has three signatures up to permutation, namely $(1, 2, 4)$, $(1, 3, 3)$ and $(2, 2, 3)$. There are 16 spanning trees with signature $(1, 2, 4)$; 32 with signature $(1, 3, 3)$; and 64 with signature $(2, 2, 3)$, for a total of $6 \cdot 16 + 3 \cdot 32 + 3 \cdot 64 = 384$ spanning trees of Q_3 .

2.4 Edge slides and the edge slide graph

For each $i \in [n]$ we define σ_i to be the automorphism of Q_n defined for each $X \in \mathcal{P}([n])$ by

$$\sigma_i(X) = X \oplus \{i\},$$

where \oplus denotes symmetric difference.

Definition 2.5 (Tuffley [12]). Let T be a spanning tree of Q_n , and let e be an edge of T in a direction $j \neq i$ such that T does not also contain $\sigma_i(e)$. We say that e is **i -slidable** or **slidable in direction i** if deleting e from T and replacing it with $\sigma_i(e)$ yields a second spanning tree T' ; that is, if $T' = T - e + \sigma_i(e)$ is a spanning tree.

Example 2.6. Figure 1 illustrates an edge slide. The tree on the right is obtained from the tree on the left by deleting the edge $e = \{\{1\}, \{1, 2\}\}$, and replacing it with the edge $\sigma_1(e) = \{\emptyset, \{2\}\}$. This constitutes an edge slide in direction 1. We visualise it as “sliding” the edge e in direction 1 across the 2-dimensional face induced by $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Edge slides are a specialisation of Goddard and Swart’s *edge move* [5] to the n -cube, in which the edges involved in the move are constrained by the structure of the cube. We visualise them as the operation of “sliding” an edge across a 2-dimensional face of the cube to get a second spanning tree, as seen in Example 2.6 and Figure 1.

Slidable edges may be characterised as follows:

Lemma 2.7. *Let T be a spanning tree of Q_n , and let e be an edge of T in direction $j \neq i$. Then e is i -slidable if and only if $\sigma_i(e)$ does not belong to T , and the cycle C in $T + \sigma_i(e)$ created by adding $\sigma_i(e)$ to T contains both e and $\sigma_i(e)$, and so is broken by deleting e .*

We define the *edge slide graph* of Q_n in terms of edge slides:

Definition 2.8 (Tuffley [12]). The **edge slide graph** of Q_n is the graph $\mathcal{E}(Q_n)$ with vertex set $\text{Tree}(Q_n)$, and an edge between trees T_1 and T_2 if and only if T_2 may be obtained from T_1 by a single edge slide.

For a connected graph G the *tree graph* [5] of G is the graph $T(G)$ on the spanning trees of G , with an edge between two trees if they’re related by an edge move. The edge slide graph $\mathcal{E}(Q_n)$ is therefore a subgraph of the tree graph $T(Q_n)$. The tree graph $T(Q_n)$ is connected, because $T(G)$ is easily shown to be connected for any connected graph G . In contrast, $\mathcal{E}(Q_n)$ is disconnected for all $n \geq 2$: edge slides do not change the signature, so the signature is constant on connected components. Accordingly, we make the following definition:

Definition 2.9. Let \mathcal{S} be a signature of Q_n . The **edge slide graph** of \mathcal{S} is the subgraph $\mathcal{E}(\mathcal{S})$ of $\mathcal{E}(Q_n)$ induced by the spanning trees with signature \mathcal{S} . If \mathcal{X} is a set of signatures, we further define

$$\mathcal{E}(\mathcal{X}) = \bigcup_{\mathcal{S} \in \mathcal{X}} \mathcal{E}(\mathcal{S}).$$

By our discussion above, for each signature \mathcal{S} the edge slide graph $\mathcal{E}(\mathcal{S})$ is a union of one or more connected components of $\mathcal{E}(Q_n)$. We say that \mathcal{S} is **connected** or **disconnected** according to whether $\mathcal{E}(\mathcal{S})$ is connected or disconnected. In Section 4 we classify signatures as irreducible, quasi-irreducible or strictly reducible. We prove in Theorem 7.1 that every strictly reducible signature is disconnected, and conjecture that \mathcal{S} is connected if and only if \mathcal{S} is irreducible or quasi-irreducible. If true, this would imply that the connected components of $\mathcal{E}(Q_n)$ can be characterised in terms of signatures of spanning trees of subcubes. By Theorem 8.3 it suffices to show that every irreducible signature is connected.

2.5 Upright trees and sections

Upright trees are a natural family of spanning trees of Q_n that are easily understood.

Definition 2.10 (Tuffley [12]). Root all spanning trees of Q_n at \emptyset . A spanning tree T of Q_n is **upright** if for each vertex X of Q_n the path in T from X to the root has length $|X|$.

Equivalently, T is upright if for every vertex X of T , the first vertex Y on the path in T from X to the root satisfies $Y \subseteq X$. Let $Y = X - \{i\}$, and set $\psi_T(X) = i$. Then ψ_T defines a function $\mathcal{P}_{\geq 1}^n \rightarrow [n]$ such that $\psi_T(X) \in X$ for all $X \in \mathcal{P}_{\geq 1}^n$. We call such a function a *section* of $\mathcal{P}_{\geq 1}^n$:

Definition 2.11 (Tuffley [12]). A function $\psi : \mathcal{P}_{\geq 1}^n \rightarrow [n]$ such that $\psi(X) \in X$ for all X is a **section** of $\mathcal{P}_{\geq 1}^n$. If ψ is a section then the signature of ψ is the n -tuple

$$\text{sig}(\psi) = (a_1, \dots, a_n)$$

such that $a_i = |\{X : \psi(X) = i\}|$ for all i .

It is clear that upright trees are equivalent to sections:

Theorem 2.12 (Tuffley [12, Lemma 11] for $n = 3$, and Al Fran [1, Lemma 2.2.27] for arbitrary n). *The correspondence $T \leftrightarrow \psi_T$ is a bijection between the set of upright spanning trees of Q_n and the set of sections of $\mathcal{P}_{\geq 1}^n$. Moreover $\text{sig}(T) = \text{sig}(\psi_T)$ for all T .*

3 Characterisation of signatures of spanning trees of Q_n

In this section we use Hall’s Theorem to prove the following characterisation of the n -tuples $\mathcal{S} = (a_1, a_2, \dots, a_n)$ that are the signature of a spanning tree of Q_n .

Theorem 3.1. *Let $\mathcal{S} = (a_1, a_2, \dots, a_n)$, where $1 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq 2^{n-1}$ and $\sum_{i=1}^n a_i = 2^n - 1$. Then \mathcal{S} is the signature of a spanning tree of Q_n if and only if $\sum_{j=1}^k a_j \geq 2^k - 1$, for all $k \leq n$.*

Remark 3.2. Since $\sum_{i=1}^n a_i = 2^n - 1$, the signature condition of Theorem 3.1 is equivalent to

$$\sum_{j=k+1}^n a_j \leq 2^n - 2^k = 2^k(2^{n-k} - 1)$$

for all $1 \leq k \leq n$.

Example 3.3 (Signatures of Q_4). Applying Theorem 3.1 with $n = 4$ we find that there are 18 ordered signatures of Q_4 :

- (1, 2, 4, 8) (1, 2, 5, 7) (1, 3, 5, 6) (2, 2, 4, 7) (2, 3, 4, 6) (3, 3, 3, 6)
- (1, 3, 3, 8) (1, 2, 6, 6) (1, 4, 4, 6) (2, 2, 5, 6) (2, 3, 5, 5) (3, 3, 4, 5)
- (2, 2, 3, 8) (1, 3, 4, 7) (1, 4, 5, 5) (2, 3, 3, 7) (2, 4, 4, 5) (3, 4, 4, 4)

We will discuss the classification of these signatures in Example 4.11, and the reason for organising them in this way will become apparent then.

The first step in the proof of Theorem 3.1 is to reduce it to the problem of characterising signatures of sections of $\mathcal{P}_{\geq 1}^n$:

Lemma 3.4. *The n -tuple $\mathcal{S} = (a_1, a_2, \dots, a_n)$ is the signature of a spanning tree of Q_n if and only if it is the signature of a section of $\mathcal{P}_{\geq 1}^n$.*

We give two independent proofs of this fact: one using Martin and Reiner’s weighted count [10] of spanning trees of Q_n , and the second via edge slides and upright trees.

Proof 1 of Lemma 3.4, via Martin and Reiner’s weighted count.

By Martin and Reiner [10] we have

$$\sum_{T \in \text{Tree}(Q_n)} q^{\text{dir}(T)} x^{\text{dd}(T)} = q_1 \cdots q_n \prod_{S \in \mathcal{P}_{\geq 2}^n} \sum_{i \in S} q_i (x_i^{-1} + x_i),$$

in which

$$q^{\text{dir}(T)} = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n} \quad \Leftrightarrow \quad \text{sig}(T) = (a_1, a_2, \dots, a_n).$$

Set $x_i = 1$ for all i to get

$$\begin{aligned} \sum_{T \in \text{Tree}(Q_n)} q^{\text{dir}(T)} &= q_1 \cdots q_n \prod_{S \in \mathcal{P}_{\geq 2}^n} \sum_{i \in S} 2q_i \\ &= 2^{2^n - n - 1} \prod_{S \in \mathcal{P}_{\geq 1}^n} \sum_{i \in S} q_i. \end{aligned}$$

Each term in the expansion corresponds to a choice of $i \in S$ for each nonempty subset S of $[n]$, and hence to a section of $\mathcal{P}_{\geq 1}^n$. □

Proof 2 of Lemma 3.4, via edge slides and upright trees. By Tuffley [12, Cor. 15], each spanning tree of Q_n is connected to an upright spanning tree by a sequence of edge slides. The signature is invariant under edge slides, so we conclude that \mathcal{S} is the signature of a spanning tree if and only if it is the signature of an upright tree. But upright spanning trees are equivalent to sections of $\mathcal{P}_{\geq 1}^n$, and the equivalence is signature-preserving. □

Recall that Hall’s Theorem may be stated as follows (e.g. see [3, Thm 11.13]):

Theorem 3.5 (Hall [7]). *Let $G = (A, B)$ be a bipartite graph with $|A| = |B|$. Then G has a perfect matching if and only if for all nonempty $Y \subseteq A$ we have $|Y| \leq |N(Y)|$, where $N(Y) \subseteq B$ is the neighbourhood of Y in G .*

If the stronger condition $|Y| < |N(Y)|$ holds for all proper nonempty Y , then for any $a \in A$ and $b \in N(A)$ one may show there exists a perfect matching such that a is matched with b . We use this idea to prove our results of Section 5.2.

We now prove Theorem 3.1. The proof is illustrated in Figure 2.

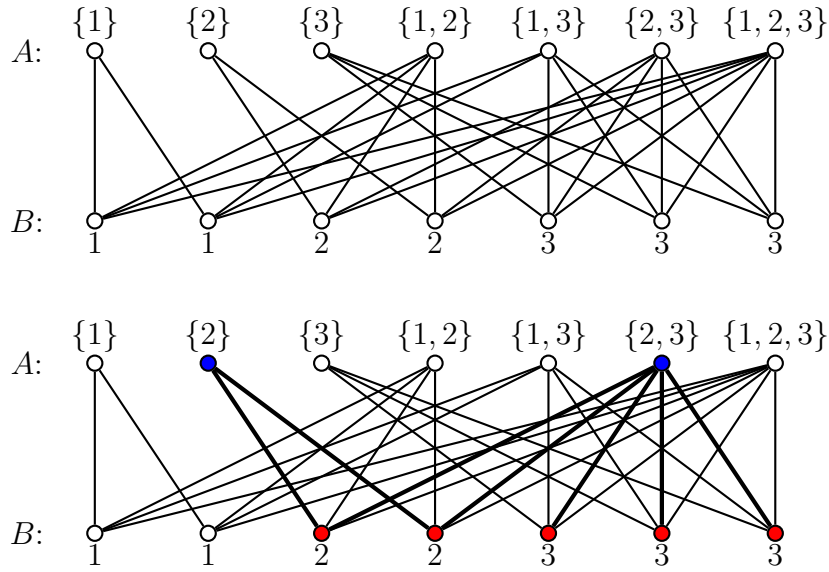


Figure 2: Illustrating the proof of Theorem 3.1, in the case $\mathcal{S} = (2, 2, 3)$. *Upper figure:* The matching graph $G_{\mathcal{S}}$. *Lower figure:* Checking the Hall condition $|N(Y)| \geq |Y|$ for $Y = \{\{2\}, \{2, 3\}\} \subseteq A$ (filled vertices in A). We have $\text{supp}(Y) = \{2\} \cup \{2, 3\} = \{2, 3\}$, so the neighbourhood of Y consists of all vertices in B labelled 2 or 3 (filled vertices in B). Consequently $|N(Y)| = \sum_{i \in \text{supp}(Y)} a_i = a_2 + a_3$. Since \mathcal{S} is ordered $|N(Y)| = a_2 + a_3 \geq a_1 + a_2 = 4$. On the other hand, Y is a nonempty subset of $\mathcal{P}_{\geq 1}(\text{supp}(Y))$, so $|Y| \leq |\mathcal{P}_{\geq 1}(\text{supp}(Y))| = 2^2 - 1 = 3$. The Hall condition for $Y = \{\{2\}, \{2, 3\}\}$ therefore follows from the condition $\sum_{i=1}^k a_i \geq 2^k - 1$ of Theorem 3.1, with $k = |\text{supp}(Y)| = 2$.

Proof of Theorem 3.1. Let A be $\mathcal{P}_{\geq 1}^n$, the set of $2^n - 1$ nonempty vertices of Q_n , and let B be a set of $2^n - 1$ vertices of which a_i are labelled i , for each $i \in [n]$. For each vertex V in A and $i \in V$ we draw an edge to every vertex in B labelled i , as shown in Figure 2 for the case $\mathcal{S} = (2, 2, 3)$. Let $G_{\mathcal{S}}$ be the resulting bipartite graph with bipartition (A, B) . A section of $\mathcal{P}_{\geq 1}^n$ with signature \mathcal{S} corresponds to a perfect matching in $G_{\mathcal{S}}$, so we show there is a perfect matching in $G_{\mathcal{S}}$ if and only if the signature condition $\sum_{j=1}^k a_j \geq 2^k - 1$ is satisfied for all $k \leq n$.

Given a nonempty subset Y of A , define the *support* of Y to be the set

$$\text{supp}(Y) = \bigcup_{V \in Y} V.$$

Suppose that $\text{supp}(Y) = \{i_1, i_2, \dots, i_k\}$, where $1 \leq i_1 < \dots < i_k \leq n$. Then $i_j \geq j$ for $1 \leq j \leq k$, which implies $a_{i_j} \geq a_j$ because \mathcal{S} is ordered. It follows that the

neighbourhood $N(Y)$ of Y in G_S satisfies

$$\begin{aligned}
 |N(Y)| &= \sum_{i \in \text{supp}(Y)} a_i = \sum_{j=1}^k a_{i_j} \\
 &\geq \sum_{j=1}^k a_j
 \end{aligned}$$

with equality if $\text{supp}(Y) = \{1, \dots, k\}$. Also

$$|Y| \leq |\mathcal{P}_{\geq 1}(\text{supp}(Y))| = 2^k - 1,$$

with equality if and only if $Y = \mathcal{P}_{\geq 1}(\text{supp}(Y))$. Then we conclude that $|N(Y)| \geq |Y|$ for all $Y \subseteq A$ if and only if $\sum_{j=1}^k a_j \geq 2^k - 1$ for all $k \leq n$. Thus by Hall’s Theorem there exists a perfect matching in G_S if and only if $\sum_{j=1}^k a_j \geq 2^k - 1$ for all k . \square

We conclude this section by proving a lower bound on the growth of an ordered signature.

Lemma 3.6. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be an ordered signature of Q_n . Then $i \leq a_i$ for all $i \in [n]$.*

Proof. We use the fact easily proved by induction that $m(m - 1) < 2^m - 1$ for all m . Let $j < i$. Since \mathcal{S} is ordered we have $a_j \leq a_i$, and therefore $2^i - 1 \leq \sum_{j=1}^i a_j \leq ia_i$.

Suppose that $i > a_i$. Then $a_i \leq i - 1$ and so $2^i - 1 \leq i(i - 1)$, contradicting the fact that $i(i - 1) < 2^i - 1$. Therefore $i > a_i$ is impossible, so $i \leq a_i$. \square

4 Classification of signatures of spanning trees of Q_n

We classify signatures of Q_n as reducible or irreducible as follows.

Definition 4.1. Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of a spanning tree of Q_n . Then \mathcal{S} is **reducible** if there exists a proper nonempty subset R of $[n]$ such that $\sum_{i \in R} a_i = 2^{|R|} - 1$. We say that R is a **reducing set** for \mathcal{S} , and that \mathcal{S} **reduces over R** . If no such set exists then \mathcal{S} is **irreducible**.

By extension, we will say that a spanning tree T is reducible or irreducible according to whether $\text{sig}(T)$ is reducible or irreducible. If $\text{sig}(T)$ is reducible with reducing set R , we will say that T **reduces over R** .

Note that if \mathcal{S} is irreducible then $a_i \geq 2$ for all i , because if $a_i = 1$ then \mathcal{S} reduces over $\{i\}$.

Remark 4.2. If \mathcal{S} is ordered and $R \subseteq [n]$ satisfies $|R| = r$ then

$$\sum_{i \in R} a_i \geq \sum_{i=1}^r a_i.$$

It follows that an ordered signature has a reducing set of size r if and only if $[r]$ itself is a reducing set. If this holds then we have $\sum_{i=1}^r a_i = 2^r - 1$, and moreover $\sum_{i=1}^k a_i \geq 2^k - 1$ for $1 \leq k \leq r$, by the signature condition for \mathcal{S} . It follows that $\mathcal{S}' = (a_1, \dots, a_r)$ is a signature of Q_r . Thus, an ordered signature is reducible if and only if it has a initial segment that is a signature of a lower dimensional cube. More generally, a not-necessarily ordered signature \mathcal{S} is reducible if and only if there is a proper nonempty subset R of $[n]$ such that the restriction of \mathcal{S} to the indices in R gives a signature of Q_R .

Example 4.3. Consider the following ordered signatures of Q_7 :

$$\begin{aligned} \mathcal{S}_1 &= (2, 2, 4, 8, 16, 32, 63), & \mathcal{S}_3 &= (2, 2, 4, 8, 15, 32, 64), \\ \mathcal{S}_2 &= (2, 2, 3, 9, 15, 33, 63), & \mathcal{S}_4 &= (2, 2, 3, 9, 15, 32, 64). \end{aligned}$$

The signature \mathcal{S}_1 is irreducible, and the rest are reducible. Signature \mathcal{S}_2 reduces over $[3]$ and $[5]$; signature \mathcal{S}_3 reduces over $[5]$ and $[6]$; and signature \mathcal{S}_4 reduces over $[3]$, $[5]$ and $[6]$.

Definition 4.4. Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n and let $1 \leq k \leq n$. We define the **excess of \mathcal{S} at k** , $\varepsilon_k^{\mathcal{S}}$, to be

$$\varepsilon_k^{\mathcal{S}} = \min_{\substack{K \subseteq [n] \\ |K|=k}} \left(\sum_{i \in K} a_i \right) - (2^k - 1).$$

Thus, the excess at k is the minimum quantity by which a set of k directions exceeds the matching condition of Hall’s Theorem. Consequently, \mathcal{S} is irreducible if and only if $\varepsilon_k^{\mathcal{S}} \geq 1$ for all $k \leq n - 1$, and is reducible if and only if $\varepsilon_k^{\mathcal{S}} = 0$ for some $k \leq n - 1$. Note that by definition $\varepsilon_n^{\mathcal{S}} = 0$, and if \mathcal{S} is ordered then the excess at k is simply given by

$$\varepsilon_k^{\mathcal{S}} = \left(\sum_{i=1}^k a_i \right) - (2^k - 1).$$

Remark 4.5. Observe that for an ordered signature $\mathcal{S} = (a_1, \dots, a_n)$ of Q_n and $r < n$, the following statements are equivalent:

1. (a_1, \dots, a_r) is a signature of Q_r .
2. \mathcal{S} reduces over $[r]$.
3. $\varepsilon_r^{\mathcal{S}} = 0$.
4. $\sum_{i=1}^r a_i = 2^r - 1$.

Note further that if $\varepsilon_{k-1}^{\mathcal{S}} = \varepsilon_k^{\mathcal{S}} = 0$, then $a_k = 2^{k-1}$.

Reducible signatures of Q_n can be divided into two types: *strictly reducible* and *quasi-irreducible* signatures. In order to define these we first introduce the notion of *saturated* and *unsaturated* signatures as follows.

Definition 4.6. Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n . If there exists $r < n$ such that $\varepsilon_k^{\mathcal{S}} = 0$ for all $r \leq k \leq n$, then \mathcal{S} is a **saturated** signature. If no such index exists then \mathcal{S} is **unsaturated**. Equivalently, \mathcal{S} is saturated if and only if it reduces over a set of size $n - 1$.

If \mathcal{S} is ordered and $\varepsilon_k^{\mathcal{S}} = 0$ for all $r \leq k \leq n$, then we further say that \mathcal{S} is **saturated above direction r** .

Note that a saturated signature is necessarily reducible. If the ordered signature \mathcal{S} is saturated above direction r then by Remark 4.5 we have $a_k = 2^{k-1}$ for $r+1 \leq k \leq n$, and moreover the k -tuple (a_1, \dots, a_k) is a signature of Q_k for $r \leq k \leq n$. We may therefore make the following definition:

Definition 4.7. Let $\mathcal{S} = (a_1, \dots, a_n)$ be an ordered signature of Q_n , and let $1 \leq s \leq n$ be the least index such that $\varepsilon_k^{\mathcal{S}} = 0$ for all $s \leq k \leq n$ (such an s exists because $\varepsilon_n^{\mathcal{S}} = 0$). Then the s -tuple $\text{unsat}(\mathcal{S})$ defined by

$$\text{unsat}(\mathcal{S}) = (a_1, \dots, a_s)$$

is necessarily an unsaturated signature of Q_s , and is the **unsaturated part** of \mathcal{S} .

If \mathcal{S} is not ordered we define $\text{unsat}(\mathcal{S})$ to be the restriction of \mathcal{S} to the entries appearing in the unsaturated part of an ordered permutation \mathcal{S}' of \mathcal{S} . Write $\mathcal{S}' = (a'_1, \dots, a'_n)$, and suppose that $\text{unsat}(\mathcal{S}') = (a'_1, \dots, a'_s)$. Then

$$\mathcal{S}' = (a'_1, \dots, a'_s, 2^s, 2^{s+1}, \dots, 2^{n-1}),$$

and $a'_i < 2^{s-1}$ for $1 \leq i \leq s$. Thus, $\text{unsat}(\mathcal{S})$ is the restriction of \mathcal{S} to the entries satisfying $a_i < 2^{s-1}$. Moreover, while there may be more than one permutation of $[n]$ that puts \mathcal{S} in nondecreasing order (where there are indices $i \neq j$ such that $a_i = a_j$), there is no ambiguity in which indices occur in the unsaturated part.

We use the unsaturated part to divide reducible signatures into quasi-irreducible and strictly reducible signatures:

Definition 4.8. Let \mathcal{S} be a reducible signature of Q_n . Then \mathcal{S} is **quasi-irreducible** if the unsaturated part $\text{unsat}(\mathcal{S})$ is irreducible. Otherwise, \mathcal{S} is **strictly reducible**.

By extension, we will say that a reducible spanning tree T of Q_n is quasi-irreducible or strictly reducible according to whether $\text{sig}(T)$ is quasi-irreducible or strictly reducible.

Example 4.9. For the signatures appearing in Example 4.3 we have

$$\begin{aligned} \text{unsat}(\mathcal{S}_1) &= \mathcal{S}_1, & \text{unsat}(\mathcal{S}_3) &= (2, 2, 4, 8, 15), \\ \text{unsat}(\mathcal{S}_2) &= \mathcal{S}_2, & \text{unsat}(\mathcal{S}_4) &= (2, 2, 3, 9, 15). \end{aligned}$$

Signatures \mathcal{S}_1 and \mathcal{S}_2 are unsaturated, while \mathcal{S}_3 and \mathcal{S}_4 are both saturated above direction 5. Signatures \mathcal{S}_2 and \mathcal{S}_4 are strictly reducible (their unsaturated parts both have $[3]$ as a reducing set), and signature \mathcal{S}_3 is quasi-irreducible.

Example 4.10 (Classification of signatures in low dimensions). The unique signature (1) of Q_1 is irreducible. Up to permutation Q_2 has the unique signature (1, 2), which is reducible and saturated, with unsaturated part (1). It is therefore quasi-irreducible. The signatures of Q_3 up to permutation are (1, 2, 4), (1, 3, 3) and (2, 2, 3), which are respectively quasi-irreducible, strictly reducible and irreducible. Of these only (1, 2, 4) is saturated.

Example 4.11 (Classification of signatures of Q_4). Consider again the signatures of Q_4 from Example 3.3:

$$\begin{array}{cccccc} (1, 2, 4, 8) & (1, 2, 5, 7) & (1, 3, 5, 6) & (2, 2, 4, 7) & (2, 3, 4, 6) & (3, 3, 3, 6) \\ (1, 3, 3, 8) & (1, 2, 6, 6) & (1, 4, 4, 6) & (2, 2, 5, 6) & (2, 3, 5, 5) & (3, 3, 4, 5) \\ (2, 2, 3, 8) & (1, 3, 4, 7) & (1, 4, 5, 5) & (2, 3, 3, 7) & (2, 4, 4, 5) & (3, 4, 4, 4) \end{array}$$

The signatures in the first column all have [3] as a reducing set, while those in the second and third columns all have [1] as a reducing set. Thus these nine signatures are reducible. The nine signatures appearing in the last three columns are all irreducible.

The signatures in the first column are obtained by appending $2^3 = 8$ to a signature of Q_3 (equivalently, have [3] as a reducing set), so are saturated. The remaining signatures are unsaturated. The reducible signatures in the second and third columns are therefore strictly reducible. For the saturated signatures, we have

$$\begin{aligned} \text{unsat}(1, 2, 4, 8) &= (1), \\ \text{unsat}(1, 3, 3, 8) &= (1, 3, 3), \\ \text{unsat}(2, 2, 3, 8) &= (2, 2, 3), \end{aligned}$$

so just (1, 3, 3, 8) is strictly reducible, and the other two are quasi-irreducible.

The signatures (1), (1, 2), (1, 2, 4) and (1, 2, 4, 8) seen above are the first four members of an infinite family of signatures:

Definition 4.12. For $n \geq 1$ let \mathcal{SS}_n be the n -tuple (a_1, \dots, a_n) defined by $a_i = 2^{i-1}$ for $1 \leq i \leq n$:

$$\mathcal{SS}_n = (1, 2, 4, 8, \dots, 2^{n-1}).$$

Then $\sum_{i=1}^k a_i = 2^k - 1$ for all $1 \leq k \leq n$, so \mathcal{SS}_n is a signature of Q_n .

Observe that \mathcal{SS}_n satisfies $\varepsilon_k^{\mathcal{SS}_n} = 0$ for $1 \leq k \leq n$. It follows that \mathcal{SS}_n is saturated above direction 1 for all $n \geq 2$, and $\text{unsat}(\mathcal{SS}_n) = (1)$ for all n . For $n \geq 2$ we will say that \mathcal{SS}_n is *supersaturated*:

Definition 4.13. Let $n \geq 2$. A signature $\mathcal{S} = (a_1, \dots, a_n)$ is **supersaturated** if $\varepsilon_k^{\mathcal{S}} = 0$ for all $1 \leq k \leq n$. Equivalently, \mathcal{S} is supersaturated if and only if it is a permutation of \mathcal{SS}_n .

5 Consequences of the classification for upright trees

In this section and the next we show that reducibility places strict structural constraints on a spanning tree of Q_n . We begin by restricting our attention to upright spanning trees, which are easily understood through their equivalence with sections of $\mathcal{P}_{\geq 1}^n$.

By identifying each upright tree T with its associated section ψ_T , we may regard an upright tree as a choice of $x \in X$ at each nonempty subset X of $[n]$. We may ask the following question:

Question 5.1. *Given a signature \mathcal{S} of Q_n , a nonempty subset X of $[n]$, and an element x of X , does there exist an upright spanning tree T with signature \mathcal{S} such that $\psi_T(X) = x$?*

Lemma 5.2 shows that, if \mathcal{S} is reducible, then it is always possible to choose a nonempty subset X of $[n]$ and an element x of X such that the answer to this question is “no”. In contrast, Corollary 5.7 shows that for irreducible \mathcal{S} , the answer to this question is always “yes”, regardless of the choice of nonempty $X \subseteq [n]$ and $x \in X$. This means that we may arbitrarily specify the value of a section with irreducible signature \mathcal{S} at any single vertex of our choice. We further show that under certain conditions (typically expressed in terms of the excess) we can specify the value of a section with signature \mathcal{S} at one or more additional vertices.

5.1 Reducible upright trees

We show that reducibility constrains the edges of an upright spanning tree:

Lemma 5.2. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a reducible signature of Q_n , and let R be a reducing set for \mathcal{S} . Let T be an upright spanning tree of Q_n with signature \mathcal{S} and let X be a nonempty vertex of Q_n . Then $\psi_T(X) \in R$ if and only if $X \subseteq R$.*

This answers Question 5.1 for reducible signatures, by showing that if $x \in [n]$ is chosen such that $x \in R$, then the answer is “yes” only if $X \subseteq R$.

Proof. The fact that $\psi_T(X) \in R$ for $X \subseteq R$ is immediate from the fact that ψ_T is a section. For the converse, observe that in total T has $\sum_{i \in R} a_i = 2^{|R|} - 1$ edges in directions belonging to R , and R has $2^{|R|} - 1$ nonempty subsets. Thus all edges of T in directions belonging to R are accounted for at the subsets of R , so we must have $\psi_T(X) \notin R$ for $X \not\subseteq R$. □

Applying Lemma 5.2 to an ordered saturated signature we get:

Corollary 5.3. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be an ordered signature. If \mathcal{S} is saturated above direction r and $X \not\subseteq [r]$, then $\psi_T(X) = \max X$.*

Proof. Since \mathcal{S} is saturated above direction r , it reduces over $[s - 1]$ for each $s > r$. If $\max X = s$, then $X \subseteq [s]$ but $X \not\subseteq [s - 1]$. Therefore $\psi_T(X)$ belongs to $[s]$ but not $[s - 1]$, and hence $\psi_T(X) = s = \max X$. □

Corollary 5.4. *Let the ordered signature $\mathcal{S} = (a_1, \dots, a_n)$ of Q_n be saturated above direction r . Then the number of upright spanning trees of Q_n with signature \mathcal{S} is equal to the number of upright spanning trees of Q_r with signature $\mathcal{S}' = (a_1, \dots, a_r)$.*

In particular, if the unsaturated part of \mathcal{S} consists of the first s entries of \mathcal{S} , then the number of upright spanning trees of Q_n with signature \mathcal{S} is equal to the number of upright spanning trees of Q_s with signature $\text{unsat}(\mathcal{S})$.

Proof. Given an upright spanning tree T of Q_n with signature \mathcal{S} let $T' = T \cap Q_r$. Then T' is an upright spanning tree of Q_r with associated section $\psi_{T'} = \psi_T|_{\mathcal{P}_{\geq 1}^r}$, the restriction of ψ_T to $\mathcal{P}_{\geq 1}^r$. Since \mathcal{S} reduces over $[r]$ Lemma 5.2 implies $\psi_T(X) \in [r]$ if and only if $X \subseteq [r]$, and it follows that $\text{sig}(T') = \mathcal{S}'$.

Conversely, given an upright spanning tree T' of Q_r with signature \mathcal{S}' , we can extend T' to an upright spanning tree T of Q_n such that $T' = T \cap Q_r$ by defining

$$\psi_T(X) = \begin{cases} \psi_{T'}(X), & \text{if } X \subseteq R, \\ \max X, & \text{otherwise.} \end{cases}$$

For each $1 \leq k \leq n$ there are 2^{k-1} subsets X of $[n]$ such that $\max X = k$, so $\text{sig}(T) = (a_1, \dots, a_r, 2^r, 2^{r+1}, \dots, 2^{n-1}) = \mathcal{S}$. Moreover, Corollary 5.3 shows that any upright spanning tree of Q_n with signature \mathcal{S} that extends T' must co-incide with T . It follows that the map $T \mapsto T \cap Q_r$ is a bijection from the set of upright spanning trees of Q_n with signature \mathcal{S} to the set of upright spanning trees of Q_r with signature \mathcal{S}' , proving the result. □

Corollary 5.5. *There is only one upright spanning tree of Q_n with the supersaturated signature $\mathcal{SS}_n = (1, 2, 4, 8, \dots, 2^{n-1})$.*

Proof. The signature \mathcal{SS}_n satisfies $\text{unsat}(\mathcal{SS}_n) = (1)$. The signature (1) has a unique upright tree, so the result follows immediately from Corollary 5.4. □

5.2 Irreducible upright trees

We now consider irreducible upright spanning trees, and show that in contrast to Lemma 5.2, for \mathcal{S} irreducible the answer to Question 5.1 is always “yes”: given nonempty $X \subseteq [n]$ and $x \in X$, there always exists an upright spanning tree T with signature \mathcal{S} such that $\psi_T(X) = x$. Since irreducible signatures satisfy $\varepsilon_k^{\mathcal{S}} \geq 1$ for all $k < n$, we deduce this as a corollary to Theorem 5.6, which loosely speaking says that if $\varepsilon_k^{\mathcal{S}} \geq \ell$ for all $k < n$, then we may arbitrarily specify the value of a section with signature \mathcal{S} at any ℓ vertices of our choice. In fact, the condition $\varepsilon_k^{\mathcal{S}} \geq \ell$ for all $k < n$ appears to be a little stronger than necessary. For $\ell = 2$ we show in Theorem 5.9 that, under certain conditions, we can specify the value of a section at two vertices even when we do not have $\varepsilon_k^{\mathcal{S}} \geq 2$ for all k . To prove this result we require Lemma 5.8, which shows that when a_k and a_{k+1} are close enough, the excess at k must be at least 2.

Theorem 5.6. *Let ℓ be a positive integer, and let \mathcal{S} be a signature of Q_n such that $\varepsilon_k^{\mathcal{S}} \geq \ell$ for $1 \leq k < n$. Let X_1, \dots, X_ℓ be distinct nonempty vertices of Q_n , and let*

$x_t \in X_t$ for each t . Then there is an upright spanning tree T of Q_n with signature \mathcal{S} such $\psi_T(X_t) = x_t$ for $1 \leq t \leq \ell$.

Proof. Let $G_{\mathcal{S}}$ be the matching graph with bipartition (A, B) constructed in the proof of Theorem 3.1. By hypothesis we have

$$\varepsilon_1^{\mathcal{S}} = \min_{i \in [n]} a_i - 1 \geq \ell,$$

so $a_i \geq \ell + 1$ for all i . It follows that there exists a partial matching M in $G_{\mathcal{S}}$ such that X_t is matched with a vertex $v_t \in B$ labelled x_t for $1 \leq t \leq \ell$. Let $G'_{\mathcal{S}}$ be the matching graph with the vertices $X_1, \dots, X_{\ell}, v_1, \dots, v_{\ell}$ and all incident edges deleted. We show that M can be extended to a perfect matching in $G_{\mathcal{S}}$ by showing that there exists a perfect matching in $G'_{\mathcal{S}}$.

Let $A' = A - \{X_1, \dots, X_{\ell}\}$, $B' = B - \{v_1, \dots, v_{\ell}\}$, and for $1 \leq i \leq n$ let a'_i be the number of vertices labelled i in B' . Given $\emptyset \neq Y \subseteq A'$ let $Z = \text{supp}(Y)$, the union of the sets in Y . If $Z = [n]$ then $N(Y) = B'$, and since $|Y| \leq |A'| = |B'|$ the Hall condition holds for Y . Otherwise we have $|Y| \leq 2^{|Z|} - 1$ and $\varepsilon_{|Z|}^{\mathcal{S}} \geq \ell$, so

$$|N(Y)| = \sum_{i \in Z} a'_i \geq \left(\sum_{i \in Z} a_i \right) - \ell \geq (2^{|Z|} + \varepsilon_{|Z|}^{\mathcal{S}} - 1) - \ell \geq 2^{|Z|} - 1 \geq |Y|.$$

Therefore the Hall condition holds for all nonempty $Y \subseteq A'$, so $G'_{\mathcal{S}}$ has a perfect matching, as required. The resulting perfect matching in $G_{\mathcal{S}}$ extending M corresponds to a section ψ of $\mathcal{P}_{\geq 1}^n$ such that $\psi(X_t) = x_t$ for $1 \leq t \leq \ell$, proving the existence of the required upright spanning tree. \square

Specialising to the case $\ell = 1$ we answer Question 5.1 in the affirmative for irreducible signatures:

Corollary 5.7. *Let \mathcal{I} be an irreducible signature of Q_n . Let X be a nonempty vertex of Q_n and let $x \in X$. Then there exists an upright spanning tree T of Q_n with signature \mathcal{I} such that $\psi_T(X) = x$.*

Proof. Since \mathcal{I} is irreducible it satisfies $\varepsilon_k^{\mathcal{I}} \geq 1$ for $1 \leq k < n$. The result therefore follows immediately from Theorem 5.6. \square

We use the following lemma to show for $\ell = 2$ that the excess condition of Theorem 5.6 can be weakened slightly under certain conditions.

Lemma 5.8. *Let $n \geq 4$ and let $\mathcal{I} = (a_1, \dots, a_n)$ be an ordered irreducible signature of Q_n . Suppose that, for some $i \in \{2, \dots, n - 1\}$, we have $a_{i+1} - a_i \leq 1$. Then $\varepsilon_i^{\mathcal{I}} \geq 2$.*

We note that the condition $n \geq 4$ is necessary in Lemma 5.8. The irreducible signature $(2, 2, 3)$ with $i = 2$ is a counterexample for $n = 3$.

Proof. Since \mathcal{I} is irreducible we necessarily have $\varepsilon_i^{\mathcal{I}} \geq 1$. Suppose that $\varepsilon_i^{\mathcal{I}} = 1$. Then since \mathcal{I} is irreducible we have

$$2^i = \sum_{j=1}^i a_j = \sum_{j=1}^{i-1} a_j + a_i \geq 2^{i-1} + a_i,$$

and therefore $a_i \leq 2^{i-1}$. If $i < n - 1$ then

$$2^{i+1} \leq \sum_{j=1}^{i+1} a_j = \sum_{j=1}^i a_j + a_{i+1} = 2^i + a_{i+1},$$

which implies $a_{i+1} \geq 2^i$. But then $a_{i+1} - a_i \geq 2^{i-1} \geq 2$, a contradiction. Similarly, if $i = n - 1$ then

$$2^{i+1} - 1 \leq \sum_{j=1}^{i+1} a_j = \sum_{j=1}^i a_j + a_{i+1} = 2^i + a_{i+1},$$

which implies $a_{i+1} \geq 2^i - 1$. Then $a_{i+1} - a_i \geq 2^{i-1} - 1 = 2^{n-2} - 1 \geq 3$, and we again reach a contradiction. Therefore it must in fact be the case that $\varepsilon_i^{\mathcal{I}} \geq 2$. \square

For $\ell = 2$ we may weaken the excess condition of Theorem 5.6 as follows:

Theorem 5.9. *Let $n \geq 4$, and let $\mathcal{I} = (a_1, \dots, a_n)$ be an ordered irreducible signature of Q_n . Let X_1, X_2 be distinct nonempty vertices of Q_n , and let $x_t \in X_t$ for $t = 1, 2$. Suppose that one of the following two conditions holds:*

1. $\varepsilon_k^{\mathcal{I}} \geq 2$ for all $k \geq \max\{x_1, x_2\}$.
2. $x_1 \neq x_2$, and either $x_1 = \max X_1$ or $x_2 = \max X_2$.

Then there exists an upright spanning tree T of Q_n with signature \mathcal{I} such $\psi_T(X_t) = x_t$ for $t = 1, 2$.

Proof. Let $G_{\mathcal{I}}$ be the matching graph with bipartition (A, B) constructed in the proof of Theorem 3.1. Since \mathcal{I} is irreducible we have $a_i \geq 2$ for all i , so there exists a partial matching M of A into B such that X_t is matched with a vertex v_t labelled x_t for $t = 1, 2$. Let $G'_{\mathcal{I}}$ be the matching graph with the vertices X_1, X_2, v_1, v_2 and all incident edges deleted. We show that M can be extended to a perfect matching in $G_{\mathcal{I}}$ by showing that there exists a perfect matching in $G'_{\mathcal{I}}$.

Let $A' = A - \{X_1, X_2\}$, $B' = B - \{v_1, v_2\}$, and for $1 \leq i \leq n$ let a'_i be the number of vertices labelled i in B' . Given $\emptyset \neq Y \subseteq A'$ let $Z = \text{supp}(Y)$, the union of the sets in Y , and set $z = |Z|$. If $z = n$ then $N(Y) = B'$, and since $|Y| \leq |A'| = |B'|$ the Hall condition holds for Y . Otherwise we have

$$|N(Y)| = \sum_{i \in Z} a'_i = \left(\sum_{i \in Z} a_i \right) - \chi_Z(x_1) - \chi_Z(x_2), \tag{1}$$

where $\chi_Z : [n] \rightarrow \{0, 1\}$ is the characteristic function of Z ; and

$$|Y| \leq 2^z - 1 - \chi_{\mathcal{P}(Z)}(X_1) - \chi_{\mathcal{P}(Z)}(X_2) \leq 2^z - 1,$$

where $\chi_{\mathcal{P}(Z)} : \mathcal{P}([n]) \rightarrow \{0, 1\}$ is the characteristic function of $\mathcal{P}(Z)$. Since \mathcal{I} is irreducible we have $\sum_{i \in Z} a_i \geq 2^z$, so

$$|N(Y)| \geq 2^z - 2,$$

with equality possible only if $\sum_{i \in Z} a_i = 2^z$ and $x_1, x_2 \in Z$. On the other hand we have $|Y| \leq 2^z - 1$, with equality possible only if $X_1, X_2 \not\subseteq Z$ and $Y = \mathcal{P}_{\geq 1}(Z)$. It follows that $|N(Y)| \geq |Y|$ except possibly when $x_1, x_2 \in Z$ and $X_1, X_2 \not\subseteq Z$.

Suppose then that $x_1, x_2 \in Z$ but $X_1, X_2 \not\subseteq Z$. We show under each of the conditions given in the theorem that we have $\sum_{i \in Z} a_i \geq 2^z + 1$, so that $|N(Y)| \geq 2^z - 1 \geq |Y|$ as needed.

1. Suppose that $\varepsilon_k^{\mathcal{I}} \geq 2$ for all $k \geq m = \max\{x_1, x_2\}$. As in the proof of Theorem 3.1 let $Z = \{i_1, i_2, \dots, i_z\}$, where $i_1 < i_2 < \dots < i_z$. If $z \geq m$ then $\varepsilon_z^{\mathcal{I}} \geq 2$, so $\sum_{i \in Z} a_i \geq 2^z + 1$ and we are done. Otherwise we have $z < m$, and then $i_z > z$, because $m \in Z$ but $|Z| < m$. Therefore $a_{i_z} \geq a_{z+1}$, because \mathcal{I} is ordered. If $\sum_{i \in Z} a_i \geq 2^z + 1$ does not hold then

$$2^z \geq \sum_{i \in Z} a_i = \sum_{s=1}^z a_{i_s} \geq \sum_{s=1}^{z-1} a_s + a_{i_z} \geq \sum_{s=1}^{z-1} a_s + a_{z+1} \geq \sum_{s=1}^z a_s \geq 2^z, \quad (2)$$

and so $a_{i_z} = a_{z+1} = a_z$.

If $z \geq 2$ then by Lemma 5.8 we have $\varepsilon_z^{\mathcal{I}} \geq 2$, and so in fact

$$\sum_{i \in Z} a_i \geq \sum_{s=1}^z a_s \geq 2^z + 1$$

after all. Otherwise, if $z = 1$ then $a_i = a_1$ for $1 \leq i \leq i_z$, and so in particular for $1 \leq i \leq m$. But then if $a_1 = 2$ we have $\sum_{i=1}^m a_i = 2m < 2^m + 1$, contradicting our hypothesis that $\varepsilon_m^{\mathcal{I}} \geq 2$. Therefore $\sum_{i \in Z} a_i = a_m \geq 3 = 2^z + 1$ in this case also.

2. Under the hypothesis that $x_1 = \max X_1$ or $x_2 = \max X_2$ we may assume without loss of generality that $x_1 = \max X_1$. As above we let $Z = \{i_1, i_2, \dots, i_z\}$, where $i_1 < i_2 < \dots < i_z$, and we note that $z \geq 2$ because $x_1 \neq x_2$ and $\{x_1, x_2\} \subseteq Z$. Then since $x_1 = \max X_1 \in Z$ and $X_1 \not\subseteq Z$ it cannot be the case that $Z = [z]$, so as in Case 1 we have $i_z > z$ and hence $a_{i_z} \geq a_{z+1}$. Arguing as in Equation (2) we therefore get $a_{z+1} = a_z$, and then since $z \geq 2$ we again have $\sum_{i \in Z} a_i \geq 2^z + 1$, by Lemma 5.8.

Therefore the Hall condition holds for all nonempty $Y \subseteq A'$, so G'_T has a perfect matching, as required. The resulting perfect matching in G_T extending M corresponds to a section ψ of $\mathcal{P}_{\geq 1}^n$ such that $\psi(X_t) = x_t$ for $t = 1, 2$, proving the existence of the required upright spanning tree. □

For completeness we consider the extent to which the hypothesis $x_1 \neq x_2$ is necessary in Case 2 of Theorem 5.9. We show that this hypothesis can in fact be eliminated except in very limited circumstances:

Proposition 5.10. *Under the hypotheses of Theorem 5.9, suppose that $x_1 = x_2 = x$, and either $\max X_1$ or $\max X_2$ is equal to x . Then the conclusion of Theorem 5.9 still holds unless $x = 2$, $a_1 = a_2 = 2$, and (perhaps after permuting them) we have $X_1 = \{1, 2\}$ and $X_2 \not\subseteq \{1, 2\}$.*

Proof. In the proof of Case 2 of Theorem 5.9, the hypothesis $x_1 \neq x_2$ is used only to rule out the possibility $z = 1$ when $x_1, x_2 \in Z$ but $X_1, X_2 \not\subseteq Z$. We therefore check when $|Y| > |N(Y)|$ can hold under these conditions.

Since $z = |Z| = 1$ and $x \in Z$ we must have $Z = \{x\}$, which in turn implies $Y = \{\{x\}\}$. From equation (1) we have $|N(Y)| = a_x - 2$, so if $|Y| > |N(Y)|$ we must have $a_x \leq 2$. Irreducibility of \mathcal{I} rules out the possibility $a_x = 1$, so we must have $a_x = 2$, and then $x \leq 2$ by Lemma 3.6. We need not consider the case $x = 1$: if $x = 1$ then the hypothesis $x = \max X_t$ for some t implies either X_1 or X_2 is equal to $\{1\}$, which means that the vertex $\{1\}$ is already matched by M and does not belong to $G'_\mathcal{I}$. So suppose that $x = 2$. Then $a_1 = a_2 = 2$ by irreducibility, and the condition $x_t = \max X_t$ for $t = 1$ or 2 together with $X_1, X_2 \not\subseteq Z$ implies that (perhaps after relabelling) we have $X_1 = \{1, 2\}$ and $X_2 \not\subseteq \{1, 2\}$, as claimed. We see moreover that in this case the required matching in $G_\mathcal{I}$ does not in fact exist, because the three distinct vertices $\{2\}$, $X_1 = \{1, 2\}$ and X_2 must all be matched with vertices in B labelled 2, and there are only two such vertices. \square

6 Structural consequences of reducibility

We now turn our attention to the consequences of reducibility for arbitrary spanning trees. We give a structural characterisation of reducible trees in Section 6.2, then use this to show in Section 6.3 that a tree that reduces over a set of size r decomposes as a sum of 2^r spanning trees of Q_{n-r} , together with a spanning tree of a certain contraction of Q_n with underlying simple graph Q_r . The constructions required to state this result are defined in Section 6.1. We then show in Section 6.4 that this decomposition is realised by a graph isomorphism between edge slide graphs. We conclude the section by applying the results to several special cases in Section 6.5.

6.1 Further definitions and notation

We present some further definitions needed for our results in this section.

6.1.1 Notation

We will be working with the sets of all spanning trees and all signatures that reduce over a given proper non-empty subset R of $[n]$. We therefore introduce the following notation:

Definition 6.1. Given a proper non-empty subset R of $[n]$, we define

$$\begin{aligned} \text{RTree}_R(Q_n) &= \{T \in \text{Tree}(Q_n) : T \text{ reduces over } R\}, \\ \text{RSig}_R(Q_n) &= \{\mathcal{S} \in \text{Sig}(Q_n) : \mathcal{S} \text{ reduces over } R\}. \end{aligned}$$

6.1.2 Partitioning the n -cube

Given a subset $R \subseteq [n]$, we partition Q_n into $2^{|R|}$ copies of $Q_{n-|R|}$ as follows:

Definition 6.2. For any $X \subseteq R$ (including the empty set), let $Q_n(R, X)$ be the induced subgraph of Q_n with vertices

$$V(Q_n(R, X)) = \{W \subseteq [n] : W \cap R = X\} = \{X \cup Y : Y \subseteq [n] - R\}.$$

The cases $R = \{1, 3\}$ and $R = \{1\}$ with $n = 3$ are illustrated in Figure 3. For any subgraph H of Q_n we further define $H(R, X) = H \cap Q_n(R, X)$. Thus $H(R, X)$ is the subgraph of H induced by the vertices $W \in V(H)$ satisfying $W \cap R = X$.

Observe that $Q_n(R, X) = (Q_{[n]-R}) \oplus X$, and so is an $(n - |R|)$ -cube; and if T is a spanning tree of Q_n , then $T(R, X)$ is a spanning forest¹ of $Q_n(R, X)$. Note further that

- every edge of Q_n in a direction $i \notin R$ belongs to $Q_n(R, X)$ for some X ; and
- every edge of Q_n in a direction $j \in R$ joins a vertex of $Q_n(R, X)$ to the corresponding vertex of $Q_n(R, X \oplus \{j\})$ for some X .

For any $X_1, X_2 \subseteq R$ such that $X_1 \neq X_2$ we have

$$Q_n(R, X_1) \cap Q_n(R, X_2) = \emptyset.$$

6.1.3 Quotienting the n -cube

Definition 6.3. Let $S \subseteq [n]$. We define Q_n/S to be the graph obtained from Q_n by contracting every edge in direction j , for all $j \in S$.

In practice we will be most interested in the case where $S = \bar{R} := [n] - R$, for some $R \subseteq [n]$. The contractions Q_3/\bar{R} for $R = \{1, 3\}$ and $R = \{1\}$ are illustrated in Figure 4. For $R \subseteq [n]$ the contraction Q_n/\bar{R} is the graph obtained from Q_n by contracting every edge in direction j , for all $j \notin R$. The construction has the effect of contracting each subcube $Q_n(R, X)$ to a single vertex, which we may label X , for each $X \subseteq R$. The resulting graph Q_n/\bar{R} is a multigraph with underlying simple graph Q_R , and $2^{n-|R|}$ parallel edges for each edge of Q_R : one for each element of $\mathcal{P}(\bar{R})$.

¹We use *spanning forest* in the sense of a spanning subgraph that is a forest, and not in the sense of a maximal spanning forest. That is, we do not require each component of a spanning forest of G to be a spanning tree of the component of G it belongs to.

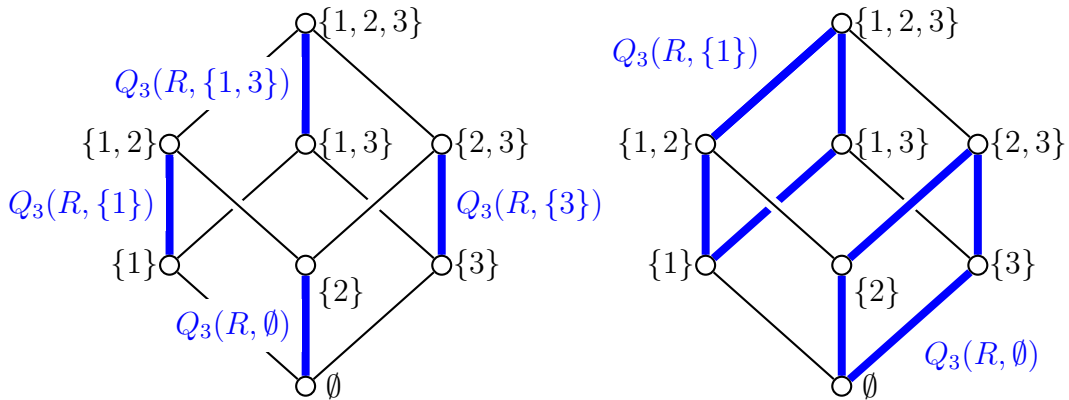


Figure 3: The subcubes $Q_3(R, X)$ for $X \subseteq R$ for $R = \{1, 3\}$ (left) and $R = \{1\}$ (right). In each case we get $2^{|R|}$ subcubes of dimension $3 - |R|$, together containing all edges of Q_n in directions not belonging to R .

We regard Q_n/\bar{R} as having vertex set $V(Q_R) = \mathcal{P}(R)$ and edge set $E(Q_R) \times \mathcal{P}(\bar{R})$, where the edge $(e, Y) \in E(Q_R) \times \mathcal{P}(\bar{R})$ joins the endpoints of e . We define

$$\pi_R : Q_n/\bar{R} \rightarrow Q_R$$

to be the projection from Q_n/\bar{R} to the underlying simple graph. This map fixes all the vertices and sends $(e, Y) \in E(Q_R) \times \mathcal{P}(\bar{R})$ to $e \in E(Q_R)$.

A spanning tree T of Q_n/\bar{R} corresponds to a choice of spanning tree $T_R = \pi_R(T)$ of the underlying simple graph Q_R , together with a choice of label $Y_e \in \mathcal{P}(\bar{R})$ for each edge e of T_R . We may define edge slides for spanning trees of Q_n/\bar{R} in an identical manner to edge slides for spanning trees of Q_n . For each $i \in [n]$ the automorphism $\sigma_i : Q_n \rightarrow Q_n$ descends to a well defined map $\sigma_i : Q_n/\bar{R} \rightarrow Q_n/\bar{R}$, and as before we may define the edge (e, Y) of T to be i -slidable if $T - (e, Y) + \sigma_i(e, Y)$ is again a spanning tree of T . For $i \in \bar{R}$ this simply corresponds to a change in label from Y to $Y \oplus \{i\}$, so every edge of T is i -slidable; while for $i \in R$ this corresponds to a label preserving edge slide in T_R , and (e, Y) is i -slidable if and only if e is i -slidable as an edge of T_R . We write $\mathcal{E}(Q_n/\bar{R})$ for the edge slide graph of Q_n/\bar{R} , and for a signature \mathcal{S} of Q_R we write $\mathcal{E}_{Q_n/\bar{R}}(\mathcal{S})$ for the edge slide graph of spanning trees of Q_n/\bar{R} with signature \mathcal{S} . Our discussion above has the following consequence:

Observation 6.4. *Let R be a proper nonempty subset of $[n]$. For any signature \mathcal{S} of Q_R , the edge slide graph $\mathcal{E}_{Q_n/\bar{R}}(\mathcal{S})$ is connected if and only if $\mathcal{E}(\mathcal{S})$ is connected.*

By a mild abuse of notation we may also regard π_R as a map from Q_n to Q_R . For each vertex U of Q_n we have

$$\pi_R(U) = U \cap R,$$

and for each edge $\{U, V\}$ of Q_n we have

$$\pi_R(\{U, V\}) = \begin{cases} \{\pi_R(U), \pi_R(V)\} = \{U \cap R, V \cap R\} & \text{if } \pi_R(U) \neq \pi_R(V), \\ \pi_R(U) = U \cap R & \text{if } \pi_R(U) = \pi_R(V). \end{cases}$$

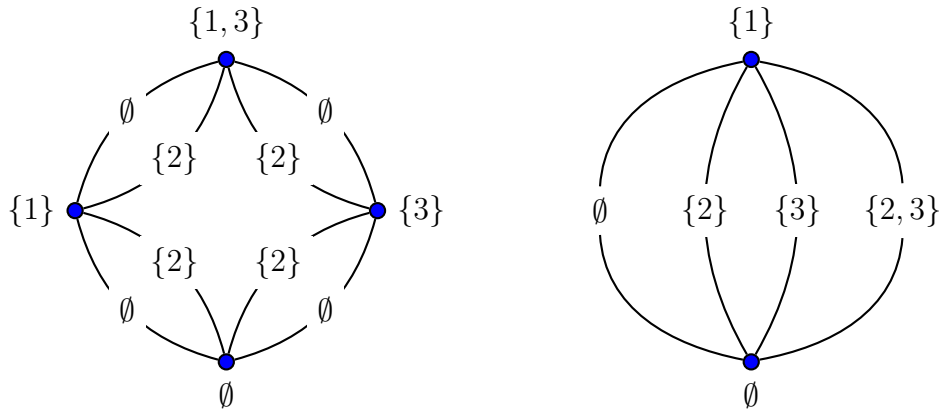


Figure 4: The graphs Q_3/\bar{R} in the cases $R = \{1, 3\}$ (left) and $R = \{1\}$ (right). The graphs are formed by contracting the bold edges in the corresponding graph of Figure 3. In each case we get a multigraph with underlying simple graph Q_R , and $2^{3-|R|}$ parallel edges for each edge of Q_R . The parallel edges may be labelled with the elements of $\mathcal{P}([3] - R)$.

If $V = U \oplus \{j\}$ then $\pi_R(\{U, V\})$ is the edge $\{U \cap R, (U \cap R) \oplus \{j\}\}$ of Q_R if $j \in R$, and is the vertex $U \cap R$ of Q_R if $j \notin R$. Thus $\pi_R : Q_n \rightarrow Q_R$ is not a graph homomorphism in the usual sense, but it is a cellular map if we regard Q_n and Q_R as 1-dimensional cell-complexes. Note that $Q_n(R, X)$ is the preimage in Q_n of $X \subseteq R$ under π_R .

6.1.4 The Cartesian product of graphs

Our edge slide graph decompositions will be expressed in terms of the Cartesian product of graphs; see for example [4, p. 30] or [6, D71 (p. 16)]. The Cartesian product may be defined as follows:

Definition 6.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs, where V_i is the vertex set of G_i and E_i is its edge set. The **Cartesian product** of G_1 and G_2 , denoted $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $(E_1 \times V_2) \cup (V_1 \times E_2)$. The incidence relation is as follows:

- If $e_1 \in E_1$ is incident with u_1, v_1 , then for all $w_2 \in V_2$, the edge (e_1, w_2) is incident with the vertices (u_1, w_2) and (v_1, w_2) of $G_1 \square G_2$.
- Similarly, if $e_2 \in E_2$ is incident with $u_2, v_2 \in V_2$, then for all $w_1 \in V_1$, the edge (w_1, e_2) is incident with the vertices (w_1, u_2) and (w_1, v_2) of $G_1 \square G_2$.

Consequently, for simple graphs G_1 and G_2 , the vertices (u_1, u_2) and (v_1, v_2) of $G_1 \square G_2$ are adjacent if and only if

- $u_2 = v_2$, and u_1, v_1 are adjacent in G_1 ; or

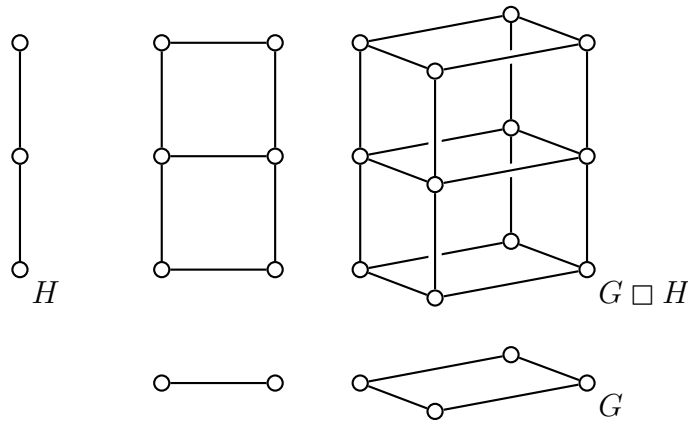


Figure 5: The Cartesian product of the graphs $G = Q_1 \amalg Q_2$ and $H = P_2$, a path of length two. Notice that $G \square H = (Q_1 \amalg Q_2) \square P_2 = (Q_1 \square P_2) \amalg (Q_2 \square P_2)$.

- $u_1 = v_1$, and u_2, v_2 are adjacent in G_2 .

We write

$$\begin{aligned} \prod_{i=1}^n G_i &= G_1 \square G_2 \square \cdots \square G_n, \\ G^{\square n} &= \prod_{i=1}^n G = \underbrace{G \square \cdots \square G}_n. \end{aligned}$$

Figure 5 illustrates the Cartesian product of $G = Q_1 \amalg Q_2$ and $H = P_2$, a path of length two.

Some sources write $G \times H$ for the Cartesian product of G and H , but others use this notation for a different graph product. The notation $G \square H$ used here and in [4] avoids this ambiguity and reflects the fact that the product of a pair of edges is a square, as can be seen in Figure 5. We note that

1. $Q_n \cong (K_2)^{\square n} = (Q_1)^{\square n}$, and so also $Q_n \square Q_m \cong Q_{n+m}$.
2. If $G = G_1 \amalg G_2$ is a disjoint union of subgraphs G_1 and G_2 , then

$$G \square H = (G_1 \amalg G_2) \square H = (G_1 \square H) \amalg (G_2 \square H),$$

as can be seen in Figure 5.

Remark 6.6. If G_1 and G_2 are regarded as 1-dimensional cell complexes, then $G_1 \square G_2$ is the 1-skeleton of the 2-dimensional cell complex $G_1 \times G_2$:

$$G_1 \square G_2 = (G_1 \times G_2)^{(1)}.$$

Here $G_1 \times G_2$ is the Cartesian product of G_1 and G_2 as cell complexes; see for example Hatcher [8, p. 8].

6.2 Structural characterisation of reducible trees

Reducible trees may be characterised as follows:

Theorem 6.7. *Let T be a spanning tree of Q_n , and let R be a proper nonempty subset of $[n]$. The following statements are equivalent:*

1. T reduces over R .
2. $T(R, X)$ is a spanning tree of $Q_n(R, X)$ for every $X \subseteq R$.
3. T/\bar{R} is a spanning tree of Q_n/\bar{R} .

Figure 6 illustrates Theorem 6.7 for a tree with signature $(1, 3, 3)$, which reduces over $R = \{1\}$.

Proof. Let $\text{sig}(T) = \mathcal{S} = (a_1, \dots, a_n)$, and let E be the set of edges of T in directions belonging to R . Then $|E| = \sum_{i \in R} a_i$. Delete all edges of T belonging to E . The resulting graph $T - E = \bigcup_{X \subseteq R} T(R, X)$ has $|E| + 1$ components and is a spanning forest² of $G = \bigcup_{X \subseteq R} Q_n(R, X)$, which is the result of deleting all edges of Q_n in directions belonging to R . As such, $T(R, X)$ is a spanning tree of $Q_n(R, X)$ for all $X \subseteq R$ if and only if $T - E$ has the same number of components as G . But G has $2^{|R|}$ components, so condition 2 holds if and only if

$$|E| = \sum_{i \in R} a_i = 2^{|R|} - 1;$$

that is, if and only if \mathcal{S} reduces over R . This proves that condition 1 of the theorem holds if and only if condition 2 does.

We now consider T/\bar{R} . This graph is the subgraph of Q_n/\bar{R} that results from T under the edge contractions transforming Q_n into Q_n/\bar{R} . Since T is a connected spanning subgraph of Q_n , the resultant T/\bar{R} is a connected spanning subgraph of Q_n/\bar{R} also. It is therefore a spanning tree if and only if it has $2^{|R|} - 1$ edges. But the edges of Q_n/\bar{R} are exactly the edges of Q_n in directions belonging to R , and so the edges of T/\bar{R} are exactly the edges of T in directions belonging to R also. Thus T/\bar{R} has $|E| = \sum_{i \in R} a_i$ edges, and so is a spanning tree if and only if $\sum_{i \in R} a_i = 2^{|R|} - 1$. This shows that condition 1 holds if and only if condition 3 does, completing the proof. □

Corollary 6.8. *Let T be a spanning tree of Q_n , and let R be a proper nonempty subset of $[n]$. If T reduces over R then $\pi_R(T)$ is a spanning tree of Q_R .*

Proof. Recall that $\pi_R : Q_n \rightarrow Q_R$ is the map given by the contraction $Q_n \rightarrow Q_n/\bar{R}$, followed by the projection $Q_n/\bar{R} \rightarrow Q_R$ to the underlying simple graph. Thus, the graph $\pi_R(T)$ is the subgraph of Q_R obtained from T/\bar{R} under the projection $Q_n/\bar{R} \rightarrow Q_R$. By Theorem 6.7 T/\bar{R} is a spanning tree of Q_n/\bar{R} , so $\pi_R(T)$ is a connected spanning subgraph of Q_R . Moreover, since it is acyclic, T/\bar{R} contains at most one edge from each family of parallel edges of Q_n/\bar{R} . Therefore the number of edges of $\pi_R(T)$ is equal to the number of edges T/\bar{R} , namely $2^{|R|} - 1$. The result follows. □

²Recall that we do not require a spanning forest to be a *maximal* spanning forest.

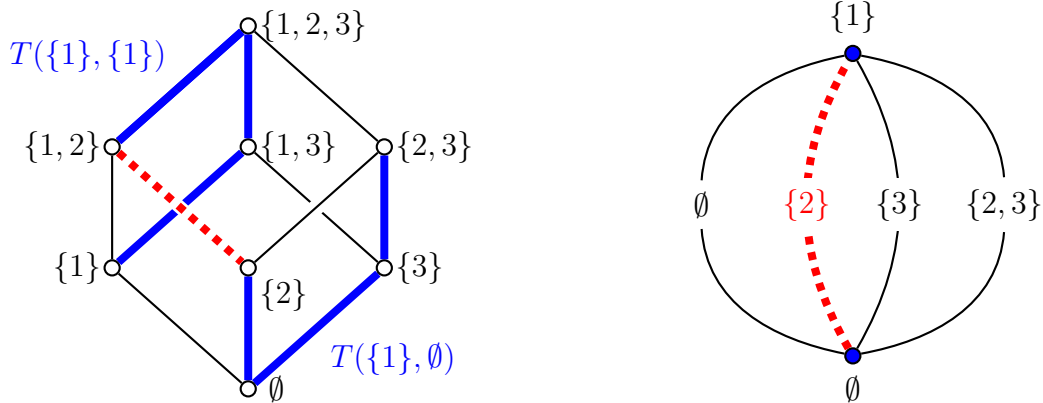


Figure 6: *Left:* A spanning tree T of Q_3 with signature $(1, 3, 3)$, which reduces over $R = \{1\}$. The solid bold edges show the spanning trees $T(R, X)$ of $Q_3(R, X)$ for $X \subseteq R$, and the bold dashed edge is the (here, unique) edge of T in a direction belonging to R . *Right:* The result of contracting the subcubes $Q_3(R, X)$ for $X \subseteq R$. The bold dashed edge $\{\{2\}, \{1, 2\}\}$ of T becomes the bold dashed spanning tree T/\bar{R} of Q_3/\bar{R} . The tree T can be completely reconstructed from the spanning trees $T(R, X)$, together with the spanning tree T/\bar{R} .

6.3 Decomposing reducible trees

In view of Theorem 6.7, we may canonically define a map

$$\Psi_R : \text{RTree}_R(Q_n) \rightarrow \text{Tree}(Q_n/\bar{R}) \times \prod_{X \subseteq R} \text{Tree}(Q_n(R, X))$$

by setting

$$\Psi_R(T) = (T/\bar{R}, (T(R, X))_{X \subseteq R}).$$

We show below in Theorem 6.9 that this map is a bijection, and then in Theorem 6.12 that it in fact defines an isomorphism of edge slide graphs.

Theorem 6.9. *Let R be a proper non-empty subset of $[n]$. The map*

$$\Psi_R : \text{RTree}_R(Q_n) \rightarrow \text{Tree}(Q_n/\bar{R}) \times \prod_{X \subseteq R} \text{Tree}(Q_n(R, X))$$

defined by

$$\Psi_R(T) = (T/\bar{R}, (T(R, X))_{X \subseteq R})$$

is a bijection.

Proof. The edges of Q_n may be naturally identified with the edges of $(Q_n/\bar{R}) \cup \bigcup_{X \subseteq R} Q_n(R, X)$. Using this identification we see that if $\Psi_R(T_1) = \Psi_R(T_2)$ then the edge set of T_1 is equal to the edge set of T_2 , so $T_1 = T_2$. Therefore Ψ_R is one-to-one.

It remains to show that Ψ_R is onto. Let

$$\mathcal{T} = (T_R, (T^X)_{X \subseteq R}) \in \text{Tree}(Q_n/\bar{R}) \times \prod_{X \subseteq R} \text{Tree}(Q_n(R, X)).$$

The edges of Q_n/\bar{R} may be canonically identified with the edges of Q_n in directions belonging to R , and using this identification we define T be the subgraph of Q_n with edge set

$$E(T) = E(T_R) \cup \bigcup_{X \subseteq R} E(T^X).$$

We claim that T is a spanning tree of Q_n that reduces over R , and that $\Psi_R(T) = \mathcal{T}$.

To see this, first note that the subcubes $Q_n(R, X)$ partition the edges of Q_n in directions belonging to \bar{R} . Thus

$$\begin{aligned} |E(T)| &= |E(T_R)| + \sum_{X \subseteq R} |E(T^X)| \\ &= (2^{|R|} - 1) + 2^{|R|}(2^{n-|R|} - 1) \\ &= 2^n - 1. \end{aligned}$$

Next, recall that Q_n/\bar{R} is obtained from Q_n by contracting each subcube $Q_n(R, X)$ to a single vertex. Since T^X is a spanning tree of $Q_n(R, X)$ for each X , and T_R is a spanning tree of Q_n/\bar{R} , it follows that T is a spanning subgraph of Q_n . Since it has $2^n - 1$ edges it is therefore a spanning tree. Moreover T has $|E(T_R)| = 2^{|R|} - 1$ edges in directions belonging to R , so T reduces over R . Thus $T \in \text{RTree}_R(Q_n)$, and it's clear by construction that we have $\Psi_R(T) = \mathcal{T}$. □

Observation 6.10. *Let \mathcal{S} be a signature belonging to $\text{RSig}_R(Q_n)$, and let $T \in \text{RTree}_R(Q_n)$ be such that $\Psi_R(T) = \mathcal{T} = (T_R, (T^X)_{X \subseteq R})$. Then*

$$\text{sig}(T) = \mathcal{S} \quad \Leftrightarrow \quad \text{sig}(T_R) = \mathcal{S}|_R \text{ and } \sum_{X \subseteq R} \text{sig}(T^X) = \mathcal{S}|_{[n]-R}.$$

Example 6.11. For the tree T of Figure 6, with $\mathcal{S} = \text{sig}(T) = (1, 3, 3)$ we have $\text{sig}(T_{\{1\}}) = (1) = \mathcal{S}|_{\{1\}}$, and

$$\text{sig}(T(\{1\}, \emptyset)) + \text{sig}(T(\{1\}, \{1\})) = (2, 1) + (1, 2) = (3, 3) = \mathcal{S}|_{\{2,3\}}.$$

6.4 The edge slide graph isomorphism theorem for reducible trees

The bijection Ψ_R of Theorem 6.9 has domain $\text{RTree}_R(Q_n)$, which is the vertex set of the edge slide graph $\mathcal{E}(\text{RSig}_R(Q_n))$. In this section we show that Ψ_R in fact defines a graph isomorphism between $\mathcal{E}(\text{RSig}_R(Q_n))$ and a suitable Cartesian product of smaller edge slide graphs:

Theorem 6.12. *Let R be a proper nonempty subset of $[n]$, and let $r = |R|$. Then*

$$\begin{aligned} \mathcal{E}(\text{RSig}_R(Q_n)) &= \bigcup_{S \in \text{RSig}_R(Q_n)} \mathcal{E}(S) \cong \mathcal{E}(Q_n/\bar{R}) \square \prod_{X \subseteq R} \mathcal{E}(Q_n(R, X)) \\ &\cong \mathcal{E}(Q_n/\bar{R}) \square (\mathcal{E}(Q_{n-r}))^{\square 2^r}. \end{aligned}$$

Theorem 6.12 follows from Theorem 6.9 and the following characterisation of edge slides in a reducible tree:

Theorem 6.13. *Let R be a proper nonempty subset of $[n]$, and let T be a spanning tree of Q_n that reduces over R . Let $e = \{Y, Y \oplus \{j\}\}$ be an edge of T in direction j , and set $X = Y \cap R$.*

1. *If $j \notin R$, then $e \in Q_n(R, X)$ and*
 - (a) *e is not slidable in any direction $i \in R$;*
 - (b) *e is slidable in direction $i \notin R$ if and only if it is i -slidable as an edge of $T(R, X)$.*
2. *If $j \in R$, then e is i -slidable if and only if it is i -slidable as an edge of T/\bar{R} . Consequently*
 - (a) *e is slidable in any direction $i \notin R$;*
 - (b) *e is slidable in direction $i \in R$ if and only if $\pi_R(e)$ is i -slidable as an edge of the tree $\pi_R(T)$.*

Proof. Let $i \in [n]$ be such that $i \neq j$. By Lemma 2.7, for e to be i -slidable we require that $\sigma_i(e)$ does not belong to T , and that the resulting cycle C in $T + \sigma_i(e)$ created by adding $\sigma_i(e)$ to T also contains e . We consider four possibilities, according to whether i and j belong to R .

1. Suppose first that $j \notin R$. Then $\sigma_i(e)$ lies in $Q_n(R, X')$, where $X' = (X \oplus \{i\}) \cap R$. Since $T(R, X')$ is a spanning tree of $Q_n(R, X')$, if $\sigma_i(e)$ does not belong to T then the cycle C lies entirely in $Q_n(R, X')$.
 - (a) If $i \in R$ then $X' = X \oplus \{i\} \neq X$, so e does not belong to C . It follows that e is not i -slidable.
 - (b) If $i \notin R$ then $X' = X$, and $\sigma_i(e)$ does not belong to T if and only if it does not belong to $T(R, X)$. If that is the case then C is the cycle in $T(R, X) + \sigma_i(e)$ created by adding $\sigma_i(e)$ to $T(R, X)$, and it follows that e is i -slidable as an edge of T if and only if it is i -slidable as an edge of $T(R, X)$.
2. Suppose now that $j \in R$. The edges of Q_n/\bar{R} in direction j may be naturally identified with the edges of Q_n in direction j , and with respect to this identification, for any $i \neq j$ the edge $\sigma_i(e)$ belongs to T if and only if it belongs to T/\bar{R} . It follows that if $\sigma_i(e)$ belongs to T then e is i -slidable in neither T nor T/\bar{R} . So suppose that $\sigma_i(e)$ does not belong to T , and let P be the path in T from one endpoint of $\sigma_i(e)$ to the other. Write $P = v_0v_1 \dots v_\ell$, and for $0 \leq a \leq \ell$ let $X_a \subseteq R$ be such that $v_a \in Q_n(R, X_a)$; that is, $X_a = v_a \cap R$. Note that $C = P + \sigma_i(e)$.

For any $0 \leq a \leq b \leq \ell$, the subpath $v_a v_{a+1} \dots v_b$ is the unique path in T from v_a to v_b . If $X_a = X_b$ then v_a and v_b both belong to $Q_n(R, X_a)$, so this path

must be the unique path from v_a to v_b inside the spanning tree $T(R, X_a)$ of $Q_n(R, X_a)$. Therefore $X_c = X_a$ for $a \leq c \leq b$. It follows that for all $X' \subseteq R$, if $P(R, X') = P \cap Q(R, X')$ is nonempty then it consists of a single path.

Consequently, when the subcubes $Q_n(R, X')$ are contracted to form Q_n/\bar{R} , the resulting subgraph C/\bar{R} is still a cycle, because it is a contraction of C in its own right. This cycle is the cycle in $T/\bar{R} + \sigma_i(e)$ that is created when $\sigma_i(e)$ is added to T/\bar{R} , and so it contains both e and $\sigma_i(e)$ if and only if both edges belong to C . It follows that e is i -slidable in T if and only if it is i -slidable in T/\bar{R} .

As an edge of T/\bar{R} the endpoints of e are X and $X \oplus \{j\}$, and the endpoints of $\sigma_i(e)$ are $X' = (X \oplus \{i\}) \cap R$ and $X' \oplus \{j\}$. We now consider two cases according to whether or not $i \in R$.

- (a) If $i \notin R$ then $X = X'$ and the end points of e and $\sigma_i(e)$ in T/\bar{R} co-incide. Therefore C/\bar{R} must consist of e and $\sigma_i(e)$ only, and so contains both edges. Therefore e is i -slidable.
- (b) If $i \in R$ then the endpoints of e and $\sigma_i(e)$ in T/\bar{R} differ. If an edge f parallel to $\sigma_i(e)$ belongs to T/\bar{R} then C/\bar{R} consists of f and $\sigma_i(e)$ only, and e is not i -slidable in T/\bar{R} . In this case $\pi_R(f) = \sigma_i(\pi_R(e))$ belongs to $\pi_R(T)$, so $\pi_R(e)$ is not i -slidable in $\pi_R(T)$ either.

Otherwise, $\pi_R(\sigma_i(e)) = \sigma_i(\pi_R(e))$ does not belong to $\pi_R(T)$, and the cycle C' created by adding $\sigma_i(\pi_R(e))$ to $\pi_R(T)$ is $\pi_R(C/\bar{R}) = \pi_R(C)$. Since $T/\bar{R} + \sigma_i(e)$ contains at most one edge from each parallel family of edges in Q_n/\bar{R} , the cycle C/\bar{R} contains both e and $\sigma_i(e)$ if and only if C' contains both $\pi_R(e)$ and $\sigma_i(\pi_R(e))$. It follows that e is i -slidable in T/\bar{R} if and only if $\pi_R(e)$ is i -slidable in $\pi_R(T)$, as claimed.

□

Proof of Theorem 6.12. The vertex set of $\mathcal{E}(\text{RSig}_R(Q_n))$ is $\text{RTree}_R(Q_n)$, and the vertex set of the product $\mathcal{E}(Q_n/\bar{R}) \square \prod_{X \subseteq R} \mathcal{E}(Q_n(R, X))$ is $\text{Tree}(Q_n/\bar{R}) \times \prod_{X \subseteq R} \text{Tree}(Q_n(R, X))$. By Theorem 6.9 the function Ψ_R is a bijection between the vertex sets of $\mathcal{E}(\text{RSig}_R(Q_n))$ and $\mathcal{E}(Q_n/\bar{R}) \square \prod_{X \subseteq R} \mathcal{E}(Q_n(R, X))$.

Let $T \in \text{RTree}_R(Q_n)$, and let e be an edge of T . Then Theorem 6.13 shows that e is slidable as an edge of T if and only if it is slidable as an edge of whichever tree T/\bar{R} or $T(R, X)$ it belongs to with respect to the decomposition $\Psi_R(T)$ of Theorem 6.9. Moreover, the proofs of these theorems show that when e is slidable, the edge slide and the decomposition commute: if T' is the result of sliding e in T , then T'/\bar{R} or $T'(R, X)$ (as applicable) is the result of sliding e in the decomposition. It follows that Ψ_R is a graph homomorphism, completing the proof. □

As a corollary to part 1a of Theorem 6.13 we obtain the following:

Corollary 6.14. *Let R be a proper nonempty subset of $[n]$, and let T be a spanning tree of Q_n that reduces over R . For all $X \subseteq R$, the signature of $T(R, X)$ is an*

invariant of the connected component of $\mathcal{E}(Q_n)$ containing T . More precisely, suppose that T' can be obtained from T by edge slides. Then $\text{sig}(T'(R, X)) = \text{sig}(T(R, X))$ for all $X \subseteq R$.

Example 6.15. Refer again to the tree T in Figure 6, which reduces over $\{1\}$. No edge of $T(\{1\}, \emptyset)$ or $T(\{1\}, \{1\})$ may be slid in direction 1. The only slidable edge of $T(\{1\}, \emptyset)$ is $\{\emptyset, \{3\}\}$, which may be slid in direction 2 only; and the only slidable edge of $T(\{1\}, \{1\})$ is $\{\{1, 3\}, \{1, 2, 3\}\}$, which may be slid in direction 3 only. The edge $\{\{2\}, \{1, 2\}\}$ in direction $1 \in R$ may be freely slid in either direction 2 or 3. These edge slides all leave $\text{sig}(T(\{1\}, \emptyset)) = (2, 1)$ and $\text{sig}(T(\{1\}, \{1\})) = (1, 2)$ unchanged.

6.5 Special cases

We now apply Theorem 6.12 in several special cases. We first consider the case $R = \{1\}$, and then use this to show that the edge slide graph of a supersaturated signature has the isomorphism type of a cube. We then apply this in turn to express the edge slide graph of a saturated signature in terms of an edge slide graph associated with its unsaturated part.

Theorem 6.16. *Let $n \geq 2$. Then*

$$\mathcal{E}(\text{RSig}_{\{1\}}(Q_n)) \cong Q_{n-1} \square (\mathcal{E}(Q_{n-1}))^{\square 2} \cong \coprod_{\mathcal{S}_1, \mathcal{S}_2 \in \text{Sig}(Q_{n-1})} Q_{n-1} \square \mathcal{E}(\mathcal{S}_1) \square \mathcal{E}(\mathcal{S}_2).$$

For $\mathcal{S} = (1, \mathcal{S}') \in \text{RSig}_{\{1\}}(Q_n)$ we have

$$\mathcal{E}(\mathcal{S}) \cong \coprod_{\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}'} Q_{n-1} \square \mathcal{E}(\mathcal{S}_1) \square \mathcal{E}(\mathcal{S}_2).$$

Proof. For compactness of notation let $R = \{1\}$. By Theorem 6.12 we have

$$\mathcal{E}(\text{RSig}_R(Q_n)) \cong \mathcal{E}(Q_n/\bar{R}) \square (\mathcal{E}(Q_{n-1}))^{\square 2}.$$

The graph Q_n/\bar{R} is a multigraph with underlying simple graph Q_1 , and 2^{n-1} parallel edges labelled with the subsets of $\bar{R} = \{2, \dots, n\}$. A spanning tree of Q_n/\bar{R} consists of a single edge, which may be canonically identified with its label in $\mathcal{P}(\bar{R})$. Two such trees are related by an edge slide in direction j precisely when their labels differ by adding or deleting j , so

$$\mathcal{E}(Q_n/\bar{R}) \cong Q_{[n]-R} \cong Q_{n-1}.$$

This gives the first isomorphism. The second then follows from the fact that

$$\mathcal{E}(Q_{n-1}) = \coprod_{\mathcal{S} \in \text{Sig}(Q_{n-1})} \mathcal{E}(\mathcal{S}).$$

For the final assertion, under the isomorphisms a vertex (Y, T_1, T_2) of $Q_{n-1} \square \mathcal{E}(\mathcal{S}_1) \square \mathcal{E}(\mathcal{S}_2)$ corresponds to a tree $T \in \mathcal{E}(\text{RSig}_R(Q_n))$ such that

$$T(R, \emptyset) = T_1, \qquad T(R, R) = T_2.$$

The signature of T is given by

$$\text{sig}(T) = (1, \text{sig}(T_1) + \text{sig}(T_2)) = (1, \mathcal{S}_1 + \mathcal{S}_2),$$

from which the claim follows. □

Example 6.17. We apply Theorem 6.16 to determine the components of $\text{RSig}_{\{1\}}(Q_3)$. We have

$$\mathcal{E}(Q_2) = \mathcal{E}(1, 2) \amalg \mathcal{E}(2, 1) \cong Q_1 \amalg Q_1,$$

so

$$\text{RSig}_{\{1\}}(Q_3) \cong Q_2 \square [\mathcal{E}(1, 2) \amalg \mathcal{E}(2, 1)]^{\square 2} \cong Q_2 \square [Q_1 \amalg Q_1]^{\square 2}.$$

Therefore $\text{RSig}_{\{1\}}(Q_3)$ has four components, each isomorphic to $Q_2 \square Q_1 \square Q_1 \cong Q_4$. Trees belonging to the component $Q_2 \square \mathcal{E}(a, b) \square \mathcal{E}(c, d)$ have signature $(1, a + c, b + d)$, so

$$\begin{aligned} \mathcal{E}(1, 2, 4) &\cong Q_2 \square (\mathcal{E}(1, 2))^{\square 2} \cong Q_4, \\ \mathcal{E}(1, 3, 3) &\cong (Q_2 \square \mathcal{E}(1, 2) \square \mathcal{E}(2, 1)) \amalg (Q_2 \square \mathcal{E}(2, 1) \square \mathcal{E}(1, 2)) \cong Q_4 \amalg Q_4, \\ \mathcal{E}(1, 4, 2) &\cong Q_2 \square (\mathcal{E}(2, 1))^{\square 2} \cong Q_4. \end{aligned}$$

Up to permutation there is just one remaining signature of Q_3 , namely the irreducible signature $(2, 2, 3)$. This signature is connected, and the structure of $\mathcal{E}(2, 2, 3)$ has been determined by Henden [9].

As a corollary to Theorem 6.16 we show that the edge slide graph of a supersaturated signature has the isomorphism type of a cube:

Corollary 6.18. *For the supersaturated signature $\mathcal{SS}_n = (1, 2, 4, \dots, 2^{n-1})$ we have*

$$\mathcal{E}(\mathcal{SS}_n) \cong Q_{2^n - n - 1}.$$

Proof. The proof is by induction on n , with the technique used to find $\mathcal{E}(1, 2, 4)$ in Example 6.17 providing the inductive step. We have previously found

$$\mathcal{E}(1) \cong Q_0, \quad \mathcal{E}(1, 2) \cong Q_1, \quad \mathcal{E}(1, 2, 4) \cong Q_4$$

so the result is already established for $n \leq 3$. We may therefore use any one of these cases as the base for the induction.

For the inductive step, suppose that the result holds for \mathcal{SS}_{n-1} . The signature \mathcal{SS}_n belongs to $\text{RSig}_{\{1\}}(Q_n)$, so by Theorem 6.16 it is a disjoint union of subgraphs of $\mathcal{E}(Q_n)$ of the form $Q_{n-1} \square \mathcal{E}(\mathcal{S}_1) \square \mathcal{E}(\mathcal{S}_2)$. Such a subgraph lies in $\mathcal{E}(\mathcal{SS}_n)$ precisely when

$$\mathcal{S}_1 + \mathcal{S}_2 = (2, 4, \dots, 2^{n-2}),$$

and it follows easily from the characterisation of signatures Theorem 3.1 that the only possibility is $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{SS}_{n-1}$. We therefore have

$$\begin{aligned} \mathcal{E}(\mathcal{SS}_n) &\cong Q_{n-1} \square \mathcal{E}(\mathcal{SS}_{n-1}) \square \mathcal{E}(\mathcal{SS}_{n-1}) \\ &\cong Q_{n-1} \square (Q_{2^{n-1} - n})^{\square 2} \\ &\cong Q_{2^n - n - 1}. \end{aligned}$$

This establishes the inductive step. □

As our final special case, we use Corollary 6.18 to show that the edge slide graph of a saturated signature may be expressed in terms of an edge slide graph associated with its unsaturated part:

Corollary 6.19. *Suppose that the ordered signature $\mathcal{S} = (a_1, \dots, a_n)$ is saturated above direction r . Let $R = [r]$ and let $\mathcal{S}' = (a_1, \dots, a_r)$. Then*

$$\begin{aligned} \mathcal{E}(\mathcal{S}) &\cong \mathcal{E}_{Q_n/\bar{R}}(\mathcal{S}') \square (Q_{2^{n-r}-(n-r)-1})^{\square 2^r} \\ &\cong \mathcal{E}_{Q_n/\bar{R}}(\mathcal{S}') \square Q_N, \end{aligned}$$

where $N = 2^r(2^{n-r} - (n - r) - 1)$.

Proof. The signature \mathcal{S} reduces over R , so $\mathcal{E}(\mathcal{S})$ consists of one or more connected components of $\mathcal{E}(\text{RSig}_R(Q_n))$. By Theorem 6.12 we have

$$\mathcal{E}(\text{RSig}_R(Q_n)) \cong \mathcal{E}(Q_n/\bar{R}) \square \bigsqcup_{X \subseteq R} \mathcal{E}(Q_n(R, X)),$$

and by Observation 6.10 a vertex $(T_R, (T^X)_{X \subseteq R})$ of this product corresponds to a tree with signature \mathcal{S} if and only if

$$\text{sig}(T_R) = \mathcal{S}|_R = \mathcal{S}' \text{ and } \sum_{X \subseteq R} \text{sig}(T^X) = \mathcal{S}|_{\bar{R}} = (2^r, 2^{r+1}, \dots, 2^{n-1}).$$

Let $\text{sig}(T^X) = (e_{r+1}^X, \dots, e_n^X)$ for each $X \subseteq R$. Then an easy induction on j using the signature condition shows that $e_{r+j}^X = 2^{j-1}$ for all $X \subseteq R$, so that $\text{sig}(T^X) = \mathcal{S}\mathcal{S}_{n-r}$ for all X . Then

$$\mathcal{E}(\mathcal{S}) \cong \mathcal{E}_{Q_n/\bar{R}}(\mathcal{S}') \square \bigsqcup_{X \subseteq R} \mathcal{E}(\mathcal{S}\mathcal{S}_{n-r}),$$

and the result now follows by Corollary 6.18. □

7 Strictly reducible signatures are disconnected

Our last major result is that the edge slide graph of a strictly reducible signature is disconnected:

Theorem 7.1. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a strictly reducible signature of Q_n . Then the edge slide graph $\mathcal{E}(\mathcal{S})$ is disconnected.*

The reason underlying Theorem 7.1 is illustrated by the edge slide graph of the strictly reducible signature $(1, 3, 3)$, which we saw in Example 6.17 breaks into two components: one consisting of the trees T such that $\text{sig}(T(\{1\}, \emptyset)) = (1, 2)$, and a second consisting of the trees T such that $\text{sig}(T(\{1\}, \emptyset)) = (2, 1)$. Recall that by Corollary 6.14, if T reduces over R , then for all $X \subseteq R$ the signature $\text{sig}(T(R, X))$ is an invariant of the connected component of $\mathcal{E}(Q_n)$ containing T . Thus, we can show that $\mathcal{E}(\mathcal{S})$ is disconnected by showing there exist $X \subseteq R$ and trees T and T' with signature \mathcal{S} such that $\text{sig}(T(R, X)) \neq \text{sig}(T'(R, X))$.

To find the required trees T, T' it suffices to prove the existence of *single* tree T for which there are subsets $X, Y \subseteq R$ such that $\text{sig}(T(R, X)) \neq \text{sig}(T(R, Y))$: given such a tree, we may obtain T' by simply exchanging $T(R, X)$ and $T(R, Y)$. In Lemma 7.2 we prove the existence of such a tree, for a suitable choice of reducing set R .

Lemma 7.2. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be an ordered strictly reducible signature with unsaturated part $\mathcal{S}' = (a_1, \dots, a_s)$, and let $r < s - 1$ be such that \mathcal{S}' reduces over $[r]$ but not $[r + 1]$. Let $R = [r]$. Then for any distinct $X, Y \subseteq R$, there exists a spanning tree T of Q_n with signature \mathcal{S} such that $T(R, X)$ and $T(R, Y)$ have different signatures.*

Remark 7.3. Note that r as required above necessarily exists. Since \mathcal{S}' is reducible it reduces over $[t]$ for some $t < s$, and since it is unsaturated it does not reduce over $[s - 1]$. Thus we may for instance take r to be the largest integer $t < s$ such that \mathcal{S}' reduces over $[t]$.

Proof of Lemma 7.2. Let T be a spanning tree of Q_n with signature \mathcal{S} . If there exists $Z \subseteq R$ such that $T(R, X)$ and $T(R, Z)$ have different signatures then we can construct the required tree by (if necessary) swapping $T(R, Y)$ and $T(R, Z)$. So suppose that this is not the case. Then the subtrees $T(R, Z)$ have the same signature for all $Z \subseteq R$. Let $\mathcal{U} = (e_{r+1}, \dots, e_n)$ be this common signature.

For any $i \in \{r + 1, \dots, n\}$ each edge of T in direction i lies in $T(R, Z)$ for some $Z \subseteq R$, and since each such tree contains e_i edges in direction i we have $a_i = 2^r e_i$. It follows that \mathcal{U} is ordered. We begin by showing under our choice of r that $e_{r+1} \geq 2$. Suppose to the contrary that $e_{r+1} = 1$. Then $a_{r+1} = 2^r$, and consequently

$$\sum_{i=1}^{r+1} a_i = \sum_{i=1}^r a_i + a_{r+1} = 2^r - 1 + 2^r = 2^{r+1} - 1,$$

so \mathcal{S}' reduces over $[r + 1]$. This contradicts the choice of r , so we must have $e_{r+1} \geq 2$ as claimed, which then forces $e_{r+2} \geq 2$ also because \mathcal{U} is ordered.

Consider

$$\begin{aligned} \mathcal{U}_1 &= (e_{r+1} - 1, e_{r+2} + 1, e_{r+3}, \dots, e_n), \\ \mathcal{U}_2 &= (e_{r+1} + 1, e_{r+2} - 1, e_{r+3}, \dots, e_n), \end{aligned}$$

and note that $\mathcal{U}_1 + \mathcal{U}_2 = 2\mathcal{U}$. We show that \mathcal{U}_1 and \mathcal{U}_2 are signatures of Q_{n-r} , so there exist spanning trees T_1 and T_2 of Q_{n-r} with signatures \mathcal{U}_1 and \mathcal{U}_2 respectively. For simplicity, we let $f_i = e_{r+i}$ for $i = 1, 2, \dots, n - r$. We consider \mathcal{U}_1 and \mathcal{U}_2 separately.

For \mathcal{U}_1 , we distinguish the following cases according to whether or not $f_2 < f_3$.

1. Suppose $f_1 \leq f_2 < f_3 \leq \dots \leq f_{n-r}$. Then we have $f_1 - 1 < f_2 + 1 \leq f_3$. Write

$$\mathcal{U}_1 = (f_1 - 1, f_2 + 1, f_3, \dots, f_{n-r}) = (f'_1, f'_2, f'_3, \dots, f'_{n-r}).$$

Then

$$f'_1 < f'_2 \leq f'_3 \leq \dots \leq f'_{n-r}$$

is in nondecreasing order; and

$$\sum_{i=1}^k f'_i = \begin{cases} f_1 - 1 \geq 1, & \text{for } k = 1, \\ \sum_{i=1}^k f_i \geq 2^k - 1, & \text{for } 2 \leq k \leq n - r, \end{cases}$$

with equality in the second case when $k = n - r$. We conclude that \mathcal{U}_1 is a signature of Q_{n-r} , by Theorem 3.1.

2. Suppose $f_1 \leq f_2 = f_3 = \dots = f_p < f_{p+1} \leq \dots \leq f_{n-r}$ for some p , with $3 \leq p \leq n - r$. Then we have

$$f_1 - 1 < f_3 = \dots = f_p < f_2 + 1 \leq f_{p+1} \leq \dots \leq f_{n-r}.$$

Let

$$\mathcal{U}'_1 = (f'_1, f'_2, \dots, f'_p, f'_{p+1}, \dots, f'_{n-r}) = (f_1 - 1, f_3, \dots, f_p, f_2 + 1, f_{p+1}, \dots, f_{n-r}).$$

Then \mathcal{U}'_1 is an ordered permutation of \mathcal{U}_1 , so it suffices to show that \mathcal{U}'_1 is a signature. The only sums $\sum_{i=1}^k f'_i$ that are not equal to the corresponding sum $\sum_{i=1}^k f_i$ are the sums

$$\sum_{i=1}^j f'_i = f_1 - 1 + (j - 1)f_2,$$

for $1 \leq j < p$. We therefore consider the value of $f_1 - 1 + (j - 1)f_2$ for $1 \leq j < p$. For all $1 \leq y \leq p$ let

$$f(y) = \sum_{i=1}^y f_i = f_1 + (y - 1)f_2,$$

and let

$$g(y) = 2^y - 1.$$

Then

$$f(1) = f_1 \geq 2 > 1 = g(1),$$

and

$$f(p) = \sum_{i=1}^p f_i \geq 2^p - 1 = g(p).$$

To verify that \mathcal{U}_1 is a signature of Q_{n-r} , it remains to show that $g(j) < f(j)$, for all $1 < j < p$. Since g is convex, for any $0 \leq t \leq 1$, we have

$$g((1 - t) + tp) \leq (1 - t)g(1) + tg(p).$$

Let $t = \frac{j-1}{p-1}$. Then for $1 \leq j \leq p$ we have $0 \leq t \leq 1$ and $1 - t + tp = j$. So

$$\begin{aligned} g(j) &\leq \left(1 - \frac{j-1}{p-1}\right)g(1) + \frac{j-1}{p-1}g(p) \\ &< \left(1 - \frac{j-1}{p-1}\right)f(1) + \frac{j-1}{p-1}f(p) \\ &= \left(1 - \frac{j-1}{p-1}\right)f_1 + \frac{j-1}{p-1}(f_1 + (p-1)f_2) \\ &= f_1 + (j-1)f_2 = f(j). \end{aligned}$$

Therefore

$$\sum_{i=1}^j f'_i = \sum_{i=1}^j f_i - 1 = f(j) - 1 \geq 2^j - 1,$$

showing that \mathcal{U}_1 satisfies the signature condition.

For \mathcal{U}_2 , we consider the following cases according to whether $f_2 = f_1$, $f_2 = f_1 + 1$ or $f_2 > f_1 + 1$.

1. If $f_2 = f_1$, then \mathcal{U}_2 is the permutation of \mathcal{U}_1 obtained by swapping the first two entries. Therefore \mathcal{U}_2 is a signature of Q_{n-r} .
2. If $f_2 = f_1 + 1$, then \mathcal{U}_2 is the permutation of \mathcal{U} obtained by swapping the first two entries. Therefore \mathcal{U}_2 is a signature of Q_{n-r} .
3. If $f_2 > f_1 + 1$, then $f_1 + 1 \leq f_2 - 1$. Let

$$\mathcal{U}_2 = (f_1 + 1, f_2 - 1, f_3, \dots, f_{n-r}) = (f''_1, f''_2, f''_3, \dots, f''_{n-r}).$$

Then

$$f''_1 \leq f''_2 < f''_3 \leq \dots \leq f''_n$$

is in nondecreasing order; and

$$\sum_{i=1}^k f''_i = \begin{cases} f_1 + 1 \geq 3 & \text{for } k = 1; \\ \sum_{i=1}^k f_i \geq 2^k - 1, & \text{for } 2 \leq k \leq n - r, \end{cases}$$

with equality in the second case when $k = n - r$. Therefore the signature condition is satisfied and we conclude that \mathcal{U}_2 is a signature of Q_{n-r} .

Since \mathcal{U}_1 and \mathcal{U}_2 are signatures of Q_{n-r} , there are spanning trees T_1 and T_2 of Q_{n-r} with signatures \mathcal{U}_1 and \mathcal{U}_2 respectively. Let T' be the tree obtained from T by replacing $T(R, X)$ with T_1 , and $T(R, Y)$ with T_2 . Then T' has signature \mathcal{S} , and the subtrees $T'(R, X)$ and $T'(R, Y)$ have different signatures, as required. \square

We now have everything we require to prove Theorem 7.1.

Proof of Theorem 7.1. Without loss of generality, we may assume \mathcal{S} is ordered with unsaturated part $\mathcal{S}' = (a_1, \dots, a_s)$. Let $1 \leq r < s$ be the largest integer such that \mathcal{S}' reduces over $R = [r]$, and choose distinct $X, Y \subseteq R$. By Remark 7.3 and Lemma 7.2, there exists a spanning tree T of Q_n with signature \mathcal{S} such that the subtrees $T(R, X)$ and $T(R, Y)$ have different signatures. Let T' be the spanning tree obtained from T by swapping $T(R, X)$ and $T(R, Y)$. Since the signatures of $T(R, X)$ and $T(R, Y)$ are invariant under edge slides by Corollary 6.14, the trees T and T' lie in different components of $\mathcal{E}(\mathcal{S})$. It follows that $\mathcal{E}(\mathcal{S})$ is disconnected, as claimed. \square

8 Discussion

Theorem 7.1 shows that strict reducibility is an obstruction to being connected. We conjecture that this is the only obstruction to connectivity:

Conjecture 8.1. *Let $\mathcal{S} = (a_1, \dots, a_n)$ be a signature of Q_n . Then the edge slide graph $\mathcal{E}(\mathcal{S})$ is connected if and only if \mathcal{S} is irreducible or quasi-irreducible.*

The “only if” direction of Conjecture 8.1 is Theorem 7.1. As discussed below the “if” direction is known to be true for $n \leq 4$, for a certain class of irreducible signatures of Q_5 , and for two infinite families of irreducible signatures. If true, the conjecture together with Observation 6.4 and Theorem 6.12 would show that connected components of the edge slide graph of Q_n are characterised in terms of signatures of spanning trees of subcubes of Q_n . We show below in Theorem 8.3 that it suffices to consider the case where \mathcal{S} is irreducible only.

The cases $n = 1$ and $n = 2$ are trivial. For $n \geq 3$ a useful approach is to reduce the problem to studying upright trees. By Tuffley [12, Cor. 15] every tree is connected to an upright tree by a sequence of edge slides, so it suffices to show that every upright tree with signature \mathcal{S} lies in a single component. Up to permutation there is a unique irreducible signature $(2, 2, 3)$ of Q_3 , and using this approach it is straightforward to show that $\mathcal{E}(2, 2, 3)$ is connected. This is done by Henden [9], who also determines the complete structure of $\mathcal{E}(2, 2, 3)$.

For $n \geq 4$ the first author’s doctoral thesis [1], completed under the supervision of the second and third authors, makes substantial partial progress towards an inductive proof of the conjecture. Al Fran [1, Defn 5.3.1] introduces the notion of a *splitting signature* of \mathcal{S} with respect to n . This is a signature \mathcal{D} of Q_{n-1} such that there exists an upright spanning tree T of Q_n such that $\text{sig}(T) = \mathcal{S}$ and $\text{sig}(T \cap Q_{n-1}) = \mathcal{D}$. As the culmination of a series of results Al Fran proves the following:

Theorem 8.2 (Al Fran [1, Thm 11.1]). *Let $n \geq 4$ and let \mathcal{I} be an ordered irreducible signature of Q_n . Suppose that every irreducible signature of Q_k is connected for all $k < n$. Suppose that \mathcal{I} has an ordered irreducible splitting signature \mathcal{D} with respect to n such that every upright spanning tree with signature \mathcal{I} and splitting signature \mathcal{D} lies in a single component of $\mathcal{E}(Q_n)$. Then the edge slide graph $\mathcal{E}(\mathcal{I})$ is connected.*

This reduces the inductive step of a proof of Conjecture 8.1 to the problem of showing that every irreducible signature has a suitable splitting signature as given. Al

Fran proves the existence of such a splitting signature for every irreducible signature of Q_4 , and (under the inductive hypothesis that every irreducible signature of Q_{n-1} is connected) for every irreducible signature $\mathcal{I} = (a_1, \dots, a_n)$ admitting a *unidirectional splitting signature*: a splitting signature $\mathcal{D} = (d_1, \dots, d_{n-1})$ such that $d_i = a_i$ for all but one index $i \leq n - 1$. This proves the “if” direction of Conjecture 8.1 for $n = 4$, and for irreducible signatures of Q_5 admitting a unidirectional splitting signature. Al Fran shows that when \mathcal{I} does not admit a unidirectional splitting signature it admits a *super rich splitting signature* (defined in terms of the excess), and conjectures such splitting signatures satisfy the requirements of Theorem 8.2.

Independently, Al Fran also proves the connectivity of two infinite families of irreducible signatures. For each $n \geq 3$ there is a unique ordered irreducible signature $\mathcal{I}_n^{(-1)}$ of Q_n such that $\varepsilon_k^{\mathcal{I}_n^{(-1)}} = 1$ for all $k < n$; and for each $n \geq 4$ there is a unique ordered irreducible signature $\mathcal{I}_{(3,n)}^{(+1,-1)}$ with excess 2 for $k = 2$, and excess 1 for $k < n$, $k \neq 2$. The first three members of these families are $(2, 2, 3)$, $(2, 2, 4, 7)$, $(2, 2, 4, 8, 15)$; and $(2, 3, 3, 7)$, $(2, 3, 3, 8, 15)$ and $(2, 3, 3, 8, 16, 31)$, respectively. By [1, Thms 10.1.1 and 10.2.1] every signature in these families has a connected edge slide graph.

We conclude the paper with Theorem 8.3, which reduces the quasi-irreducible case of Conjecture 8.1 to the irreducible case.

Theorem 8.3. *Let the ordered signature $\mathcal{S} = (a_1, \dots, a_n)$ be saturated above direction r . Then $\mathcal{E}(\mathcal{S})$ is connected if and only if $\mathcal{E}(a_1, \dots, a_r)$ is connected.*

In particular, $\mathcal{E}(\mathcal{S})$ is connected if and only if $\mathcal{E}(\text{unsat}(\mathcal{S}))$ is connected.

Proof. We may write $\mathcal{S} = (\mathcal{S}', 2^{n-1})$, where $\mathcal{S}' = (a_1, \dots, a_{n-1})$. Inductively, it suffices to show that $\mathcal{E}(\mathcal{S})$ is connected if and only if $\mathcal{E}(\mathcal{S}')$ is connected.

A spanning tree T of Q_n with signature \mathcal{S} contains every edge of Q_n in direction n , and under the isomorphism

$$\Psi_{[n-1]} : \mathcal{E}(\text{RSig}_{[n-1]}(Q_n)) \rightarrow \mathcal{E}(Q_n/\{n\}) \square (\mathcal{E}(Q_1))^{\square 2^{n-1}} \cong \mathcal{E}(Q_n/\{n\})$$

it corresponds to the spanning tree $T/\{n\}$ of $Q_n/\{n\}$ with signature \mathcal{S}' obtained by contracting these edges. The graph $Q_n/\{n\}$ has underlying simple graph Q_{n-1} , with two parallel edges labelled \emptyset and $\{n\}$ for each edge of Q_{n-1} . Thus, $T/\{n\}$ in turn corresponds to the spanning tree $T' = \pi_{[n-1]}(T)$ of Q_{n-1} with signature \mathcal{S}' , together with a choice of label \emptyset or $\{n\}$ on every edge. Moreover, by Theorem 6.13 an edge e of T or $T/\{n\}$ can be slid in direction $i \in [n - 1]$ if and only if the corresponding edge $\pi_{[n-1]}(e)$ of T' can be, and the label \emptyset or $\{n\}$ can be freely changed at any time.

Suppose that $\mathcal{E}(\mathcal{S}')$ is connected, and let $T_1, T_2 \in \mathcal{E}(\mathcal{S})$. Since $\mathcal{E}(\mathcal{S}')$ is connected there is a sequence of edge slides transforming $\pi_{[n-1]}(T_1)$ into $\pi_{[n-1]}(T_2)$. These edge slides may all be carried out in Q_n , to give a sequence of edge slides from T_1 to a tree T'_2 such that $\pi_{[n-1]}(T'_2) = \pi_{[n-1]}(T_2)$. The trees T_2, T'_2 may differ only in the edge labels \emptyset or $\{n\}$, and after a further series of edge slides in direction n only these can be brought into agreement. Therefore $\mathcal{E}(\mathcal{S})$ is connected.

Conversely, suppose $\mathcal{E}(\mathcal{S})$ is connected, and let $T_1, T_2 \in \mathcal{E}(\mathcal{S}')$. Choose spanning trees T'_1, T'_2 of Q_n such that $\pi_{[n-1]}(T'_i) = T_i$ for each i (for example, by regarding Q_{n-1} as a subgraph of Q_n , and adding all edges of Q_n in direction n to T_i for each

i). There is a sequence of edge slides in Q_n transforming T'_1 into T'_2 , and applying $\pi_{[n-1]}$, these may all be carried out in Q_{n-1} to transform T_1 into T_2 . Therefore $\mathcal{E}(\mathcal{S}')$ is connected also. \square

References

- [1] H. Al Fran, *The edge slide graph of the n -dimensional cube*, PhD thesis, Massey University, New Zealand (2017).
- [2] O. Bernardi, On the spanning trees of the hypercube and other products of graphs, *Electron. J. Combin.* **19(4)** (2012), #P51, 16pp.
- [3] M. Bóna, *A walk through combinatorics: An introduction to enumeration and graph theory*, 4th edition, World Scientific, 2017.
- [4] J. A. Bondy and U. S. R. Murty, *Graph theory*, vol. 244 of *Graduate Texts in Mathematics*, Springer, N.Y., 2008.
- [5] W. Goddard and H. C. Swart, Distances between graphs under edge operations, *Discrete Math.* **161(1-3)** (1996), 121–132.
- [6] J. L. Gross, J. Yellen, and P. Zhang, editors, *Handbook of graph theory*, 2nd edition, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2014.
- [7] P. Hall, On representatives of subsets, *J. London Math.* **10(2)** (1935), 26–30.
- [8] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [9] L. Henden, The edge slide graph of the 3-cube, *Rose-Hulman Undergrad. Math J.* **12(2)** (2011), 67–90.
- [10] J. L. Martin and V. Reiner, Factorization of some weighted spanning tree enumerators, *J. Combin. Theory Ser. A* **104(2)** (2003), 287–300.
- [11] R. P. Stanley, *Enumerative combinatorics, Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1999.
- [12] C. Tuffley, Counting the spanning trees of the 3-cube using edge slides, *Australas. J. Combin.* **54** (2012), 189–206.

(Received 30 July 2018; revised 4 Oct 2019)