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Graceful pairings

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Abstract

A graceful labeling of a graph G with n edges is an injection from the set of vertices of G to $\{0, 1, \ldots, n\}$ such that if each edge of G is labeled by the absolute value of the difference of the labels of its incident vertices, then every edge has a distinct label in $\{1, \ldots, n\}$. The famously unsettled graceful labeling conjecture proposes that every tree has a graceful labeling. A graceful labeling θ of a graph G is said to be gracious if for each vertex v of G either all adjacent vertices have larger labels than $\theta(v)$ or all adjacent vertices have smaller labels than $\theta(v)$. We introduce novel machinery for combining graceful bipartite graphs to produce new graceful graphs. If the constituent graphs have a gracious labeling then our methods produce a gracious labeling. Infinite families of gracious trees are produced and new classes of graceful trees are introduced. Along the way we offer a partial solution to a question posed in 1979.

1 Introduction

Graceful labelings of graphs were introduced by A. Rosa in 1967 [19]. These labelings have practical as well as theoretical applications. For example, graceful labelings played a central role in the solution of the two-table Oberwolfach problems [21]. A graceful graph with n edges cyclically decomposes the complete graph on $2n + 1$

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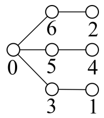

Figure 1: A gracious tree which cannot be α-labeled

vertices [19]. There are graphs which are not graceful but it is unknown whether or not every tree is graceful. The graceful tree conjecture proposes that every tree is graceful. While many papers have been written on this conjecture [9] it remains unresolved.

Some general methods for generating graceful labelings have been discovered [9]; for some previous constructions, see [5] and [13]. The "product" method presented in [13] (in fact, it appears earlier in [20]) has been subsequently modified and extended many times (see [3, 14, 15]). In this paper we present new methods for combining certain families of gracefully labeled graphs to produce new gracefully labeled graphs. These theorems are of particular interest since our construction is fundamentally different from previous constructions.

A *graph* G is a set of vertices $V(G)$ together with a set of edges $E(G) \subseteq V(G) \times V(G)$ whose elements are denoted uv. All graphs are assumed to be simple and connected. If G is a graph and v is a vertex of G, the *neighborhood* of v in G, denoted by $N(v)$, is the set of all vertices adjacent to v. A *tree* is a connected graph with no cycles. We refer the reader to West [22] for definitions omitted.

A *labeling* of a graph G is an injection $\theta : V(G) \to \{0, 1, 2, \ldots\}$. A *labeled graph* is a pair (G, θ) where G is a graph and θ is a labeling of G. When θ is clear from context, we may refer to the labeled graph (G, θ) as G and the vertex v of G which is labeled k as the k-vertex of G. Every labeling θ of a graph G induces a labeling on $E(G)$ in the following manner: if uv is an edge of G, assign the *induced edge label* $\theta(uv) := |\theta(u) - \theta(v)|$ to the edge uv.

A labeling θ of a graph G with n edges is *graceful* if the range of θ is contained in the set $\{0, 1, \ldots, n\}$ and the set of induced edge labels is precisely $\{1, 2, \ldots, n\}$. If a graph G admits a graceful labeling then G is said to be *graceful*. All trees with at most 35 vertices are claimed to be graceful in [8]. For a comprehensive survey of known results on the topic of graceful labeling see [9].

A labeling θ of a graph G with n edges is said to be an *α-labeling* (originally termed an α-valuation by Rosa [19]) if there exists a natural number s such that for all edges uv of G, either $\theta(u) \leq s < \theta(v)$ or $\theta(v) \leq s < \theta(u)$. The number s is the *boundary value* of the α-labeling.

In this paper P_n will denote the path with n edges and $n + 1$ vertices. All paths and caterpillar graphs (paths with any number of additional pendant vertices) can be α-labeled [19]. However, there are graceful trees for which an α-labeling does not exist. The smallest example of such a tree is the "spider" with 3 legs, each of which has two edges: see Figure 1.

A labeling θ of a graph G is said to be *ordered* if for each $v \in V(G)$, either for all

$u \in N(v)$, $\theta(u) < \theta(v)$ or for all $u \in N(v)$, $\theta(u) > \theta(v)$. This definition appears to be due to Cahit [6]; however the concept is also introduced in [10] in which it is called a *gracious labeling* and a graph admitting a gracious labeling is termed *gracious*. It is also rediscovered in [7], therein referred to as a *near-α labeling*. In this paper we will use the term *gracious*. Every α-labeled tree is also a gracious tree, but not conversely; see Figure 1. It is well-known that the tree in Figure 1 has no α-labeling.

Every tree with at most 20 vertices has been verified to have a gracious labeling [10]. The ordered (or near-α or gracious) labeling conjecture proposes that every tree has an ordered labeling [6, 7, 10]. Every graph with k edges with a gracious labeling cyclically decomposes the complete graph on $2kn + 1$ vertices [7].

A graph G is *bipartite* if there exists a partition (A, B) of $V(G)$ such that for each edge e of G, $e = uv$ where $u \in A$ and $v \in B$. If G admits an α-labeling or a gracious labeling then G is bipartite. Not all bipartite graphs are graceful; for example the cycle graph C_6 is bipartite yet fails to be graceful [19].

If (G, θ) is a gracefully labeled graph with n edges, the *complementary labeling* $\bar{\theta}$ of G is defined for each vertex v of G by $\bar{\theta}(v) = n - \theta(v)$. If θ is a graceful labeling, an α-labeling, or a gracious labeling, then so is $\bar{\theta}$. If f is a function, ran$f$ will denote the range of f. If (G, θ) is a labeled graph, a *relabeling function* is a function $f : \text{ran}\theta \to \{0, 1, 2, \ldots\}$. We may abuse notation and denote $(G, f \circ \theta)$ by (G, f).

Suppose (G, θ) is a gracefully labeled bipartite graph with bipartition (A, B). We will say an edge e of G is *positively oriented* (with respect to the bipartition (A, B)) provided $e = uv$ where $u \in A$, $v \in B$ and $\theta(u) > \theta(v)$. The labeling θ is gracious if and only if the set of positively oriented edges of G is either empty or is equal to $E(G)$.

Definition 1.1. Suppose $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is an indexed collection of disjoint bipartite gracefully labeled graphs each with n edges. $\mathscr{C}$ is a *compatible collection* if there exists a partition (A, B) of $\{0, 1, \ldots, n\}$ and for each i with $1 \leq i \leq m$ there exists a bipartition (A'_i, B'_i) of $V(G_i)$ so that if $A_i = \{\theta_i(v) : v \in A'_i\}$, $B_i = \{\theta_i(v) : v \in B'_i\}$ and

$$P_i = \{\theta_i(uv) : uv \text{ is a positively oriented edge of } G_i\},$$

then for each i, $A_i \subseteq A$, $B_i \subseteq B$ and $P_i = P_1$.

If $\mathscr{C}$ is a compatible collection in which all graphs are graciously labeled, then $\mathscr{C}$ is a *gracious compatible collection*. See Figure 2 for an example of a gracious compatible collection for which one may choose $A = \{3, 4, 6, 7\}$, $B = \{0, 1, 2, 5\}$, and B'_i to be the set of vertices of G_i colored white.

Definition 1.2. Suppose $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a compatible collection of graphs. For each i with $1 \leq i \leq m$, let w_i be the 0-vertex of G_i and let $Z_i = \{\theta_i(vw_i) : vw_i \in E(G_i)\}$. $\mathscr{C}$ is a 0-*compatible collection* if for each i, $Z_i = Z_1$.

See Figure 3 for an example of a 0-compatible collection of graphs each with 7 edges for which $Z_1 = \{5, 7\}$.

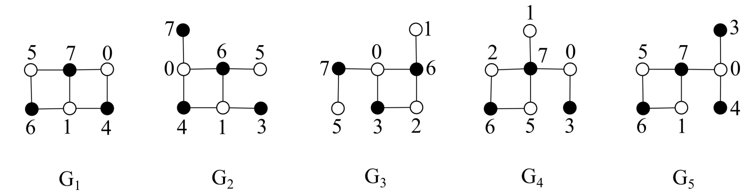

Figure 2: A gracious compatible collection

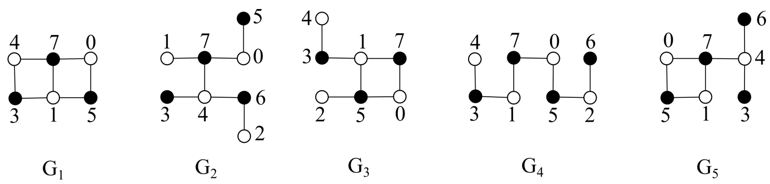

Figure 3: A 0-compatible collection

We introduce *graceful pairings* in Section 2. In Section 3 we prove that if $m = 2k+1$, $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a (gracious) 0-compatible collection, and G is a (graciously) gracefully labeled graph with k edges, then the graph formed by identifying the graphs $G, G_1, \ldots, G_m$ at their 0-vertices is (gracious) graceful. Using this result we provide a partial answer to a question posed in [13] as well as produce a new family of graceful trees. We also prove that if $m = 2k+1$, $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a (gracious) compatible collection, and G is a (graciously) gracefully labeled graph with $k-1$ edges, then the graph formed by attaching the 0-vertices of the graphs $G, G_1, \ldots, G_m$ to a single new vertex v is (gracious) graceful.

A *spider tree* is a tree with one root for which all branches (or *legs*) are paths; these trees are also known as subdivided stars or star-like graphs. An l-spider is a spider tree all of whose legs have l edges. See Figure 1 for an example of a 2-spider. In Section 4 we show that for each l there are infinitely many gracious l-spiders. As a further application we produce a gracious labeling of a spider tree which was not previously known to be graceful.

2 Graceful pairings

Our product theorems will require new machinery which is introduced in this section.

Definition 2.1. Let k be a natural number. A *graceful pairing* (of order k) is a bijection $f : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$ such that the function $g(x) := x - f(x)$ is also a bijection $g : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$.

Notice that if f and g are as in Definition 2.1, then g is also a graceful pairing.

Lemma 2.2. *There exists a graceful pairing of order k for each natural number k.*

Proof. Let k be a natural number. Define $f : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$ as follows:

$$f(i) = \begin{cases} -(k+i) & \text{if } i \in \{-k, \ldots, 0\} \\ 1 + k - i & \text{if } i \in \{1, \ldots, k\}. \end{cases}$$

The minimum value assumed by f is $-k$ (when $i = 0$) and the maximum value of f is k (when $i = 1$). Therefore the range of f is contained in $\{-k, \ldots, k\}$. We show that f is a graceful pairing.

Clearly f is one-to-one when its domain is restricted to either $\{-k, \ldots, 0\}$ or to $\{1, \ldots, k\}$. So suppose $i \geq 1$ and $j \leq 0$. Then

$$f(i) - f(j) = 2k + 1 + (j - i) \neq 0$$

since $j - i \geq -2k$. Therefore f is a bijection.

Now let $g(x) = x - f(x)$. Then

$$g(i) = \begin{cases} 2i + k & \text{if } i \in \{-k, \ldots, 0\} \\ 2i - (k+1) & \text{if } i \in \{1, \ldots, k\}. \end{cases}$$

The range of g is contained in $\{-k, \ldots, k\}$. Clearly g is injective on $\{-k, \ldots, 0\}$ and on $\{1, \ldots, k\}$. If $i \leq 0$ and $j \geq 1$, then $g(i) \not\equiv g(j) \bmod 2$ so that $g(i) \neq g(j)$. Therefore g is a bijection. □

We will need the following technical lemma. The functions defined in this lemma depend on more than just i; however to simplify notation this information is suppressed.

Lemma 2.3. *Let $m = 2k + 1$ be any odd natural number, let n be any natural number, and let (W, X, Y) be any partition of $\{1, 2, \ldots, n\}$. Let $f : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$ be a bijection, and for each $i \in \{-k, \ldots, k\}$ define functions $l_i : \{1, 2, \ldots, n\} \to \{k+1, \ldots, mn + k\}$ by*

$$l_i(x) = \begin{cases} mx + i & \text{if } x \in W \\ mx + f(i) & \text{if } x \in X \\ mx - f(i) & \text{if } x \in Y. \end{cases}$$

Then $\{\operatorname{ran} l_i : -k \leq i \leq k\}$ forms a partition of $\{k+1, \ldots, mn + k\}$.

Proof. It is straightforward to verify that the range of each function l_i is contained in the set $\{k+1, \ldots, mn+k\}$. Note that if $-k \leq i \leq k$ and $-k \leq j \leq k$ then since $-k \leq f(j) \leq k$ we have $|i - f(j)| \leq 2k$. If $a \neq b$ and $a, b \in \{1, 2, \ldots, n\}$ then $|m\,(a - b)| \geq m = 2k + 1$.

First we show each l_i is an injection. Fix i with $-k \leq i \leq k$; then l_i is clearly injective on W, on X and on Y. Suppose $w \in W$ and $x \in X$; then $w \neq x$ and

$$l_i(w) - l_i(x) = m(w - x) + i - f(i) \neq 0$$

since $|i - f(i)| \leq 2k$ and $|m\,(x-w)| \geq 2k+1$. The other two cases are argued similarly.

Now suppose $i \neq j$; then since f is a bijection, $f(i) \neq f(j)$. We show $\mathrm{ran} l_i \cap \mathrm{ran} l_j = \varnothing$. There are six cases to consider. If $w_1 \in W$ and $w_2 \in W$, then

$$l_i(w_1) - l_j(w_2) = m(w_1 - w_2) + i - j \neq 0$$

since $i \not\equiv j \bmod m$. If $x_1 \in X$ and $x_2 \in X$, then

$$l_i(x_1) - l_j(x_2) = m(x_1 - x_2) + f(i) - f(j) \neq 0$$

since $f(i) \not\equiv f(j) \bmod m$. Similarly, if $y_1 \in Y$ and $y_2 \in Y$ then $l_i(y_1) \neq l_j(y_2)$.

Now suppose $w \in W$ and $x \in X$. Then $w \neq x$ and

$$l_i(w) - l_j(x) = m(w - x) + i - f(j) \neq 0$$

since $|m\,(w-x)| \geq 2k+1$ and $|i - f(j)| \leq 2k$.

If $w \in W$ and $y \in Y$, then $w \neq y$ and

$$l_i(w) - l_j(y) = m(w - y) + i + f(j) \neq 0$$

since $|m\,(w-y)| \geq 2k+1$ and $|i + f(j)| \leq 2k$. Similarly, if $x \in X$ and $y \in Y$ then $x \neq y$ and

$$l_i(x) - l_j(y) = m(x - y) + f(i) + f(j) \neq 0$$

since $|m\,(x-y)| \geq 2k+1$ and $|f(i) + f(j)| \leq 2k$.

Since for each i the function l_i is injective, $|\mathrm{ran} l_i| = n$. Since the range of each l_i is contained in the set $\{k+1, \ldots, mn+k\}$, and if $i \neq j$ the range of l_i is disjoint from the range of l_j, and since $|\{-k, \ldots, k\}| = 2k+1 = m$ and $|\{k+1, \ldots, mn+k\}| = mn$, we have

$$\bigcup_{-k \leq i \leq k} \mathrm{ran} l_i = \{k+1, \ldots, mn+k\}.$$

□

3 Product theorems

In this section we describe a new type of product of graceful graphs. Our construction is different from those in [3, 13, 14]. One notable difference is that our construction preserves the gracious property. The assumption of 0-compatibility is weaker than the technical assumption on the labelings in the constructions of [3, 13]. Our construction allows for graphs other than trees to be combined, not just trees as in [13, 14]. However, the number of graphs to be combined is more restrictive.

Theorem 3.1. *Suppose $m = 2k+1$, $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a (gracious) 0-compatible collection, and G is a (graciously) gracefully labeled graph with k edges. Then the graph formed by identifying the graphs $G, G_1, \ldots, G_m$ at their 0-vertices is (gracious) graceful.*

Proof. Let m and $\mathscr{C}$ be as in the statement of the theorem and assume each graph in $\mathscr{C}$ has n edges. For convenience re-index the sets in $\mathscr{C}$ so that $\mathscr{C} = \{(G_{-k}, \theta_{-k}), \ldots, (G_k, \theta_k)\}$. Since $\mathscr{C}$ is a compatible collection, there exists a partition (A, B) of $\{0, 1, \ldots, n\}$ so that $0 \in B$ and bipartitions (A'_i, B'_i) of $V(G_i)$ for $-k \le i \le k$ so that if $A_i = \{\theta_i(v) : v \in A'_i\}$ and $B_i = \{\theta_i(v) : v \in B'_i\}$ then $A_i \subseteq A$ and $B_i \subseteq B$.

Fix a graceful pairing $f : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$ of order k. Then f is a bijection. For each i with $-k \le i \le k$, let $l_i : \{1, \ldots, n\} \to \{k+1, \ldots, mn+k\}$ be defined as follows:

$$l_i\,(x) = \begin{cases} mx + i & \text{if } x \in A \\ mx + f(i) & \text{if } x \in B \backslash \{0\}. \end{cases}$$

It follows from Lemma 2.3 (taking $W = A$, $X = B\backslash\{0\}$, and $Y = \emptyset$) that each l_i is an injection, and if $i \ne j$ then $\mathrm{ran} l_i \cap \mathrm{ran} l_j = \emptyset$.

Let w_i be the 0-vertex of G_i. Let

$$P'_i = \{\theta_i\,(uv) : uv \,\text{is a positively oriented edge of}\, G_i\},$$

$Z_i = \{\theta_i\,(vw_i) : vw_i \in E(G_i)\}$ and $Q_i = \{1, 2, \ldots, n\}\,\backslash P'_i$. Since $0 \in B$ we have $w_i \in B'_i$, so that if $vw_i \in E(G)$ then vw_i is positively oriented. Thus $Z_i \subseteq P'_i$. Let $P_i = P'_i \backslash Z_i$ and $Z = Z_0$, $P = P_0$ and $Q = Q_0$. Then $\{Z, P, Q\}$ is a partition of $\{1, 2, \ldots, n\}$. Since $\mathscr{C}$ is a 0-compatible family, for each i with $-k \le i \le k$ we have $Z_i = Z$, $P_i = P$ and $Q_i = Q$.

The function $g : \{-k, \ldots, k\} \to \{-k, \ldots, k\}$ defined by $g(x) = x - f(x)$ is a bijection since f is a graceful pairing. For $-k \le i \le k$ define $q_i : \{1, \ldots, n\} \to \{k+1, \ldots, mn+k\}$ as follows:

$$q_i\,(x) = \begin{cases} mx + i & \text{if } x \in Z \\ mx + g(i) & \text{if } x \in P \\ mx - g(i) & \text{if } x \in Q. \end{cases}$$

Then since $\{Z, P, Q\}$ partitions $\{1, 2, \ldots, n\}$ and g is a bijection, the family of functions q_i for $-k \le i \le k$ satisfy the hypotheses of Lemma 2.3, with $f = g$, $W = Z$, $X = P$, and $Y = Q$.

Next we relabel each G_i. For $-k \le i \le k$ define

$$f_i : A_i \cup B_i \to \{0\} \cup \{k+1, \ldots, mn+k\}$$

as follows:

$$f_i\,(x) = \begin{cases} 0 & x = 0 \\ l_i\,(x) & x \ne 0 \end{cases} = \begin{cases} 0 & x = 0 \\ mx + i & x \in A_i \\ mx + f(i) & x \in B_i \backslash \{0\}. \end{cases} \tag{1}$$

Consider $(G_i, f_i) = G'_i$. We show that for each edge $e = uv$ of G'_i

$$|f_i\,(\theta_i\,(u)) - f_i\,(\theta_i\,(v))| = q_i\,(\theta_i\,(uv))\,.$$

First suppose e is an edge of G_i such that $e = vw_i$ where w_i is the 0-vertex of (G_i, θ). Then $\theta_i(vw_i) \in Z$ and $v \in A_i'$. Since $\theta_i(v) > \theta_i(w_i) = 0$ and $\theta_i(v) \in A_i$, $\theta_i(vw_i) = \theta_i(v)$. Therefore

$$f_i(\theta_i(v)) - f_i(\theta_i(w_i)) = f_i(\theta_i(v)) - f_i(0) = m \cdot \theta_i(v) + i - 0 = q_i(\theta_i(vw_i)).$$

Next, suppose $e = uv$ is an edge of G_i so that $u \in A_i'$, $v \in B_i'$, $v \neq w_i$ and $\theta_i(u) > \theta_i(v)$. Then $\theta_i(uv) \in P$ and

$$\begin{aligned} f_i(\theta_i(u)) - f_i(\theta_i(v)) &= m \cdot \theta_i(u) + i - (m \cdot \theta_i(v) + f(i)) \\ &= m(\theta_i(u) - \theta_i(v)) + (i - f(i)) \\ &= m \cdot \theta_i(uv) + g(i) \\ &= q_i(\theta_i(uv)). \end{aligned}$$

Finally, suppose $e = uv$ is an edge of G_i so that $u \in A_i'$ and $v \in B_i'$ and $\theta_i(v) > \theta_i(u)$. Then $\theta_i(uv) \in Q$ and

$$\begin{aligned} f_i(\theta_i(v)) - f_i(\theta_i(u)) &= m \cdot \theta_i(v) + f(i) - (m \cdot \theta_i(u) + i) \\ &= m(\theta_i(v) - \theta_i(u)) - (i - f(i)) \\ &= m \cdot \theta_i(uv) - g(i) \\ &= q_i(\theta_i(uv)). \end{aligned}$$

Let H' be the tree with mn edges formed by identifying the 0-vertices of $G'_{-k}, \ldots, G'_k$. It follows from Lemma 2.3 that the set of vertex labels of H' is contained in the set $\{0\} \cup \{k+1, \ldots, mn+k\}$, no two vertices have the same label, and the set of induced edge labels is precisely the set $\{k+1, \ldots, mn+k\}$.

Now let G be as in the statement of the theorem and let θ be a graceful labeling of G. Let H be the graph formed by identifying the 0-vertex of G with the 0-vertex of H'. Define a graceful labeling h of H by

$$h(x) = \begin{cases} \theta(x) & x \in V(G) \\ f_i(\theta_i(x)) & x \in V(G_i). \end{cases}$$

Next, we show that each f_i defined as in Equation 1 is increasing. Suppose $0 < x_1 < x_2$. If x_1, x_2 are both in A_i or both in B_i then $f_i(x_1) < f_i(x_2)$. So suppose $x_1 \in A_i$ and $x_2 \in B_i$. Then

$$\begin{aligned} f(x_2) - f(x_1) &= m(x_2 - x_1) + f(i) - i \\ &\geq m - 2k = 1. \end{aligned}$$

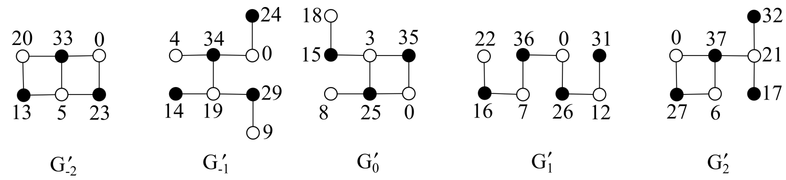

Figure 4: The collection in Figure 3 relabeled as in Theorem 3.1

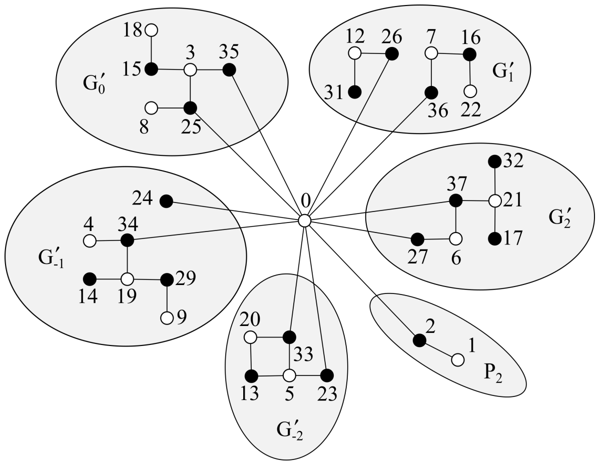

Figure 5: Theorem 3.1 applied to the collection in Figure 3 and P_2

The case $x_1 \in B_i$ and $x_2 \in A_i$ is similar. If $x_1 = 0$ and $0 < x_2$ then $f_i(x_1) = 0 < f_i(x_2)$ since $f_i(x_2) \geq k+1$.

Since each f_i is an increasing function, the set of positively oriented edges of (G_i, θ_i) is equal to the set of positively oriented edges of (G_i, f_i) (both with respect to the bipartition (A_i', B_i')). Therefore if $\mathscr{C}$ is a gracious 0-compatible collection and (G, θ) is gracious then (H, h) is gracious. □

The 0-compatible collection in Figure 3 relabeled as in the proof of Theorem 3.1 appears in Figure 4. See Figure 5 for the result of applying Theorem 3.1 to the 0-compatible collection in Figure 3 and the path P_2, using the graceful pairing appearing in the proof of Lemma 2.2.

Corollary 3.2. *If G is a (graciously) gracefully labeled graph with k edges, then the graph formed by taking $2k+2$ copies of G and identifying them at their 0-vertices is (gracious) graceful.*

Proof. Let $\mathscr{C}$ consist of $2k+1$ copies of G; then $\mathscr{C}$ is a (gracious) 0-compatible collection. Apply Theorem 3.1 to $\mathscr{C}$ and G to obtain the result. □

Although Corollary 3.2 is similar to Theorem 2 of [13] if the graph G is a graceful tree, in fact Corollary 3.2 generates a new class of graceful trees. This is because Theorem 2 of [13] additionally requires the labeled tree (G, θ) to satisfy the following condition: if G has n edges and w is the 0-vertex of G, then if $v \in N(w)$ and $\theta(v) \neq n$

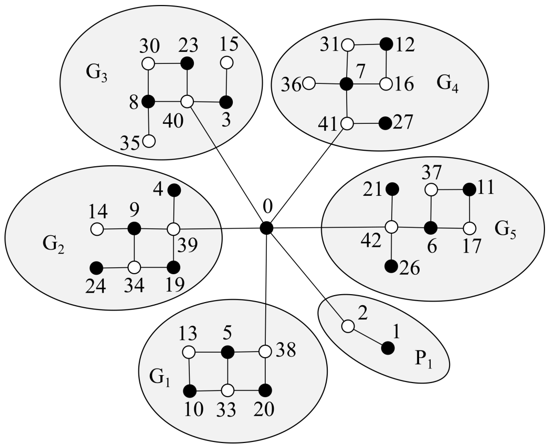

Figure 6: Theorem 3.3 applied to the gracious collection in Figure 2 and P_1

then there is $u \in N(v)$ with $\theta(u) = n - \theta(v)$. Weakening this condition appears to be difficult. In fact, in [13] the authors pose the question of whether there always exists a graceful labeling on this type of product of copies of (G, θ) which can be defined in terms of the original labeling θ, regardless of whether or not θ satisfies the additional requirement. Our Corollary 3.2 is a partial answer to this question: yes, if the number of copies of G is as described in Corollary 3.2.

Theorem 3.3. *Suppose $m = 2k + 1$, $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a (gracious) compatible collection, and G is a (graciously) gracefully labeled graph with $k-1$ edges. Then the graph formed by attaching the 0-vertices of the graphs $G, G_1, \ldots, G_m$ to a single new vertex v is (gracious) graceful.*

Proof. Let $\mathscr{C}$, m, and k be as in the statement of the theorem and suppose each G_i has n edges. For each i with $1 \leq i \leq m$, let (H_i, f_i) be the labeled graph obtained by appending a single vertex labeled $n + 1$ to the 0-vertex of G_i. Next let $\mathscr{C}' = \left\{\left(H_i, \bar{f}_i\right) : 1 \leq i \leq m\right\}$, where $\bar{f}_i$ denotes the complementary labeling of f_i. Then $\mathscr{C}'$ is a (gracious) 0-compatible collection. Let G be as in the statement of the theorem, and let (H, f) be the labeled graph resulting from appending a new vertex labeled k to the 0-vertex of G. Applying Theorem 3.1 to $\mathscr{C}'$ and $\left(H, \bar{f}\right)$ yields the result. □

See Figure 6 for the result of applying Theorem 3.3 to the gracious compatible collection in Figure 2 and the path P_1, with the graceful pairing as in the proof of Lemma 2.2.

Corollary 3.4. *Suppose $m = 2k+2$ and $\mathscr{C} = \{(G_1, \theta_1), \ldots, (G_m, \theta_m)\}$ is a gracious compatible family so that each G_i has $k-1$ edges. Then the graph formed by attaching the 0-vertex of each G_i to a single new vertex has a gracious labeling.*

Proof. $\mathscr{C}' = \mathscr{C} \setminus \{G_1\}$ is a gracious compatible collection with $2k+1$ members; apply Theorem 3.3 to $\mathscr{C}$ and G_1. □

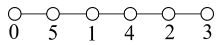

Figure 7: The path P_5 with an α-labeling

4 Spider trees

As an application of our product theorems, in this section we produce labelings of spider trees. A spider is a tree with at most one vertex of degree larger than two. We allow paths to be considered spiders. It is well-known that every path has an α-labeling in which the 0-vertex is a degree 1 vertex [18]; see Figure 7. If a spider is not a path, we refer to the highest degree vertex of the spider as its *center*.

An l-spider is a spider for which every leg has l edges. Since l-spiders are symmetrical trees, every l-spider is graceful [4]. Every 2-spider has a gracious labeling (proven in both [7, 16]) however it is unknown whether or not every l-spider is gracious.

Theorem 4.1. *For each natural number $l \geq 2$ there are infinitely many gracious l-spiders.*

Proof. Fix a natural number $l \geq 2$. There is at least one gracious l-spider since the path with l edges has an α-labeling. We can choose this labeling so that the 0-vertex is a degree 1 vertex, and we will refer (somewhat unusually) to this 0-vertex as the center of the l-spider.

Suppose S is a graciously labeled l-spider whose center vertex is labeled 0 and suppose the number of edges of S is k. Apply Corollary 3.2 to S to obtain a graciously labeled graph S' with $(2k+2)\,k$ edges. Since S' was obtained by identifying the 0-vertices of $(2k+2)$ copies of S, S' is also a graciously labeled l-spider whose center vertex labeled 0. This completes the proof. □

We use our product method to create large classes of gracious spiders, for some of which the existence of a graceful labeling was previously unknown. We will need the following fact, due to Rosa [18]:

Fact 4.2. *[18] If P is a path with n edges, then if $n \neq 4$, for any vertex v of P there exists an α-labeling of P in which the vertex v is labeled 0.*

We use these labeled paths to build gracious spiders.

Corollary 4.3. *Suppose P is a graciously labeled path and S is a graciously labeled spider with k edges whose center is labeled 0. Then the spider tree formed by identifying the 0-vertices of S and $2k+1$ copies of P is gracious.*

If G is an (α-labeled) gracefully labeled graph and H is an α-labeled graph, then the graph formed by identifying the 0-vertex of G with the 0-vertex of H has an (α-labeling) graceful labeling; see Theorem 3.1 of [23] (see also [2]) and Lemma 2.1 of [12]. This result extends to gracious labelings, as shown in the following proposition.

Proposition 4.4. *If (H, f) is an α-labeled graph and (G, θ) is a graciously labeled graph then the graph formed by identifying the* 0*-vertices of G and H is gracious.*

Proof. Assume (H, f) is an α-labeled graph with n edges and with boundary value s and that (G, θ) is a graciously labeled graph with m edges. Let W be the graph formed by identifying the 0-vertex of G with the 0-vertex of H. Label W as follows:

$$w(v) = \begin{cases} s - f(v) & v \in V(H),\ f(v) \le s \\ s - f(v) + n + m + 1 & v \in V(H),\ f(v) > s \\ \theta(v) + s & v \in V(G). \end{cases}$$

We verify that w is a gracious labeling. Let u_H and u_G be the 0-vertices of H and G, respectively, and note that $w(u_H) = s = w(u_G)$. If $v \in V(G)$ and $v \ne u_G$ then $s+1 \le w(v) \le m+s$. If $v \in V(H)$ with $v \ne u_H$ and $f(v) \le s$, then $0 \le w(v) \le s-1$. If $v \in V(H)$ and $f(v) > s$ then $m + s + 1 \le w(v) \le n + m$. These facts together show that w is an injection from $V(G) \cup V(H)$ to $\{0, 1, \dots, n + m\}$.

Next, notice that if uv is an edge of G then $\theta(uv) = w(uv)$ and so the labeled graph $\left(G, w \restriction_{V(G)}\right)$ has induced edge labels $\{1, 2, \dots, m\}$. If uv is an edge of H, then since f is an α-labeling, we can suppose $f(u) \le s < f(v)$. Then

$$n + m + 1 - f(uv) = w(v) - w(u) = w(uv)$$

so the edge labels of $\left(H, w \restriction_{V(H)}\right)$ are $\{m + 1, \dots, m + n\}$. Therefore w is graceful. Moreover, it is clear that w preserves the orientation of the edges of H, and w preserves the orientation of the edges of G. Therefore since θ is gracious then so is w. □

We note that the labeled spiders created as in Theorem 4.1 and Corollary 4.3 have their centers labeled 0, and so one may add two new legs of any length to such a spider, provided the new legs do not both have two edges, and obtain a new gracious spider using Proposition 4.4 and Fact 4.2.

We conclude this paper by providing a gracious labeling for a spider tree S which was previously unknown to be graceful. This spider tree S has 13 legs of length 5 and 3 legs of length 2. The spider tree S has more than 4 legs [12], diameter greater than 5 [11], more than 35 vertices (it has 72) [8], is not an olive tree [17], contains two legs which differ in length by more than 1 [1], and cannot be composed of equally-sized smaller rooted trees that are identified at their roots (the construction in [13]) since 71 is prime.

Example 4.5. We begin by taking 13 copies of P_5 labeled as in Figure 7 and relabeling them as in the proof of Theorem 3.1; the 13 copies of P_5 form a gracious 0-compatible family. Next, take the gracious 2-spider labeled as in Figure 1 and attach the 13 relabeled paths and the 2-spider at their 0-vertices, as in the proof of Theorem 3.1. See Figure 8.

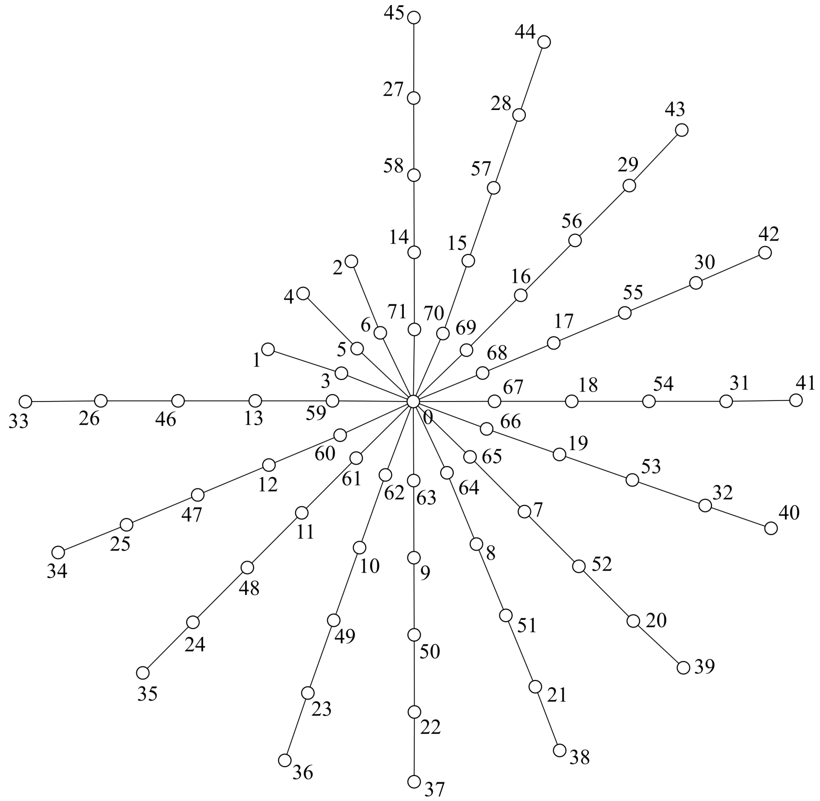

Figure 8: A new graceful spider with a gracious labeling

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