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# The Erdős-Gallai theorem modulo $k$

Richard A. Brualdi

*Department of Mathematics*
*University of Wisconsin, Madison, WI 53706*
*U.S.A.*
brualdi@math.wisc.edu

Seth A. Meyer

*Mathematics Discipline*
*St. Norbert College, De Pere, WI 54115*
*U.S.A.*
seth.meyer@snc.edu

**Abstract**

We obtain a mod $k$ analogue of the Erdős-Gallai theorem for the existence of a graph with a prescribed degree sequence. It is obtained as a corollary of a theorem about degree sequences of graphs that are interval-constrained.

## 1 Introduction

Two famous theorems in discrete mathematics are the Gale-Ryser theorem for the existence of a $(0,1)$-matrix with prescribed row sum vector $R$ and column sum vector $S$ (bipartite graphs with prescribed degree sequences $R$ and $S$ for the two parts of its bipartition) and the Erdős-Gallai theorem for the existence of a graph with a prescribed degree sequence $R$ (a symmetric $(0,1)$-matrix with zero trace and a prescribed row sum vector $R$). These results can be found in many publications [1, 2, 5, 7]. In [4] we gave a mod $k$ analogue (entries of $R$ and $S$ computed mod $k$) of the Gale-Ryser theorem. In this note, we obtain a mod $k$ analogue of the Erdős-Gallai theorem.

The Erdős-Gallai theorem is the following.

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**Theorem 1.1.** *Let $R = (r_1, r_2, \ldots, r_n)$ be an integral vector with $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$. There is a graph with $n$ vertices whose degree sequence is $R$ if and only if*

$$\sum_{i=1}^{t} r_i \leq t(t-1) + \sum_{i=t+1}^{n} \min\{t, r_i\} \quad (1 \leq t \leq n) \tag{1}$$

*and*

$$\sum_{i=1}^{n} r_i \text{ is even.} \tag{2}$$

Here by a graph we mean a graph without multiple edges or loops. That $\sum_{i=1}^{n} r_i$ is required to be even is a consequence of the fact that each edge has two vertices. If we allow loops (an edge joining a vertex to itself) where, contrary to the standard convention in graph theory, *a loop contributes* 1 *to the degree of its vertex*, we have the following corollary. See the explanation following its proof.

**Corollary 1.2.** *Let $R = (r_1, r_2, \ldots, r_n)$ be an integral vector with $r_1 \geq r_2 \geq \cdots \geq r_n \geq 1$. There is a graph with $n$ vertices and exactly one loop whose degree sequence is $R$ if and only if*

$$\sum_{i=1}^{t} r_i \leq t(t-1) + 1 + \sum_{i=t+1}^{n} \min\{t, r_i\} \quad (1 \leq t \leq n) \tag{3}$$

*and*

$$\sum_{i=1}^{n} r_i \text{ is odd.} \tag{4}$$

*Proof.* Let $R' = (r_1, r_2, \ldots, r_n, r_{n+1} = 1)$. Then $R'$ satisfies the Erdős and Gallai conditions and hence there is a graph $G'$ of $n+1$ vertices with degree sequence $R'$. The vertex $n+1$ has degree equal to 1 and is joined to vertex $p$ for some $p$ with $1 \leq p \leq n$. Deleting vertex $n+1$ and its incident edge and putting a loop at vertex $p$ gives the required graph. The converse is obvious. □

The justification for loops contributing 1 to the degree of a vertex comes from consideration of the adjacency matrix $A$. The Erdős-Gallai theorem gives necessary and sufficient conditions for the existence of an $n \times n$ symmetric $(0,1)$-matrix $A$ with all 0's on the main diagonal and with row sum vector $R$. It is natural to allow 1's on the main diagonal of $A$ and then to regard $A$ as the adjacency matrix of a loopy graph in which loops add 1 to the degree of its vertex. A general existence theorem for a symmetric nonnegative integral matrix with entries at most some integer $p$ (so $p = 1$ corresponds to a loopy graph in which loops add 1 to the degree of its vertex) is given in Theorem 6.3.2 of [3].

We recall the definition of majorization. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two non-increasing sequences of nonnegative integers. Then $X$ is *majorized* by $Y$, equivalently, $Y$ *majorizes* $X$, provided that

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad (1 \leq k \leq n) \text{ with equality for } k = n.$$

Majorization is denoted by $X \preceq Y$.

Theorem 1.1 immediately implies the following corollary.

**Corollary 1.3.** *Let $S = (s_1, s_2, \ldots, s_n)$ and $R = (r_1, r_2, \ldots, r_n)$ be non-increasing nonnegative integral vectors where $S \preceq R$. Then if $R$ is the degree sequence of a graph, that is,* (1) *and* (2) *hold, then $S$ is also the degree sequence of a graph.*

In the next section, we show that if the proposed degrees of a graph with $n$ vertices belong to a certain interval, then the inequalities (1) of Theorem 1.1 are automatically satisfied. This result is then used in the last section to give a mod $k$ analogue of the Erdős-Gallai theorem which mirrors the mod $k$ analogue of the Gale-Ryser theorem given in [4].

## 2 Interval-constrained Degree Sequences

First we consider the following question: Given positive integers $a$, $b$, and $n$, what conditions on those integers guarantee that every integral vector $R$ of size $n$ whose entries are in the closed interval $[a, a+b]$ and have even sum is the degree sequence of a graph.

**Theorem 2.1.** *Let $a$, $b$ and $n$ be positive integers with $n > a + b$, and let $R = (r_1, r_2, \ldots, r_n)$ be a non-increasing integral vector of size $n$ with entries in the interval $[a, a+b]$. If $n \geq a + b + 1 + \frac{(b+1)^2}{4a}$, then $R$ satisfies* (1).

*Proof.* This theorem is equivalent to a result of Zverovich and Zverovich [7] by making the substitutions $b \leftarrow a, a \leftarrow a + b$ in their Theorem 6. □

**Corollary 2.2.** *Assume that the hypotheses of Theorem* 2.1 *are satisfied. Then*

(i) *If the sum of the components of $R$ is even, then there exists a graph with degree sequence $R$.*

(ii) *If the sum of the components of $R$ is odd, then there exists a loopy graph with degree sequence $R$ and exactly one loop.*

*Proof.* The corollary is an immediate consequence of Theorem 1.1, Theorem 2.1, and Corollary 1.2. □

By Theorem 2.1, if the components of $R$ lie in the interval $[a, a+b]$ and $n \geq a+b+1+\frac{(b+1)^2}{4a}$, then all of the Erdős-Gallai inequalities are satisfied. We show this lower bound on $n$ is within 1 of the best possible, provided one is required to check all the inequalities (1). But note that, in general not all of the inequalities (1) need to be checked to know that all the inequalities are satisfied ([1, 5, 7]).

Suppose $n < a+b+1+\frac{(b+1)^2}{4a}$. This is equivalent to $4an < 4a^2+4ab+4a+(b+1)^2$. Consider the quadratic in $t$ given by $f(t) = t^2+(-1-b-2a)t+na$. Its discriminant is $(-1-b-2a)^2-4na = 4a^2+4ab+4a+(b+1)^2-4na$ and so $f$ has real solutions. In particular, if $t$ is an integer with $f(t) < 0$, then the vector $R$ formed by $t$ entries of $a+b$ and $n-t$ entries of $a$ does not satisfy the $t^{th}$ Erdős-Gallai inequality. Thus the bound given in the theorem is sharp up to the existence of *integral* solutions. In [5], Cairns, Mendan, and Nikolayevsky determined exactly when this differs from the integrally obtainable values and thus solved the sharpness condition exactly. To see this for specific numbers, consider the following example.

**Example 2.3.** Suppose that we consider all integral vectors of size 8 with entries in the range $[4, 7]$. This corresponds to $a = 4$, $b = 3$, and $n = 8$ above. Then $n$ is less than $a+b+1+\frac{(b+1)^2}{4a} = 4+3+1+\frac{4^2}{4\times 4} = 9$. Notice that the vertex in $f(t) = t^2+(-1-b-2a)t+na = t^2-12t+32$ occurs at $(6, -4)$. Consider the specific integral vector $R = (7, 7, 7, 7, 7, 7, 4, 4)$ (6 entries of maximum size and $n-6$ entries of minimum size). To satisfy the $6^{th}$ Erdős-Gallai inequality, we need $\sum_{i=1}^{6} r_i \leq 6(6-1)+\sum_{i=7}^{8}\min(r_i, 6)$, but $\sum_{i=1}^{6} r_i = 6\times 7 = 42$ while $6(6-1)+\sum_{i=7}^{8}\min(r_i, 6) = 30+2\times 4 = 38$. Notice that this fails by exactly the value of $f(6)$, which follows from the derivation used in the proof. Thus, not all integral vectors of the given type are graphical.

## 3 Degree sequences modulo $k$

Let $k$ be an integer with $k \geq 2$, and let $(\mathbb{Z}_k, +_k)$ be the additive group of integers modulo $k$. The set of elements of $\mathbb{Z}_k$ is taken to be $\{0, 1, \ldots, k-1\}$. For a graph $G$, define its *mod $k$ degree sequence* to be its degree sequence $R$ calculated mod $k$ then sorted to be non-increasing. Because of this sorting, we say two integral vectors $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots, s_n)$ of length $n$ are congruent mod $k$, written $R \equiv S \bmod k$, if there is a permutation of their entries so that $r_i \equiv s_i \bmod k$ for all $1 \leq i \leq n$. This is easily checked by counting the number of entries of $R$ and $S$ in the same congruence class modulo $k$. We consider the question of whether or not a graph exists with a prescribed mod $k$ degree sequence, that is, a mod $k$ Erdős-Gallai theorem. Mod $k$ consideration of some graphical concepts have previously been considered, e.g. [6].

Let $\mathcal{G}(R)$ be the set of all graphs with degree sequence $R = (r_1, r_2, \ldots, r_n)$. For a vector $S = (s_1, s_2, \ldots, s_n)$ with entries in $\mathbb{Z}_k$, let $\mathcal{G}_k(S)$ be the set of all graphs whose degree sequence modulo $k$ is $S$. If $\mathcal{G}(R) \neq \emptyset$ and $S \equiv R \bmod k$, then $\mathcal{G}_k(S) \neq \emptyset$. In this note we consider the inverse question of determining when $\mathcal{G}_k(S) \neq \emptyset$ for a given $S$. We shall use Theorem 2.1 to show that if $n$ is large enough relative to $k$ (a linear bound), then there always exists an $R$ such that $R \equiv S \bmod k$ with $\mathcal{G}(R) \neq \emptyset$, and thus $\mathcal{G}_k(S) \neq \emptyset$. In some instances we may require a loop at one vertex.

We first give an algorithm on which our subsequent discussion is based.

**$EG_k(S)$: Algorithm for the Existence of a Matrix in $\mathcal{G}_k(S)$**

Let $n$ and $k \geq 2$ be positive integers. The algorithm takes a non-increasing vector $S = (s_1, s_2, \ldots, s_n)$ with entries in $\mathbb{Z}_k$ as input and either returns an integral vector $\hat{R}$ with $S \equiv \hat{R} \bmod k$ and $\mathcal{G}(\hat{R}) \neq \emptyset$, or returns failure.

(i) Initialize by setting $R = (r_1, r_2, \ldots, r_n)$ equal to $S$.

(ii) If $R$ satisfies both (1) and (2), then go to (v).

(iii) Otherwise, compute $r_n + k$; if $r_n + k \geq n$, then go to (iv). Otherwise update $R$ as the non-increasing vector $(r_n + k, r_1, \ldots, r_{n-1})$, and go back to (ii).

(iv) Terminate the algorithm with Failure.

(v) Output $\hat{R} = R$ (satisfying (1) and (2)).

**Theorem 3.1.** *Let $n \geq 1$ and $k \geq 2$ be integers, and let $S = (s_1, s_2, \ldots, s_n)$ be a non-increasing vector with entries in $\mathbb{Z}_k$. The $EG_k(S)$ algorithm outputs a vector $\hat{R} = (\hat{r_1}, \hat{r_2}, \ldots, \hat{r_n})$ satisfing both* (1) *and* (2) *if and only if $\mathcal{G}_k(S) \neq \emptyset$.*

*Proof.* Suppose the algorithm returns an $\hat{R}$ satisfing both (1) and (2). Thus, $\mathcal{G}(\hat{R}) \neq \emptyset$. Since $\hat{R}$ is obtained from $S$ by successively adding $k$'s to its components, $\hat{R} \equiv S \bmod k$, and so any graph in $\mathcal{G}(\hat{R})$ is also a graph in $\mathcal{G}_k(S)$. Hence $\mathcal{G}_k(S) \neq \emptyset$.

Now suppose that $\mathcal{G}_k(S) \neq \emptyset$. We need to show that the algorithm does not terminate with Failure. Since $\mathcal{G}_k(S) \neq \emptyset$, after reordering components if necessary, there exists a non-increasing vector $R' = (r'_1, r'_2, \ldots, r'_n)$ with $R' \equiv S \bmod k$ such that $\mathcal{G}(R') \neq \emptyset$, where therefore $r'_1 \leq n-1$. Since $R' \equiv S \bmod k$, there exists an integer $t$ with $tk + \sum_{i=1}^n s_i = \sum_{i=1}^n r'_i$. Since the algorithm sequentially adds $k$ to the smallest component of the current $R$ and since $r'_1 \leq n-1$, the algorithm cannot fail when $t$ iterations of step (iii) are completed. Since in constructing $R$, $k$ is always added to the smallest component, after $t$ iterations, the current vector $R$ is necessarily majorized by $R'$. Since $\mathcal{G}(R') \neq \emptyset$, $R'$ satisfies (1) and (2) so $R$ also satisfies (1) by Corollary 1.3. Since $\sum_{i=1}^n r'_i = \sum_{i=1}^n r_i$, $R$ also satisfies (2). Thus, the algorithm does not return Failure (although it could have returned a "smaller" $\hat{R}$ vector earlier). $\square$

**Remark 3.2.** By (ii) the output vector $\hat{R}$ in the $EG_k(S)$ algorithm has the property that the sum of its components is even. Without such an assumption we would not be able to conclude the existence of a simple graph in $\mathcal{G}(\hat{R})$ from the algorithm.

In the algorithm we make no such assumption on the sum of the components of $S = (s_1, s_2, \ldots, s_n)$.

For example, if $k$ is even and $\sum_{i=1}^n s_i$ is not even, the algorithm will terminate with Failure as the sum of the components of each $R$ produced is odd. When $k$ is odd, adding $k$ to one component of the vector $S$ changes the parity of the sum of its entries and so it is possible for there to exist simple graphs and loopy graphs with one loop which have $S$ as their mod $k$ degree sequence. Because of Corollary 1.2, if we change step (ii) in the algorithm so that we go to step (v) when either (1) and (2) or (3) and (4) hold, and change step (iii) to go to step (iv) only when $r_n+k > n$ (now there could be a vertex with degree $n$ if it has a loop), then the proof of Theorem 3.1 follows except that the algorithm may output an integral vector $\hat{R}$ which admits either a simple graph or a loopy graph with one loop (depending on the parity of the sum of the entries of $\hat{R}$), and will only return Failure when neither is possible. ☐

We now derive a mod $k$ analogue of the Erdős-Gallai theorem by combining Algorithm $EG_k(S)$ with Theorem 2.1. First, we notice that using the algorithm produces a vector at each stage whose entries differ by at most $k$. If we agree to add $k$ to each component of smallest size, then the entries can be made to differ by at most $k-1$. Thus, we can maximize the usefulness of Theorem 2.1 by finding the $a$ value which minimizes the value of $a+b+1+\frac{(b+1)^2}{4a}$ for a fixed value of $b = k-1$.

**Lemma 3.3.** *For a fixed value of $b$, let $f(a) = a+b+1+\frac{(b+1)^2}{4a}$ for positive integers $a$. The function $f$ is minimized when $a = \frac{b+1}{2}$ and its minimum value is $2(b+1)$. When only integer inputs are considered, the smallest integer greater than or equal to the minimum output is $2b+2$ when $b$ is odd, and $2b+3$ when $b$ is even.*

*Proof.* This is a simple optimization problem from calculus. The only critical point for $a \geq 0$ is $a = \frac{b+1}{2}$, which is a minimum, and $f(\frac{b+1}{2}) = 2(b+1)$. When $b$ is even, checking the points $a = \frac{b}{2}$ and $a = \frac{b}{2}+1$ gives that the minimum output is $2b+2+\frac{1}{2b}$. So the minimum integers are as stated. ☐

Combining this with the previous discussion gives the following theorem.

**Theorem 3.4.** *Let $n \geq 1$ and $k \geq 2$ be integers, and let $R = (r_1, r_2, \ldots, r_n)$ be a non-increasing vector where $0 \leq r_i \leq k-1$. If $k$ is odd and $n \geq 2k+1$ or if $k$ is even and $n \geq 2k$, then there exists a simple graph with mod $k$ degree sequence $R$ or there exists a loopy graph with one loop[1] with mod $k$ degree sequence $R$.*

*Proof.* We begin by adding $k$ to every component of $R$ smaller than $\frac{k}{2}$ to create a new vector $R'$ with $R' \equiv R \bmod k$. Notice that the components of $R'$ now fall in the interval $[\frac{k}{2}, \frac{3k}{2}-1]$ when $k$ is even and $[\frac{k+1}{2}, \frac{3k-1}{2}]$ when $k$ is odd. In the notation of Theorem 2.1, this corresponds to $b = k-1$ and $a = \frac{k}{2}$ or $a = \frac{k+1}{2}$. Thus, by Lemma 3.3 and Theorem 2.1, we have that (1) is satisfied for $R'$ when $n \geq 2b+2 = 2k$ when

[1]Recall we are considering here that a loop contributes 1, not 2, to its vertex.

$k$ is even or $n \geq 2b+3 = 2k+1$ when $k$ is odd. The statement about simple and loopy graphs for $R$ follows from Remark 3.2. □

We end by noting that computer simulation suggests that the bound given in Theorem 3.4 is not the best possible result. For example, when $k = 6$, every possible vector $S$ with at least $n = 9$ components with entries from $\mathbb{Z}_6$ satisfies $\mathcal{G}_6(S) \neq \emptyset$. The smallest bound seems to be roughly $n \geq \frac{3}{2}k$. However, for $n$ values between $\frac{3}{2}k$ and $2k$, giving different $S$ vectors as input to the $EG_k(S)$ algorithm returns output vectors whose entries are not in a consistent range, so an approach following Theorem 2.1 does not seem possible.

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