AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **77(1)** (2020), Pages 9–34

Hypercube packings and coverings with higher dimensional rooks

Mehtaab Sawhney

Massachusetts Institute of Technology
Cambridge, MA 02139
U.S.A.
msawhney@mit.edu

David Stoner

Department of Mathematics, Stanford University
Stanford, CA 94305
U.S.A.
dwstoner@stanford.edu

Abstract

We introduce a generalization of classical q-ary codes by allowing points to cover other points that are Hamming distance 1 or 2 in a freely chosen subset of all directions. More specifically, we generalize the notion of 1-covering, 1-packing, and 2-packing in the case of q-ary codes. In the covering case, we establish the analog of the sphere-packing bound while for the packing case, we establish an analog of the singleton bound. Given these bounds, in the covering case we establish that the sphere-packing bound is asymptotically never tight except in trivial cases. This is analogous to a seminal result of Rodemich regarding q-ary codes. We then establish, in contrast, for 1-packing and 2-packing, that the analog of the singleton bound is sharp in several possible cases and conjecture that these bounds are sharp in general.

1 Introduction

Consider a set of n football matches which each end in either a win, a draw, or a loss. How many bets are necessary for an individual to guarantee that they predict at least $n-1$ of the outcomes of the games correctly? What about having at least $n-k$ outcomes correct?

The above problem is the classical Football Pool Problem that has been extremely well studied for small specific values of n, as well as generalizations allowing for more possible outcomes [1, 2, 5–11]. In particular, consider $H_{n,k}$, the k-dimensional

ISSN: 2202-3518

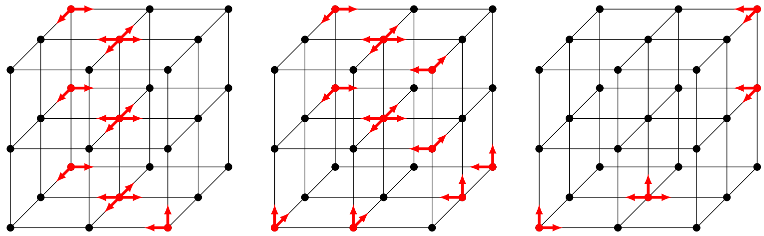

Figure 1: $a_{3,3,2} = 7, b_{3,3,2} = 10, c_{3,3,2} = 4$

hypercube with side length $n-1$. Now $H_{n,k}$ can be placed with the lattice $\{0, \ldots, n-1\}^k$, and define the distance between two points in $H_{n,k}$ to be the Hamming distance between their coordinate representations. The Hamming distance between two points is the number of coordinates on which they differ. Following [11], an *R-covering* is defined as a set of points S such that every point in the hypercube is within distance R of a point in S. In the literature, the minimum possible size of an R-covering has been the primary subject of interest, in particular when $R = 1$. Similarly a set T is an *R'-packing* if no two points of T are within distance R' of each other [11]. In this case the primary object of study is the largest possible R'-packing. For the remainder of this paper, we will focus on generalizations of the well studied cases where $R = 1$ and $R' = 1, 2$.

Given the extensive research in the case where points can cover all directions parallel to the axes, we instead consider the generalization where each point can cover in only a subset of these directions. Define an *ℓ-rook* to be a rook which can cover in ℓ directions. More precisely, an ℓ-rook is a point in $\mathbb{Z}^k$ along with a selection of ℓ out of k coordinates and this point covers exactly the points which differ in one of the ℓ chosen coordinates. For example a 2-rook in two dimensional space is a regular planar rook. Given this close relation with the chessboard piece, we use the terms "attack" and "cover" interchangeably. With this notion, we can now define the primary objects of study for this paper.

Definition 1.1. Let n, k, ℓ be positive integers with $k \geq \ell$. Define $a_{n,k,\ell}$ to be the minimum number of ℓ-rooks that can cover $H_{n,k}$, define $b_{n,k,\ell}$ to be the maximum number of ℓ-rooks in $H_{n,k}$ with no rooks attacking another, and define $c_{n,k,\ell}$ to be the maximum number of ℓ-rooks that can be placed in $H_{n,k}$ so that no two rooks attack the same point. In each of these three cases we do not allow multiple rooks at a single point. Furthermore in each case a rook attacks its own square.

For concreteness, the three grids in Figure 1 demonstrate an optimal constructions for $a_{n,k,\ell}$, $b_{n,k,\ell}$, and $c_{n,k,\ell}$ in the case $(n, k, \ell) = (3, 3, 2)$.

Previous research studies the case when $k = \ell$, that is when a rook can "attack" in the full set of dimensions. In particular, $a_{n,k,k}$ corresponds to a 1-covering while $b_{n,k,k}$ and $c_{n,k,k}$ correspond to a 1-packing and 2-packing, respectively. Furthermore $c_{q,k,k}$ corresponds to generalized q-ary Hamming-distance-3 subsets of $H_{q,k}$, which are useful for error correcting codes. The most classical bound in the case of coverings is the sphere-packing bound and we prove an analog of this bound with a proof closely related to the classical one. This determines $a_{n,k,\ell}$ to within a constant depending on ℓ.

Theorem 1.2. *We have*

$$\frac{n^k}{\ell\,(n-1)+1} \le a_{n,k,\ell} \le n^{k-1}.$$

Proof. Since each rook covers at most $\ell\,(n-1)+1$ points, and every point is covered, it follows that $|S|\,(\ell\,(n-1)+1) \ge n^k$ points. This implies the lower bound. To prove the upper bound, let S be the set of all points with first coordinate 0. Allow each point in S to attack in the direction of the first coordinate, and arbitrarily choose the other $\ell-1$ directions in which it may attack. These rooks collectively cover the cube, proving the upper bound. Note in this construction that the last $\ell-1$ dimensions are essentially unused. ☐

Since the above theorem holds for $\ell = 1$, it implies that $a_{n,k,1} = n^{k-1}$. Given the relative simplicity of this case, we consider $\ell \ge 2$ for the remainder of the paper. The analogous lower bounds for $b_{n,k,k}$ and $c_{n,k,k}$ comes from the classical Singleton bound [10]. The proof presented in the classical case can be adapted to this situation as well, however we rely on a more geometrical argument.

Theorem 1.3. *For all positive integers n, k, and ℓ with $k \ge \ell$, we have*

$$b_{n,k,\ell} \le \frac{kn^{k-1}}{\ell}.$$

Furthermore, if $k \ge \ell \ge 2$ then

$$c_{n,k,\ell} \le \frac{\binom{k}{2}n^{k-2}}{\binom{\ell}{2}}.$$

Proof. For $b_{n,k,\ell}$, consider all lines parallel to edges of the $H_{n,k}$ containing n points in $H_{n,k}$. Note that there are kn^{k-1} such lines by choosing a direction and letting the remaining coordinates vary over all possibilities within the cube. Furthermore, no two ℓ-rooks can cover the same axis. Since each ℓ-rook covers ℓ lines, it follows that $b_{n,k,\ell} \le \frac{kn^{k-1}}{\ell}$. Similarly, for $c_{n,k,\ell}$ consider all planes passing through $H_{n,k}$, parallel to one of the faces. Note there are $\binom{k}{2}n^{k-2}$ of these faces and each ℓ-rook intersects $\binom{\ell}{2}$ planes. If two rooks have the same associated plane, then they cover a common point, and it follows that $c_{n,k,\ell} \le \frac{\binom{k}{2}n^{k-2}}{\binom{\ell}{2}}$ for $\ell \ge 2$. (If $\ell = 1$ note the ℓ-rook does not determine a plane and therefore the proof does not follow.) ☐

Remark. Note that trivially $b_{n,k,\ell} \ge b_{n,k,k}$ and $c_{n,k,\ell} \ge c_{n,k,k}$. Theorems 4.3 and 4.4 will complement these upper bounds and prove that they are the correct order of growth in n.

Note that $c_{n,k,1} \leq n^{k-1}$ as each 1-rook covers n points and the points these rooks cover are distinct. This can be achieved by putting a 1-rook on all points with the first coordinate 0 and having all rooks point in the direction of the first coordinate. Given this difference in behavior between $\ell \geq 2$ and $\ell = 1$ for $c_{n,k,\ell}$, we assume that $\ell \geq 2$ for the remainder of the paper in this case.

In the remainder of the paper, we focus on the asymptotic growth rates of $a_{n,k,\ell}$, $b_{n,k,\ell}$, and $c_{n,k,\ell}$ when k and ℓ are fixed and n increases.

Definition 1.4. Let $a_{k,\ell} = \lim_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}}$, $b_{k,\ell} = \lim_{n\to\infty} \frac{b_{n,k,\ell}}{n^{k-1}}$, and $c_{k,\ell} = \lim_{n\to\infty} \frac{c_{n,k,\ell}}{n^{k-2}}$.

We now briefly summarize how the remainder of the paper is organized. Section 2 establishes the existence of the limits in Definition 1.4 for all k and ℓ (with $\ell \geq 2$ for $c_{k,\ell}$). Section 3 focuses on covering bounds and demonstrates that for $\ell \neq 1$, the lower sphere-packing bound in Theorem 1.2 is never asymptotically tight. Furthermore, Section 3 proves that for fixed ℓ, $a_{k,\ell} \to \frac{1}{\ell}$ as $k \to \infty$. Section 4 focuses on the packing bounds and demonstrates that $b_{k,\ell}$ and $c_{k,\ell}$ achieve the bounds in Theorem 1.3 in several possible cases. Finally, Section 5 presents a series of open problems regarding $a_{k,\ell}, b_{k,\ell}$, and $c_{k,\ell}$.

2 Limit Existence Results

The general procedure for proofs in this section is to demonstrate that $a_{nm,k,\ell} \leq m^{k-1}a_{n,k,\ell}$ for all positive integers m and then show that adjacent terms are sufficiently close. (The inequality is reversed for $b_{n,k,\ell}$, and an analogous inequality holds for $c_{n,k,\ell}$ but only when m is a prime number.) These two facts, combined with arguments closely related to the proof of Fekete's lemma, establish that the necessary limit exists. For $a_{n,k,\ell}$ and $b_{n,k,\ell}$, we make use of a construction of Blokhuis and Lam [1] whereas for $c_{n,k,\ell}$ we rely on a different construction in order to establish the first inequality.

Theorem 2.1. *For positive integers k and ℓ with $k \geq \ell$, the limits*

$$a_{k,\ell} = \lim_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}},$$

$$b_{k,\ell} = \lim_{n\to\infty} \frac{b_{n,k,\ell}}{n^{k-1}}$$

exist.

Proof. We first consider $a_{k,\ell}$. For $\ell = 1$, $a_{n,k,1} = n^{k-1}$ and the result is trivial. Therefore it suffices to assume that $\ell \geq 2$. Using Theorem 1.2, it follows that

$$\frac{1}{\ell} \leq \liminf_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}} \leq \limsup_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}} \leq 1.$$

Now suppose that $L = \liminf_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}}$. Then for every $\epsilon > 0$, there exists an integer m such that $\frac{a_{m,k,\ell}}{m^{k-1}} \leq L + \frac{\epsilon}{2}$. Now consider the points $(x_1, \ldots, x_k)$ in $\{0, 1, \ldots, n-1\}^k$ such that

$$x_1 + \cdots + x_k \equiv 0 \mod n.$$

(This is the construction presented in Blokhuis and Lam [1].) Note that if a k-rook is placed at every point in this construction, all points are covered and every point of an outer face of the hypercube has an axis "protruding" out of it. Therefore we can essentially blowup every point in $H_{m,k}$ to a copy of $H_{n,k}$ to create an $H_{mn,k}$, mark all the corresponding $H_{n,k}$ in $H_{mn,k}$ that correspond to rooks from the construction of $a_{m,k,\ell}$, and place ℓ-rooks within the $H_{n,k}$ corresponding to the points from the Blokhuis and Lam construction. Note that the ℓ axes for each of these ℓ-rooks in this construction match the orientation for the ℓ-rook in the original construction $a_{m,k,\ell}$ ℓ-rooks in $H_{m,k}$. This construction can easily be seen to give a covering for $H_{mn,k}$, and therefore it follows that $a_{nm,k,\ell} \leq n^{k-1} a_{m,k,\ell}$.

Now consider $a_{n+1,k,\ell}$ and $a_{n,k,\ell}$. If we let $H_{n,k} = \{0, \ldots, n-1\}^k$ and $H_{n+1,k} = \{0, \ldots, n\}^k$ then we place the construction for $a_{n,k,\ell}$ in $\{1, \ldots, n\}^k$. In order to cover the rest of the cube, place ℓ-rooks at every point with at least two coordinates being 0 and choose the directions of the points arbitrarily. For the remaining $k\,(n-1)^{k-1}$ points with exactly one 0, we break into cases of the form $(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_k)$. In order to cover these point we take all such points with $a_i = 0$ and place one axis of the ℓ possible in the direction of $(i+1)^{st}$ coordinate where indices are taken mod n. These points together cover $H_{n+1,k}$ and we have added at most $kn^{k-2} + \sum_{i=2}^{k} \binom{k}{i} n^{k-i} \leq \sum_{i=1}^{k} \binom{k}{i} n^{k-2} \leq 2^k n^{k-2}$ additional points. Therefore it follows that $a_{n+1,k,\ell} \leq 2^k n^{k-2} + a_{n,k,\ell}$ and

$$\frac{a_{n+1,k,\ell}}{(n+1)^{k-1}} \leq \frac{2^k n^{k-2} + a_{n,k,\ell}}{(n+1)^{k-1}} \leq \frac{2^k n^{k-2} + a_{n,k,\ell}}{n^{k-1}} = \frac{2^k}{n} + \frac{a_{n,k,\ell}}{n^{k-1}}.$$

Taking n sufficiently large, the following inequality holds:

$$\sum_{i=mn}^{mn+m-1} \frac{2^k}{i} < \frac{\epsilon}{2}.$$

Thus for $i \geq mn$ it follows that $\frac{a_{i,k,\ell}}{i^{k-1}} \leq L + \epsilon$. Therefore

$$\limsup_{n\to\infty} \frac{a_{n,k,\ell}}{n^{k-1}} \leq L + \epsilon$$

and since ϵ was an arbitrary constant greater than 0, the result follows.

For $b_{k,\ell}$, an identical procedure demonstrates that $\frac{b_{mn,k,\ell}}{(mn)^{k-1}} \geq \frac{b_{n,k,\ell}}{n^{k-1}}$ for all positive integers m and n. Furthermore, the sequence $\frac{b_{n,k,\ell}}{n^{k-1}}$ is bounded due to Theorem 1.3 and note that

$$\frac{b_{n+1,k,\ell}}{(n+1)^{k-1}} \geq \frac{b_{n,k,\ell}}{(n+1)^{k-1}} = \frac{n^{k-1}}{(n+1)^{k-1}} \left(\frac{b_{n,k,\ell}}{n^{k-1}} \right).$$

Thus taking $L = \limsup_{n\to\infty} \frac{b_{n,k,\ell}}{n^{k-1}}$ and choosing $\epsilon > 0$ arbitrarily there exists an m such that $\frac{b_{m,k,\ell}}{m^{k-1}} > L - \frac{\epsilon}{2}$. Since n is fixed, we can choose m such that $\left(\frac{mn}{mn+n-1}\right)^{k-1} > \frac{L-\epsilon}{L-\epsilon/2}$. Then for all $i \geq mn$ it follows that $\frac{b_{i,k,\ell}}{i^{k-1}} > L - \epsilon$. Therefore,

$$\liminf_{n\to\infty} \frac{b_{n,k,\ell}}{n^{k-1}} \geq L - \epsilon$$

and since $\epsilon > 0$ was arbitrary the result follows. $\square$

For the existence of $c_{k,\ell}$, we follow a similar strategy except that we rely on a different construction for the initial inequality that allows only for prime "blowup" factors. This construction is motivated by the construction of general q-ary codes.

Theorem 2.2. *For positive integers $k \geq \ell \geq 2$, the limit*

$$c_{k,\ell} = \lim_{n\to\infty} \frac{c_{n,k,\ell}}{n^{k-2}}$$

exists.

Proof. Suppose p is prime and $p \geq k$. Consider the set S of points $(x_1, \ldots, x_k)$ in $H_{p,k}$ that satisfy $x_{k-1} \equiv x_1 + \cdots + x_{k-2} \mod p$ and $x_k \equiv x_1 + 2x_2 + 3x_3 + \cdots + (k-2)\, x_{k-2} \mod p$. We will show that in this construction no two points are less than distance 3 apart. Suppose for sake of contradiction that there are two points $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ such that the distance between A and B is at most 2. If $a_i = b_i$ for all t with $1 \leq t \leq k-2$, then $A = B$. If $a_t = b_t$ for $1 \leq t \leq k-2$ except for $i \in \{1, \ldots, k-2\}$ where $a_i \neq b_i$, then $a_{k-1} \neq b_{k-1}$ and $a_k \neq b_k$. Finally, we consider the case where $a_t = b_t$ for $1 \leq t \leq k-2$ except for $i, j \in \{1, \ldots, k-2\}$ where $a_i \neq b_i$ and $a_j \neq b_j$. If both of the last two digits match then $a_i + a_j \equiv b_i + b_j \mod p$ and $ia_i + ja_j \equiv ib_i + jb_j \mod p$. Subtracting i times the first equation from the second yields $(j-i)\, a_j \equiv (j-i)\, b_j \mod p$ or $a_j \equiv b_j \mod p$, which is impossible. Thus each pair of points in S differ on at least 3 coordinates. Hence, we can place ℓ-rooks at the points in $S \subset H_{p,k}$, and no pair of rooks will attack a common point. Furthermore note the set S has exactly p^{k-2} points.

Now given a construction for $c_{n,k,l}$ in $H_{n,k}$, we can blow up each point to a copy of $H_{p,k}$ (for $p > k$ and p prime). Then place the construction given above into each $H_{p,k}$ corresponding to marked points in the original set. Orienting the set of points in each $H_{p,k}$ to match the original orientation of the corresponding points in $H_{n,k}$, it follows that $\frac{c_{np,k,\ell}}{(np)^{k-2}} \geq \frac{c_{n,k,\ell}}{n^{k-2}}$ for all primes greater than k. Furthermore, note that

$$\frac{c_{n+1,k,\ell}}{(n+1)^{k-2}} \geq \frac{c_{n,k,\ell}}{(n+1)^{k-2}} = \frac{n^{k-2}}{(n+1)^{k-2}} \left(\frac{c_{n,k,\ell}}{n^{k-2}}\right).$$

Now $\frac{c_{n,k,l}}{n^{k-2}}$ is bounded above due to Theorem 1.3 and bounded below as it is nonnegative. Let $L = \limsup_{n\to\infty} \frac{c_{n,k,\ell}}{n^{k-2}}$ and thus for every $\epsilon > 0$ there is an m such that $\frac{c_{m,k,\ell}}{m^{k-2}} > L - \frac{\epsilon}{2}$. Now order the primes $2 = p_1 < p_2 < \cdots$. Since $\lim_{i\to\infty} \frac{p_{i+1}}{p_i} = 1$ it follows that there exists j such that for $i \geq j$, $\frac{p_{i+1}}{p_i} < \left(\frac{L-\frac{\epsilon}{2}}{L-\epsilon}\right)^{\frac{1}{k-2}}$. For every integer $t > p_j n$ it follows that there exists $i \geq j$ such that $t \in [p_i n, p_{i+1}n - 1]$ and therefore $\frac{c_{t,k,\ell}}{t^{k-2}} > \left(\frac{p_i n}{t}\right)^{k-2} \frac{c_{p_i n,k,\ell}}{(p_i n)^{k-2}} > L - \epsilon$. Therefore $\liminf_{n\to\infty} \frac{c_{n,k,\ell}}{n^{k-2}} > L - \epsilon$, and since ϵ was arbitrary the result follows. □

Given these limit existence results we now turn our attention to establishing a variety of bounds on these values and begin by analyzing $a_{n,k,\ell}$.

3 Bounds for Covering

Given the initial bounds from Theorem 1.2, it follows that $\frac{1}{\ell} \leq a_{k,\ell} \leq 1$. In this section we demonstrate that $a_{k,\ell} \neq \frac{1}{\ell}$, except for the trivial case $a_{k,1} = 1$. To do this it is necessary to "amortize" a result of Rodemich [9] which is equivalent to $a_{n,k,k} \geq \frac{n^{k-1}}{k-1}$. However, the original proof given by Rodemich can be adapted for this situation, and we provide a complete proof below for the reader's convenience.

Theorem 3.1. *Suppose that $N \leq n^{k-1}$. Then N k-rooks on a $H_{n,k}$ cover at most $kNn - \frac{(k-1)N^2}{n^{k-2}}$ points.*

Proof. The bound is clear when $k = 1$. For $k = 2$, note that N 2-rooks cover at most $n^2 - (n-N)^2 = 2Nn - N^2$ points since at least $n - N$ rows and columns are uncovered. Therefore it suffices to consider $k \geq 3$. Furthermore, when $N \in [\frac{n^{k-1}}{k-1}, n^{k-1}]$, we have $kNn - \frac{(k-1)N^2}{n^{k-2}} \geq n^k$ so the bound holds trivially in these cases. Hence, it suffices to consider $N \leq \frac{n^{k-1}}{k-1}$.

Now consider any set S of k-rooks with N k-rooks. For any point $P \in H_{n,k}$ define $c_j(P)$ to be the number of times that the point P is attacked in the j^{th} direction. Furthermore define $q(P)$ to be the number of directions that P is attacked in and define

$$m(P) = \sum_{1 \leq j \leq k} c_j(P) = q(P) + \sum_{c_j(P)>0} (c_j(P) - 1).$$

Next, define $e_{i,j}(P)$ to be 1 if P is covered in the i and j directions and 0 otherwise. Then note that

$$\sum_{1 \leq i < j \leq k} e_{i,j}(P) = \frac{q(P)(q(P)-1)}{2} \leq \frac{k(q(P)-1)}{2}$$

for points P that are attacked and therefore

$$q(P) \geq 1 + \frac{2}{k} \sum_{1 \leq i < j \leq k} e_{i,j}(P).$$

Finally define $n_j(P) = c_j(P) - 1$ if $c_j(P)$ is positive and 0 otherwise. Therefore

$$\begin{aligned} m(P) &= q(P) + \sum_{1 \le j \le k} n_j(P) \\ &\ge 1 + \sum_{1 \le j \le k} n_j(P) + \frac{2}{k} \sum_{1 \le i < j \le k} e_{i,j}(P) \end{aligned}$$

for points P that are attacked and suppose that S attacks the points $T \subset H_{n,k}$. Summing over $P \in T$ yields

$$\begin{aligned} kNn &\ge |T| + \sum_{1 \le j \le k} \sum_{P \in T} n_j(P) + \frac{2}{k} \sum_{1 \le i < j \le k} \sum_{P \in T} e_{i,j}(P) \\ &= |T| + \sum_{1 \le j \le k} n_j + \frac{2}{k} \sum_{1 \le i < j \le k} e_{i,j} \end{aligned}$$

where we have defined

$$n_j = \sum_{P \in T} n_j(P)$$

and

$$e_{i,j} = \sum_{P \in T} e_{i,j}(P).$$

Now we arbitrarily order the n^{k-2} planes in the (i, j) direction. For r^{th} plane suppose there are a_r rows in the i^{th} direction with a point of S in them, b_r rows in the j^{th} direction with a point of S in them, and d_r total points in this plane. Furthermore, for convenience define $\alpha_r = d_r - a_r$ and $\beta_r = d_r - b_r$ and note by definition that $\alpha_r, \beta_r \ge 0$. Then it follows that

$$\begin{aligned} e_{i,j} &= \sum_{1 \le r \le n^{k-2}} a_r b_r \\ &= \sum_{1 \le r \le n^{k-2}} (d_r - \alpha_r)(d_r - \beta_r) \\ &= \sum_{1 \le r \le n^{k-2}} \left(\left(d_r - \frac{\alpha_r + \beta_r}{2} \right)^2 - \left(\frac{\alpha_r - \beta_r}{2} \right)^2 \right) \end{aligned}$$

Using the Cauchy-Schwarz inequality in the form $\sum_{i=1}^{n} x_i^2 \ge \frac{1}{n} \left(\sum_{i=1}^{n} x_i \right)^2$ along with the trivial inequality that $|\alpha_r - \beta_r| \le n$ it follows that

$$\begin{aligned} e_{i,j} &\ge \frac{1}{n^{k-2}} \left(\sum_{1 \le r \le n^{k-2}} \left(d_r - \frac{\alpha_r + \beta_r}{2} \right) \right)^2 - \frac{n}{2} \sum_{1 \le r \le n^{k-2}} \frac{|\alpha_r - \beta_r|}{2} \\ &\ge \frac{1}{n^{k-2}} \left(\sum_{1 \le r \le n^{k-2}} \left(d_r - \frac{\alpha_r + \beta_r}{2} \right) \right)^2 - \frac{n}{2} \sum_{1 \le r \le n^{k-2}} \frac{\alpha_r + \beta_r}{2} \\ &= \frac{1}{n^{k-2}} \left(N - \frac{n_i + n_j}{2n} \right)^2 - \frac{n_i + n_j}{4}. \end{aligned}$$

Here we have used the fact that

$$n \sum_{1 \le r \le n^{k-2}} \Big(\alpha_r + \beta_r \Big) = n_i + n_j,$$

which follows from counting the number of points covered multiple times in the i-th and j-th directions. Summing over all i, j it follows that

$$\sum_{1 \le i < j \le k} e_{i,j} \ge \frac{k(k-1)N^2}{2n^{k-2}} - i\frac{(k-1)N}{n^{k-1}} \sum_{1 \le j \le k} n_j - \frac{k-1}{4} \sum_{1 \le j \le k} n_j + \frac{1}{4n^k} \sum_{1 \le i < j \le k} (n_i + n_j)^2 .$$

Applying this inequality it follows that

$$\begin{aligned} kNn &\ge |T| + \sum_{1 \le j \le k} n_j + \frac{2}{k} \sum_{1 \le i < j \le k} e_{i,j} \\ &\ge |T| + \left(1 - \frac{2(k-1)N}{kn^{k-1}} - \frac{k-1}{2k}\right) \sum_{1 \le j \le k} n_j + \frac{(k-1)N^2}{n^{k-2}} \\ &\quad + \frac{1}{2kn^k} \sum_{1 \le i < j \le k} (n_i + n_j)^2 \\ &\ge |T| + \left(1 - \frac{2(k-1)N}{kn^{k-1}} - \frac{k-1}{2k}\right) \sum_{1 \le j \le k} n_j + \frac{(k-1)N^2}{n^{k-2}}. \end{aligned}$$

Using $N \le \frac{n^{k-1}}{k-1}$ it then follows that

$$\begin{aligned} kNn &\ge |T| + \left(1 - \frac{2}{k} - \frac{k-1}{2k}\right) \sum_{1 \le j \le k} n_j + \frac{(k-1)N^2}{n^{k-2}} \\ &= |T| + \left(\frac{k-3}{2k}\right) \sum_{1 \le j \le k} n_j + \frac{(k-1)N^2}{n^{k-2}} \\ &\ge |T| + \frac{(k-1)N^2}{n^{k-2}} \end{aligned}$$

and therefore it follows that

$$|T| \le kNn - \frac{(k-1)N^2}{n^{k-2}}$$

as desired. □

Note that Theorem 3.1 in general cannot be improved. In particular for $N = n^{k-1}/(k-1)$ and k being a prime power the result is sharp due to the existence of perfect codes [1]. Using this amortized version of Rodemich's result, we now prove a better lower bound for $a_{k,\ell}$. Note that the $\ell = k$ case of Theorem 3.2 is established by the Rodemich result that $a_{k,k} \ge \frac{1}{k-1}$ [9].

Theorem 3.2. *For every pair of positive integers (ℓ, k) with $\ell \le k$, we have*

$$a_{k,\ell} \ge \frac{2}{\ell\left(1+\sqrt{1-\frac{4(\ell-1)}{\ell^2\binom{k}{\ell}}}\right)}.$$

Proof. Note that when $\ell = 1$ the above follows from Theorem 1.2, therefore for the remainder of the proof we consider $\ell > 1$. Suppose we have a configuration of N ℓ-rooks that covers $H_{n,k}$. Since $\binom{k}{\ell} \ge 1$, it follows that

$$\frac{2}{\ell\left(1+\sqrt{1-\frac{4(\ell-1)}{\ell^2\binom{k}{\ell}}}\right)} \le \frac{2}{\ell\left(1+\sqrt{1-\frac{4(\ell-1)}{\ell^2}}\right)} = \frac{1}{\ell-1}.$$

Therefore, it suffices to consider the case $N \le \frac{n^{k-1}}{\ell-1}$. We first prove the following lemma:

Lemma 3.3. *Suppose that $a_1, \dots, a_{n^{k-\ell}}$ are nonnegative reals that satisfy $\sum_{i=1}^{n^{k-\ell}} a_i = A \le \dfrac{n^{k-1}}{\ell-1}$. Then*

$$\sum_{i:\ a_i \le \frac{n^{\ell-1}}{\ell-1}} \left(\ell n a_i - \frac{\ell-1}{n^{\ell-2}} a_i^2\right) + \sum_{i:\ a_i > \frac{n^{\ell-1}}{\ell-1}} n^\ell \le \ell n A - \frac{\ell-1}{n^{k-2}} A^2.$$

Proof. Consider the piecewise function $g(x)$ defined by

$$g(x) = \begin{cases} \ell n x - \frac{\ell-1}{n^{\ell-2}} x^2 & x \le \frac{n^{\ell-1}}{\ell-1}; \\ n^\ell & x > \frac{n^{\ell-1}}{\ell-1}. \end{cases}$$

Then $g(x)$ is continuous and concave on the region $[0, A]$. It follows that for $A = \sum_{i=1}^{n^{k-\ell}} a_i$ fixed, the left-hand side achieves its maximum when the a_i are all equal to $\frac{A}{n^{k-\ell}}$. Since $\frac{A}{n^{k-\ell}} \le \frac{n^{\ell-1}}{\ell-1}$, it follows that

$$\sum_{i, a_i \le \frac{n^{\ell-1}}{\ell-1}} \left(\ell n a_i - \frac{\ell-1}{n^{\ell-2}} a_i^2\right) + \sum_{i, a_i > \frac{n^{\ell-1}}{\ell-1}} n^\ell = \sum_i f(a_i)$$

$$\le n^{k-\ell} g\left(\frac{A}{n^{k-\ell}}\right) = \ell n A - \frac{\ell-1}{n^{k-2}} A^2$$

as required. □

Now we proceed with the proof of Theorem 3.2. We consider the $\binom{k}{\ell}$ possible choices of directions for the ℓ-points separately. Label these directions $1, 2, \dots, \binom{k}{\ell}$ arbitrarily. Note that each direction spans some dimension-ℓ hypercube. Each choice of direction corresponds to a choice of ℓ out of k coordinates, so there are $n^{k-\ell}$ distinct dimension-ℓ hypercubes for each direction, and these collectively form a partition of the full $H_{n,k}$. Order these ℓ-dimensional hypercubes arbitrarily and let $a_{i,j}$ denote the number of ℓ-points in the j^{th} hypercube of the i^{th} direction which attack in that direction. Furthermore let $A_i = \sum_{j=1}^{n^{k-\ell}} a_{i,j}$. Since the $\binom{k}{\ell}$ directions contain all rooks exactly once between them, $\sum_{i=1}^{\binom{k}{l}} A_i = N$. Also, since $N \le \frac{n^{k-1}}{\ell-1}$, we have $A_i \le \frac{n^{k-1}}{\ell-1}$ for each i. Now invoking Theorem 3.1, the total number of points covered is bounded above by

$$\sum_{i=1}^{\binom{k}{\ell}} \left(\sum_{j:\ a_{i,j} \le \frac{n^{\ell-1}}{\ell-1}} \left(\ell n a_{i,j} - \frac{\ell-1}{n^{\ell-2}} a_{i,j}^2 \right) + \sum_{j:\ a_{i,j} > \frac{n^{\ell-1}}{\ell-1}} n^{\ell} \right).$$

It follows that

$$\begin{aligned} n^k &\le \sum_{i=1}^{\binom{k}{\ell}} \left(\sum_{j:\ a_{i,j} \le \frac{n^{\ell-1}}{\ell-1}} \left(\ell n a_{i,j} - \frac{\ell-1}{n^{\ell-2}} a_{i,j}^2 \right) + \sum_{j:\ a_{i,j} > \frac{n^{\ell-1}}{\ell-1}} n^{\ell} \right) \\ &\le \sum_{i=1}^{\binom{k}{\ell}} \left(\ell n A_i - \frac{\ell-1}{n^{k-2}} A_i^2 \right) \\ &= \ell n N - \frac{\ell-1}{n^{k-2}} \sum_{i=1}^{\binom{k}{\ell}} A_i^2 \\ &\le \ell n N - \frac{\ell-1}{\binom{k}{\ell} n^{k-2}} N^2 \end{aligned}$$

where we have used Lemma 3.3 and then the Cauchy-Schwarz inequality. Dividing through by n^k and rearranging gives

$$(\ell-1)\left(\frac{N}{n^{k-1}}\right)^2 - \binom{k}{\ell}\left(\frac{\ell N}{n^{k-1}} - 1\right) \le 0.$$

Specializing to when $N = a_{n,k,\ell}$, we have that

$$(\ell-1)\left(\frac{a_{n,k,\ell}}{n^{k-1}}\right)^2 - \binom{k}{\ell}\left(\frac{\ell a_{n,k,\ell}}{n^{k-1}} - 1\right) \le 0.$$

It follows that for all n,

$$a_{n,k,\ell} \ge \frac{2n^{k-1}}{\ell\left(1 + \sqrt{1 - \frac{4(\ell-1)}{\ell^2\binom{k}{\ell}}}\right)},$$

and the result follows. $\square$

Corollary 3.4. *For $k \geq \ell \geq 2$, $a_{k,\ell} \neq \frac{1}{\ell}$. Therefore, in the limit, $a_{n,k,\ell}$ never achieves the lower bound of the sphere-packing bound.*

However, despite the fact that $a_{k,\ell} \neq \frac{1}{\ell}$ for $\ell \geq 2$, we can show that as k gets large $a_{k,\ell}$ in fact approaches $\frac{1}{\ell}$. In particular the portion of forced "overlapping" of the attacking rooks goes to 0. For convenience, define $f(k)$ to be the largest prime power less than or equal to k.

Theorem 3.5. *For every pair of positive integers (k, ℓ), with $k \geq 2$ and $f(k) \geq \ell$,*

$$a_{k,\ell} \leq \frac{1}{f(k)-1} \left\lceil \frac{f(k)}{\ell} \right\rceil .$$

Proof. The idea is to use the existence of a q-ary covering code for $q = f(k)$, and consider large blowups of this code. Take an integer $n_1 > \frac{f(k)}{\ell}$. We first construct a size-n_1, dimension-k block from $\left\lceil \frac{f(k)}{\ell} \right\rceil$ ℓ-rooks. In particular, consider the points that satisfy $x_1 + x_2 + \cdots + x_k \equiv i \mod n_1$ for $0 \leq i \leq \lceil \frac{f(k)}{\ell} \rceil - 1$ and then choose the $(i\ell + 1)^{st}$ through $((i+1)\,\ell)^{th}$ directions to attack for the points whose coordinate sum is equivalent to i where we take the specified direction mod n. Note that this block has an attacking line in every possible axis.

Since perfect q-ary covering codes exist for prime powers q (see [11] for example) and $f(k) \leq k$, we have $a_{k,k} \leq a_{f(k),f(k)} = \frac{1}{f(k)-1}$. Now we note that, using the size n_1 scaled blocks in place of points for a construction of $a_{n_2,k,k}$, an $H_{n_1 n_2,k}$ can be tiled with at most

$$\left(\left\lceil \frac{f(k)}{\ell} \right\rceil n_1^{k-1} \right) (a_{n_2,k,k})$$

ℓ-rooks, and the result follows from $\lim_{n\to\infty} \frac{a_{n,k,k}}{n^{k-1}} = a_{k,k}$. □

Corollary 3.6. *For every positive integer ℓ, $\lim_{k\to\infty} a_{k,\ell} = \frac{1}{\ell}$.*

The rest of this section is devoted to the specific case $a_{3,2}$ that demonstrates that the bounds in the previous two theorems are not tight in general. Furthermore, this case corresponds to asymptotically understanding the number of chess rooks needed so that every square in an n by n by n grid is attacked and therefore is of particular interest.

Theorem 3.7. *The following holds:*

$$a_{3,2} \leq \frac{1}{\sqrt{2}}.$$

Note that this bound is less than 1, the bound achieved by Theorem 3.5.

Proof. Let (a, b) be any pair of positive integers that satisfies $2 < \frac{a}{b} < \sqrt{2}+1$, so that $\frac{4ab}{2a-2b} \geq a+b$. Consider a construction on $H_{2a+2b,3}$. For $0 \leq i \leq 2a-1, 0 \leq j \leq 2b-1$ we place a 2-rook at $\left(i, j, \lfloor \frac{2bi+j}{2a-2b} \rfloor\right)$ that covers along the second and third coordinates and place a 2-rook at $\left(2a+2b-j-1, 2a+2b-i-1, 2a+2b-1-\lfloor\frac{2bi+j}{2a-2b}\rfloor\right)$ which attacks along the first and third coordinates. We note that the points between these groups are distinct since the first coordinates between them never coincide.

Now we claim that the uncovered squares in the plane $z=k$ are contained in the union of $2b$ columns and $2b$ rows. Indeed, in each plane of this form, either $2a-2b$ rooks of the first type are covering in the second coordinate, or $2a-2b$ rooks of the second type are covering the third direction. Since the corresponding rooks are in distinct rows, the remaining plane can be covered via at most $2b$ 2-rooks, so this construction yields a covering of $H_{2a+2b,3}$ with at most $8ab+2b\,(2a+2b)=4b^2+12ab$ 2-rooks. This yields an upper bound $a_{3,2}\le\frac{4b^2+12ab}{4(a+b)^2}=\frac{1+3t}{(1+t)^2}$ where $t=\frac{a}{b}$ as the proof of Theorem 2.1 implies $\frac{a_{n,3,2}}{n^2}\ge a_{3,2}$. Taking $t=\frac{a}{b}$ to be an arbitrarily precise rational approximation of $\sqrt{2}+1$ from below, we obtain

$$a_{3,2}\le\frac{4+3\sqrt{2}}{\left(2+\sqrt{2}\right)^2}=\frac{1}{\sqrt{2}}$$

as required. □

Theorem 3.8. *The following holds:*

$$a_{3,2}\ge\alpha,$$

where $\alpha\approx 0.583567$ *is*

$$\min_{0\le c\le 1}\frac{1-c+2c^2+c^3-c^4}{1+2c-c^2}.$$

Remark. The optimal value of c in the above statement is the root of $2c^5-7c^4+6c^2+6c-3$ near .378235.

Proof. In order to prove this lower bound, we first prove the following crucial algebraic lemma.

Lemma 3.9. *Suppose that c_i, x_i are nonnegative reals with $c_i\in[0,n], x_i\in[0,2n]$ for $1\le i\le n$. Let $C=\sum_{i=1}^n c_i$ and $X=\sum_{i=1}^n x_i$, and $(n-c_i-x_i)(n-c_i)\le X$ for each i. Further suppose that $\sum_{i=1}^n\left(X-(n-c_i-x_i)(n-c_i)\right)\ge\frac{X^2}{2n-\frac{X}{n}}$. Then*

$$C+X\ge\alpha n^2-o\left(n^2\right).$$

where $\alpha\approx 0.583567$ *is*

$$\min_{0\le c\le 1}\frac{1-c+2c^2+c^3-c^4}{1+2c-c^2}.$$

Remark. Here $o\left(n^2\right)$ denotes a function $g(n)$ such that $\lim_{n\to\infty}\frac{g(n)}{n^2}=0$ and we will similarly use $h(n)=O(g(n))$ to mean that there exist absolute constant C such that $h(n)\le Cg(n)$ for all positive integers n. Since we use a compactness argument the function $o\left(n^2\right)$ in theory is not quantitative but with additional work can be shown to be $O(n)$.

Proof. Let $t = \frac{X}{n^2}$, and define $S_i = (n - c_i)(n - x_i - c_i)$. If $t \geq 1$ then the conclusion is immediate and hence we assume that $t \leq 1$.

Let $y(x_1, c_1, \ldots, x_n, c_n) = C + X$, and $z(x_1, c_1, \ldots, x_n, c_n) = \sum_{i=1}^{n} S_i$. The conditions describe a closed and bounded subset of $\mathbb{R}^{2n}$, which is hence compact. Since y, z are continuous, it follows that the pair (y, z) achieves its lexicographical minimum if this domain is nonempty. That is, there is an assignment of x_i, c_i which yields a pair (y_0, z_0), for which any other achievable (y, z) satisfies either $y_0 < y$, or $y_0 = y, z_0 \leq z$. To prove this note that the set of values which achieve the minimum for $y(x_1, c_1, \ldots, x_n, c_n)$ in this domain is once again a compact subset. Using that a continuous function achieves a minimum on compact subsets for $z(x_1, c_1, \ldots, x_n, c_n)$ we have proved that a lexicographical minimum is achieved. Since we wish to find a lower bound for $C + X$, it suffices to consider an assignment of (x_i, c_i) which achieves the lexicographical minimum.

We now deduce a number of properties regarding this optimal value through a number of smoothing procedures which demonstrate that the optimum exhibits certain characteristics.

Suppose there exist distinct indices i, j with $0 < x_i < 2n, 0 < x_j < 2n, S_i < X, S_j < X$, and $c_i \geq c_j$. Let ε be the smallest positive real such that the replacement $x_i \to x_i - \varepsilon$ and $x_j \to x_j + \varepsilon$ hits one of these six boundaries (referring to $0 \leq x_i$, $x_i \leq 2n$, $0 \leq x_j$, $x_j \leq 2n$, $S_i \leq X$, $S_j \leq X$), and make that replacement.

Each application of this procedure keeps $y(x_1, c_1, \ldots, x_n, c_n)$ constant and we claim that it does not increase z. Indeed, the change in $S_i + S_j$ is $\varepsilon(c_j - c_i) \leq 0$. The procedure also clearly preserves all of the conditions necessary in order to maintain that we are within the desired domain. Furthermore, each application results in an additional index k for which $x_k = 0, x_k = 2n$, or $S_k = X$. Hence, after finitely many applications, all but possibly one index is in one of these three categories, and (y, z) achieves its lexicographical minimum. Let A be the set of indices for which $x_k = 0$, let B be the set of indices for which $x_k = 2n$, and let D be the set of indices for which $S_k = X$ and $x_k \neq 0$. Let E denote the set of remaining indices, which has size at most one currently.

We furthermore claim that all indices $k \in D$, except for possibly one, must satisfy $(x_k, c_k) = (n - X/n, 0)$. Indeed, suppose otherwise; then there is a pair of indices $i, j \in D$ with $c_i > 0, c_j > 0, 0 < x_i \leq x_j < n - \frac{X}{n}$. Let

$$g(t) = \frac{2n - t - \sqrt{t^2 + 4X}}{2}.$$

Then the condition $S_k = X$ implies that $c_k = g(x_k)$ for $k \in D$. In particular, $c_i = g(x_i)$ and $c_j = g(x_j)$. Since $x_i, x_j > 0$, we have $X > 0$, which means that the function $g(t)$ is strictly concave. This is immediate as

$$g''(t) = -2X\left(t^2 + 4X\right)^{-3/2} < 0$$

for all of $t \in \mathbb{R}$. Given this we can pick $0 < \varepsilon < \min\{x_i, n - \frac{X}{n} - x_j\}$, and replace $(x_i, c_i), (x_j, c_j)$ with $(x_i - \varepsilon, g(x_i - \varepsilon)), (x_j + \varepsilon, g(x_j + \varepsilon))$. By the strict concavity of g, we have $g(x_i) + g(x_j) > g(x_i - \varepsilon) + g(x_j + \varepsilon)$, which means this operation would strictly decrease y. This violates the assumption that our assignment attains the lexicographical minimum of (y, z). Hence there is at most one $k \in D$ with $(x_k, c_k) \neq (n - X/n, 0)$. If that extraneous k does exist, move it to E and note that E has size at most 2.

Finally we claim that $c_i = c_j$ for all $i, j \in A$. Indeed, suppose otherwise. Then we may replace $(0, c_i)$ and $(0, c_j)$ with two copies of $\left(0, \frac{c_i + c_j}{2}\right)$. Since $a^2 + b^2 > 2\left(\frac{a+b}{2}\right)^2$ for $a \neq b$, this transformation would strictly decrease z (and hold y constant), which contradicts (y, z) being at a lexographical minimum. The same argument shows that $c_i = c_j$ for $i, j \in B$. Let $c_i = c_A$ for $i \in A$, and $c_i = c_B$ for $i \in B$. If either set is empty, define the respective constant to be 0.

We now define the *special* setting to be where all but at most two indices are either of the form $x_i = nd, c_i = 0, d \in [0, 2]$, or $x_i = 0, c_i = nc, c \in [0, 1]$. We now take two cases based on whether D is empty and reduce each of the cases to this special scenario.

Case 1: D is empty. If B is empty, then only at most two indices k can satisfy $x_k > 0$; namely, $k \in E$. Then $X \leq 4n$. For the $\geq n-2$ indices k satisfying $x_k = 0$, we then have $(n - c_k)^2 \leq 4n$, which means $c_k = n - o(n)$. In particular, $C = n^2 - o(n^2)$ and the result follows in this case. If A is empty, then $x_k = 2n$ for all but at most one k, which means $X = 2n^2 - o(n)$ and this case is also immediate. Hence, it suffices to consider A, B nonempty. We claim $c_A = n$ or $c_B = 0$. Otherwise we can replace a pair of indices $(2n, c_B), (0, c_A)$ with $(2n, c_B - \varepsilon), (0, c_A + \varepsilon)$, which strictly decreases z and holds y constant while preserving all conditions which contradicts being at a lexicographical minimum. (For ε small the change is $\varepsilon(-2c_B + 2c_A - 2n)$ and note that $(-2c_A + 2c_B - 2n) < 0$ by assumption.) In the case when $c_A = n$, we have $c_k + x_k \geq n$ for all but at most two indices k, so $C + X \geq n^2 + o(n^2)$. Finally in the case where we have $c_B = 0$, we are in the special setting with $(c, d) = \left(\frac{c_A}{n}, 2\right)$.

Case 2: D is nonempty. If B is empty, then we are again in the special setting mentioned previously. Otherwise, we claim $c_B = 0$. Indeed, else, for a pair of indices we may swap $(n - X/n, 0), (2n, c_B)$ with $(n - X/n, c_B), (2n, 0)$, which would hold y the same, preserve the given conditions, and strictly decrease z. Note that $(n - X/n, c_B)$ satisfies the constraint as $(n - c_i - x_i)(n - c_i) = (X/n - c_B)(n - c_B) \leq X$ and to see that z strictly decreases, note the change can explicitly be computed to be $c_B n + c_B(X/n)$. Therefore we have reduced to the case $c_B = 0$. Then we may merge groups B and D and average together all of the x coordinates. S_k is linear as a function in x_k for fixed c_k, so this operation preserves the conditions as well as the values y, z. This averaging also lands us into the special setting, since everything in the merged group is of the form $(dn, 0), d \in [0, 2]$, and everything in A is of the form $(0, cn), c \in [0, 1]$.

We now analyze the special setting with indices in set L which are $(nd, 0)$ and indices in the set M which are $(0, nc)$. Let L have ℓn indices. Since E has size less than 2, M has $(1 - \ell)n + O(1)$ indices. Furthermore it follows that $X/n^2 = (\ell \cdot d) + O(1/n) = t$ as $|x_i| \leq 2n$ for the at most two coordinates in E. The constraint that

$(n - x_i - c_i)(n - c_i) \leq X$ for all indices in M and L become $(1-c)^2 \leq t + O(1/n)$ and $(1-d) \leq t + O(1/n)$. Finally the constraint that

$$\sum_{i=1}^{n} (X - (n - c_i - x_i)(n - c_i)) \geq \frac{X^2}{2n - \frac{X}{n}}$$

becomes

$$t - (\ell)(1-d) - (1-\ell)(1-c)^2 \geq t^2/(2-t) - O(1/n).$$

Note that $nc \in [0, n]$ becomes $c \in [0, 1]$ and $nd \in [0, 2n]$ becomes $d \in [0, 2]$ and that $\ell \in [0, 1]$. Finally we may assume that $t \in [0, 1]$ due to the first reduction in the proof. The objective now translates to $C + X = \ell dn^2 + (1-\ell)cn^2 + O(n)$ and thus to determine α in the theorem statement it suffices to solve the following optimization problem:

$$\min(\ell d + (1-\ell)c)$$

subject to the constraints

- $t \in [0, 1]$
- $c \in [0, 1]$
- $d \in [0, 2]$
- $\ell \in [0, 1]$
- $t - (\ell)(1-d) - (1-\ell)(1-c)^2 \geq t^2/(2-t)$
- $(1-c)^2 \leq t$
- $(1-d) \leq t$
- $t = \ell \cdot d$

and this is addressed precisely by Lemma 5.5 in the appendix to give the theorem's statement.

Note here we are using the real analysis that the optimum value of a continuous function over a set of nesting compact domains approaches the obvious limit. In particular let D_i be a set of compact domains such that $D_1 \supseteq D_2 \supseteq \ldots$, $D = \cap_{i=1}^{\infty} D_i$, and Opt_i be the optimum over a continous function h defined on D_1 and hence all D_i. Then we have that Opt_i approaches the optimum value of h over the domain D. The proof is simply to take a convergent subsequence of the sequence of optima. Note that the limit must lie in D_k for all k and hence in D itself. In our case, D_i is the compact set obtained when all of the conditions above are relaxed by an additive lower order $O\left(\frac{1}{n}\right)$ term, and D is exactly the compact set given by the above conditions. $\square$

Lemma 3.10. *Suppose that $0 < X \leq n^2$ squares are marked in an $n \times n$ grid, and in each marked square is written either the number of marked squares in the same row, or the number of marked squares in the same column. Then the sum of the written numbers is at least $\frac{X^2}{2n-\frac{X}{n}}$.*

Proof. We claim that the sum of the reciprocal of the written numbers is at most $2n - \frac{X}{n}$. Indeed, for a marked square m_i, let c_i, n_i denote the number of marked squares in the chosen and not chosen direction of m_i respectively. Then:

$$\begin{aligned}\sum_{i=1}^{X} \frac{1}{c_i} &= \sum_{i=1}^{X} \left(\frac{1}{c_i} + \frac{1}{n_i}\right) - \sum_{i=1}^{X} \frac{1}{n_i} \\ &\leq 2n - \sum_{i=1}^{X} \frac{1}{n_i} \\ &\leq 2n - \frac{X}{n}\end{aligned}$$

Here, we have used estimate $\sum_{i=1}^{X} \left(\frac{1}{c_i} + \frac{1}{n_i}\right) \leq 2n$, which is true since the sum of the column-directional terms along any column is at most 1, and similarly for row-directional terms.

Then the result follows from the Cauchy-Schwarz inequality, since

$$\left(\sum_{i=1}^{X} \frac{1}{c_i}\right)\left(\sum_{i=1}^{X} c_i\right) \geq X^2.$$

□

Now we proceed to the proof of the lower bound in Theorem 3.8. Suppose that there is a configuration of 2-rooks which covers an $H_{n,3}$ situated at $1 \leq x, y, z \leq n$ in coordinate space. We first claim that this configuration may be transformed into one in which no two rooks cover each other; equivalently, no two rooks share the same axis. Indeed, consider some instance of this, with rooks R_1 and R_2, covering axis A jointly. Let B be the other axis covered by R_2. Then there are two cases. If every other space on B is covered by another rook already, then removing R_2 entirely will still result in a valid covering. Otherwise, moving B to the uncovered space results in a valid covering, and decreases the number of pairs of rooks which share an axis. So, after finitely many iterations of this, we may assume no two rooks share an axis.

We call rooks *cross rooks* if they cover in the x and y directions, and all other rooks *axis rooks.* We call the set of axes covered by axis rooks the *strong axes.*

For each $i, 1 \leq i \leq n$, we denote by x_i the number of axis rooks which lie in the plane $z = i$, and by c_i the number of cross rooks which lie in this plane. Let $C = \sum_{i=1}^{n} c_i, X = \sum_{i=1}^{n} x_i$. Note that we may assume $c_i \leq n$ since n cross rooks can already cover a plane. We claim that, in the i-th plane, at most $n^2 - (n - c_i)(n - c_i - x_i)$ points are covered by rooks in that plane. Indeed, suppose that

h_i of the x_i axis rooks choose to cover a row in $z = i$, and v_i choose to cover a column, so that $v_i + h_i = x_i$. Then at most $c_i + v_i$ rows are covered by the rooks in plane i, and at most $c_i + h_i$ columns are covered. Hence in total, at most:

$$\begin{aligned} n^2 - (n - c_i - h_i)(n - c_i - v_i) &= n^2 - (n - c_i)(n - c_i - x_i) - h_i v_i \\ &\leq n^2 - (n - c_i)(n - c_i - x_i) \end{aligned}$$

points in plane i are covered by rooks in that plane, as required. It follows that the remaining points in plane i must be covered by axis rooks from other planes, so that in particular $X \geq (n - c_i)(n - c_i - x_i)$ for every i.

Let P_i denote the number of strong axes covered by axis rooks in the ith plane, and let C_i denote the number of total points covered by rooks outside the plane, so that $X \geq C_i + P_i$ for each i. Note that $C_i \geq (n - c_i)(n - c_i - x_i) + h_i v_i$ from before.

We translate to the setting of Lemma 3.10. Project all of the axis rooks onto the xy plane. By assumption, no two axis rooks will project to the same coordinate. For each axis rook, write in its square the number of strong axes covered by that rook in its corresponding plane. Then according to Lemma 3.10, the sum of the written number will be at least $\frac{X^2}{2n - \frac{X}{n}}$. On the other hand, consider only those axis rooks in the ith plane. The sum of these numbers is P_i plus the number of strong axes covered twice in the ith plane. Note that no axis can be covered more than twice in a plane, since no pair of rooks cover each other by assumption. The number of strong axes which are in both the row and column of some rook in the ith plane is at most $v_i h_i$, since that is the number of points in the ith plane which are covered twice by axis rooks in that plane. Hence

$$\sum_{i=1}^{n} (P_i + v_i h_i) \geq \frac{X^2}{2n - \frac{X}{n}}.$$

Combining this with previous estimates gives

$$\sum_{i=1}^{n} (X - (n - c_i)(n - c_i - x_i)) \geq \sum_{i=1}^{n} (X - C_i + v_i h_i) \geq \sum_{i=1}^{n} (P_i + v_i h_i) \geq \frac{X^2}{2n - \frac{X}{n}}.$$

Therefore, the x_i, c_i satisfy the conditions given in Lemma 3.9, so it follows that the total number of rooks used is

$$C + X \geq \alpha n^2 - o\left(n^2\right)$$

Hence $a_{3,2} \geq \alpha$ as required. □

4 Bounds for Packing

In this section we prove that $b_{k,\ell} = \frac{k}{\ell}$ and $c_{k,\ell} = \frac{\binom{k}{2}}{\binom{\ell}{2}}$ for certain special values of k and ℓ. We begin by demonstrating that this first equality holds when ℓ divides k.

Theorem 4.1. *For positive integers k, t, we have $b_{kt,t} = k$.*

Proof. By Theorem 1.3, it follows that $b_{kt,t} \leq k$. Therefore, it suffices to demonstrate that $b_{kt,t} \geq k$. We prove this by demonstrating that $b_{n,kt,t} \geq kn^{k(t-1)} (n-1)^{k-1}$ through explicit construction.

Consider points of the form $(x_1, \ldots, x_t, x_{t+1}, \ldots, x_{2t}, \ldots, x_{kt})$ with $0 \leq x_i \leq n-1$ for $1 \leq i \leq kt$. Define the L_j block of points as the set of points that satisfy

$$\sum_{i=0}^{t-1} x_{jt-i} \equiv 0 \mod n$$

and satisfy for $m \in \{1, \ldots, \ell\}$ and $m \neq j$,

$$\sum_{i=0}^{t-1} x_{mt-i} \not\equiv 0 \mod n.$$

For each point in the L_j block, place an ℓ-rook that attacks in the direction of the $((j-1)\,t+1)^{th}$ coordinate to the jt^{th} coordinate. Note that

$$|L_j| = n^{t-1} \left(n^{t-1} (n-1)\right)^{k-1} = n^{k(t-1)} (n-1)^{k-1}$$

and thus taking the union of these rooks for $1 \leq j \leq k$ it follows that

$$\left| \bigcup_{j=1}^{k} L_j \right| = kn^{k(t-1)} (n-1)^{k-1}.$$

Now we demonstrate that no rook attacks another in the above constructions. Suppose for the sake of contradiction that R_1 attacks R_2 with $R_1 \in L_i$ and $R_2 \in L_j$. If $i \neq j$ then note that R_1 and R_2 differ in at least one coordinate in $x_{(i-1)t+1}, \ldots, x_{it}$ and at least one coordinate in $x_{(j-1)t+1}, \ldots, x_{jt}$. Since attacking rooks differ by at most 1 coordinate, such rooks R_1 and R_2 do not exist. Otherwise R_1 and R_2 both lie in L_i. If these rooks differ in the coordinates $x_{(i-1)t+1}, \ldots, x_{it}$ then they differ in at least two positions and therefore they cannot attack each other. Otherwise R_1 and R_2 differ outside of the coordinates $x_{(i-1)t+1}, \ldots, x_{it}$, and since R_1 and R_2 attack in these coordinates, R_1 does not attack R_2 and the result follows. $\square$

We can now establish a crude lower bound for $b_{k,\ell}$.

Corollary 4.2. *For positive integers k, ℓ, we have $b_{k,\ell} \geq \lfloor \frac{k}{\ell} \rfloor$.*

Proof. Note that $nb_{n,k,\ell} \leq b_{n,k+1,\ell}$ as we can stack n constructions of $b_{n,k,\ell}$ in the $(k+1)^{st}$ dimension. Therefore it follows that $b_{k,\ell} \leq b_{k+1,\ell}$ and that $b_{k,\ell} \geq b_{\ell\lfloor \frac{k}{\ell} \rfloor,\ell} = \lfloor \frac{k}{\ell} \rfloor$ where we have used Theorem 4.1 in the final step. $\square$

The last bound we establish for $b_{k,\ell}$ is that in fact $b_{k,2} = \frac{k}{2}$ for all integers $k \geq 2$.

Theorem 4.3. *For integers $k \geq 2$, we have $b_k = \frac{k}{2}$.*

Proof. We will provide an inductive construction based on constructions in Section 2. In particular, we show the following:

Claim: For every integer $k \geq 2$, there exists a nonnegative constant c_k such that $b_{n,k,2} \geq \frac{k}{2}n^{k-1} - c_k n^{k-2}$.

For the base case $k = 2$, we may take $c_2 = 0$, and place 2-rooks along the main diagonal of $H_{n,2}$ which is a size-n square. Suppose the claim holds for all $j \leq k-1$ and that $(k-1)!^2$ divides n. Then there exists some set S of $\frac{k-1}{2}n^{k-2} - c_{k-1}n^{k-3}$ 2-rooks that tiles $H_{n,k-1}$.

We now describe a way to pack

$$\frac{(k-1)^k \left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1} - k + 1\right)!}$$

labeled 1-rooks into a $H_{n,k-1}$ so that no two 1-rooks of the same label attack each other. For this, we first group some of the n^{k-1} points in the hypercube into

$$\frac{\left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1} - k + 1\right)!}$$

buckets of size $(k-1)^{k-1}$. We do this by sending the point $(x_1, \ldots, x_{k-1})$ to a bucket labeled $\left(\lfloor \frac{x_1}{k-1} \rfloor, \ldots, \lfloor \frac{x_k}{k-1} \rfloor\right)$ if and only if the $\lfloor \frac{x_i}{k-1} \rfloor$ are distinct.

Notice that the points in each bucket form a $H_{k-1,k-1}$. Within each bucket, we partition the points into $k-1$ parts of the form $\sum_{i=1}^{k-1} x_i \equiv j \mod k-1$, each of which has $(k-1)^{k-2}$ points. We then label each point in the j-th part of such a partition with the label $\lfloor \frac{x_j}{k-1} \rfloor$.

When this is done, there are

$$\frac{(k-1)^k \left(\frac{n}{k-1}\right)!}{n\left(\frac{n}{k-1} - k + 1\right)!}$$

points of label i for each $i \in \{1, \ldots, \frac{n}{k-1}\}$. All points of label i have $\lfloor \frac{x_j}{k-1} \rfloor = i$ for exactly one index j. Therefore, assigning all points of label i to attack in the direction corresponding to this direction yields a packing of $\frac{(k-1)^k \left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1} - k + 1\right)!}$ labeled 1-rooks into a $H_{n,k-1}$ so that no two 1-rooks of the same label attack each other, as required.

Now we combine this partition P with the set S. For $1 \leq x_k \leq \frac{n}{k-1}$, we let the last coordinate act as the label for P and have all of these rooks attack in the direction of the last coordinate in addition to their normal direction. For $\frac{n}{k-1} + 1 \leq x_k \leq n$, we fill each layer corresponding to a fixed x_k according to S. The number of points in the construction at this point is at least

$$\frac{(k-1)^k\left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1}-k+1\right)!}+\left(\frac{k-2}{k-1}n\right)\left(\frac{k-1}{2}n^{k-1}-c_kn^{k-2}\right)$$
$$\geq \frac{k}{2}n^{k-1}-\left(\frac{k-2}{k-1}c_k+\frac{k\,(k-1)^2}{2}\right)n^{k-2}.$$

Here we have used the estimate that

$$\frac{\left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1}-k+1\right)!}\geq\left(\frac{n}{k-1}\right)^{k-1}-\frac{(k-1)\,(k-2)}{2}\left(\frac{n}{k-1}\right)^{k-2}.$$

For a proof of this estimate, see the appendix. At this point, the only pairs of 2-rooks that attack other 2-rooks are the rooks from P that attack the rooks from S. But there are at most $\frac{k-1}{2}n^{k-2}$ rooks in S and each rook in S lead to at most 1 offending point in P. Therefore we may simply remove these rooks to obtain a configuration of at least $\frac{k}{2}n^{k-1}-\left(\frac{k-2}{k-1}c_k+\frac{k(k-1)^2}{2}+\frac{k-1}{2}\right)n^{k-2}$ 2-rooks, none of which attack each other.

It follows that $b_{k,2}=\lim_{t\to\infty}b_{(k-1)!^2t,k,2}=\frac{k}{2}$, as desired. □

Transitioning, we now determine $c_{k,2}$ and $c_{k,k}$. The second constant is known implicitly in the literature, but the proof is included in the following theorem.

Theorem 4.4. *For all positive integers k, $c_{k,k}=1$ and for $k\geq 2$, $c_{k,2}=\binom{k}{2}$.*

Proof. We begin by proving that $c_{k,k}=1$. By Theorem 1.3, $c_{k,k}\leq 1$. The construction that $c_{k,k}\geq 1$ is exactly the one given in Theorem 2.2 as this demonstrates $c_{p,k,k}=p^{k-2}$ for primes $p>k$. The result follows.

For the second part of the proof, note that $c_{k,2}\leq\binom{k}{2}$ by Theorem 1.3. Therefore, it suffices to demonstrate that $c_{k,2}\geq\binom{k}{2}$. To demonstrate this we prove that $c_{n,k,2}\geq\binom{k}{2}\left(n-2\binom{k}{2}\right)^{k-2}$ for $n>2\binom{k}{2}$. In particular, for $i<j$, let $A_{i,j}$ be the set of points with i-th and j-th coordinates being $2i-2+(j-1)\,(j-2)$ and $2i-1+(j-1)\,(j-2)$ respectively, and other coordinates varying in the range $[k\,(k-1)+1,n-1]$. Note that $2i-2+(j-1)\,(j-2)$ and $2i-1+(j-1)\,(j-2)$ lie between $[0,k\,(k-1)]$ and take each value in this range exactly once. Now for each point in $A_{i,j}$, orient the corresponding 2-rook to attack in the direction of the i-th and j-th axes. No two rooks within a set attack each other as they differ outside the i-th and j-th coordinates, and any pair of rooks from different sets differ in at least 3 coordinates and therefore cannot attack each other. Thus, the result follows by taking all $A_{i,j}$ where the 2-rooks in $A_{i,j}$ are directed to attack along the i-th and j-th dimension. □

5 Conclusion and Open Questions

There are several natural questions and conjectures regarding the values of $a_{k,\ell}$, $b_{k,\ell}$, and $c_{k,\ell}$. The most surprising open question is the following.

Conjecture 5.1. *For integers $k \geq 2$, $a_{k,k} = \frac{1}{k-1}$.*

Note that the above conjecture is known when k is one more than a power of a prime [1], and the construction is essentially that of perfect codes. This construction implies that the first open case of this conjecture is $a_{7,7}$. The most natural case when $k \neq \ell$ and that is not covered by our result is $a_{3,2}$.

Question 5.2. What is the exact value of $a_{3,2}$?

We end with a pair of conjectures based on the results of the previous section, and in contrast to $a_{k,\ell}$ we conjecture exact values for $b_{k,\ell}$ and $c_{k,\ell}$ that are upper bounds from Theorem 1.3.

Conjecture 5.3. *For positive integers $k \geq \ell$, $b_{k,\ell} = \frac{k}{\ell}$.*

Conjecture 5.4. *For positive integers $k \geq \ell \geq 2$, $c_{k,\ell} = \frac{\binom{k}{2}}{\binom{\ell}{2}}$.*

Appendix

Lemma 5.5. *The optimum for the optimization problem*

$$\min \ell d + (1-\ell)\, c$$

subject to the constraints

- $c \in [0,1]$
- $d \in [0,2]$
- $\ell \in [0,1]$
- $t - (\ell)(1-d) - (1-\ell)(1-c)^2 \geq t^2/(2-t)$
- $(1-c)^2 \leq t$
- $(1-d) \leq t$
- $t = \ell \cdot d$

is larger than the optimum for the optimization problem

$$\min_{0 \leq c \leq 1} \frac{1 - c + 2c^2 - c^3 + c^4}{1 + 2c - c^2}$$

Proof. We begin by dropping the constraints that $(1-d) \leq t$ and $\ell \in [0,1]$. Furthermore the variable d can be entirely eliminated at the cost of dropping the constraint $d \in [0,2]$, rewriting the objective as $t + (1-\ell)\, c$, and substituting into the fourth constraint. In particular we have reduced to the optimization problem of

$$\min\,(t + (1-\ell)\, c)$$

subject to the constraints

- $c \in [0, 1]$
- $t \in [0, 1]$
- $2t - \ell - (1 - \ell)(1 - c)^2 \geq t^2 / (2 - t)$
- $(1 - c)^2 \leq t$

Now note that the second constraint can be rewritten as

$$\left(\frac{t(4-3t)}{2-t} - (1-c)^2 \right) \geq \ell \left(1 - (1-c)^2 \right).$$

Since the function we are optimizing is decreasing in ℓ for $c > 0$, we find that we may assume that

$$\ell = \left(\frac{t(4-3t)}{2-t} - (1-c)^2 \right) / \left(1 - (1-c)^2 \right).$$

The endpoint case that $c = 0$ is trivial as we are now simply minimizing t and $(1 - c)^2 \leq t$ implies $t \geq 1$. Note that

$$1 - \ell = \frac{(1-t)(2-3t)}{c(2-c)(2-t)}$$

upon simplifying. Thus our optimization problem reduces to

$$\min \left(t + \frac{(1-t)(2-3t)}{(2-c)(2-t)} \right)$$

subject to the constraints

- $c \in [0, 1]$
- $t \in [0, 1]$
- $(1 - c)^2 \leq t$

At this point we treat c as a constant and note that the endpoint $t = 1$ trivially gives 1. Now taking a derivative in t we find that

$$\frac{d}{dt}\left(t + \frac{(1-t)(2-3t)}{(2-c)(2-t)} \right) = \frac{c(t-2)^2 + (t-4)t}{(c-2)(t-2)^2}.$$

Setting equal to 0 the possible critical points are $t = \frac{2\left(1+c\pm\sqrt{1+c}\right)}{1+c}$ and only $t = \frac{2\left(1+c-\sqrt{1+c}\right)}{1+c}$ can ever lie in the interva l $[(1 - c)^2, 1]$ and for $c \leq 3$ the value is always ≤ 1. Since the second derivative is

$$\frac{d^2}{dt^2}\left(t + \frac{(1-t)(2-3t)}{(2-c)(2-t)} \right) = \frac{8}{(2-c)(2-t)^3} \geq 0$$

it suffices to consider $t = \max\{(1-c)^2, \frac{2(1+c-\sqrt{1+c})}{1+c}\}$. At this point this is a numerical verification; note for $0 \le c \le 17/40$ we have $(1-c)^2 \ge \frac{2(1+c-\sqrt{1+c})}{1+c}$. Thus we can relax the optimization to the minimum of

$$\min\left(t + \frac{(1-t)(2-3t)}{(2-c)(2-t)}\right)$$

subject to the constraints

- $c \in [17/40, 1]$
- $t = 2\left(1 + c - \sqrt{1+c}\right)/(1+c)$

and

$$\min\left(t + \frac{(1-t)(2-3t)}{(2-c)(2-t)}\right)$$

subject to the constraints

- $c \in [0, 1]$
- $t = (1-c)^2$

These are both explicit 1-variable optimization problems and in the first problem substituting $t = 2\left(1 + c - \sqrt{1+c}\right)/(1+c)$ we need to optimize

$$\frac{4\sqrt{1+c} - 3 - 2c}{2-c}.$$

The derivative in c is

$$\frac{8 + 2c - 7\sqrt{1+c}}{(2-c)^2\sqrt{1+c}} \ge 0$$

for $c \le 5/4$ and thus it suffices to take $c = 17/40$ which gives a value of $\ge .587$. The second optimization, upon substituting, is equivalent to

$$\min_{0 \le c \le 1} \frac{1 - c + 2c^2 + c^3 - c^4}{1 + 2c - c^2}$$

and taking $c = .378235$ this is less than $.587$. This completes the proof. (Note that this minimum is less than 1; this fact is used implicitly several times in the proof when discarding cases.) $\square$

Lemma 5.6. *Suppose that $k \ge 2$, and $(k-1)^2 | n$. Then*

$$\frac{\left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1} - k + 1\right)!} \ge \left(\frac{n}{k-1}\right)^{k-1} - \frac{k(k-1)}{2}\left(\frac{n}{k-1}\right)^{k-2}.$$

Proof. Let $a = \frac{n}{k-1} \geq k-1$. Then we calculate:

$$\begin{aligned}\frac{\left(\frac{n}{k-1}\right)!}{\left(\frac{n}{k-1}-k+1\right)!} &= a^{k-1}\prod_{i=0}^{k-2}\left(1-\frac{i}{a}\right)\\ &\geq a^{k-1}\left(1-\sum_{i=0}^{k-2}\frac{i}{a}\right)\\ &= a^{k-1}-\frac{(k-1)(k-2)}{2}a^{k-2}\end{aligned}$$

as desired. Here we have used the classical inequality $\prod_{i=0}^{n}(1-x_i) \geq 1-\sum_{i=0}^{n} x_i$ for $n = k-2$ and $x_i = \frac{i}{a} \in [0,1]$. □

Acknowledgements

This research was conducted at the University of Minnesota Duluth REU and was supported by NSF grant 1659047. We would like to thank Ashwin Sah, Joe Gallian, Colin Defant, and Ben Gunby for reading over the manuscript and providing valuable comments. We would especially like to thank Ben Gunby for finding a critical error in an earlier version of the paper. Finally we would like to thank the anonymous referees who significantly improved the exposition of this paper.

References

[1] A. Blokhuis and C. W. Lam, More coverings by rook domains, *J. Combin. Theory Ser. A* 36 (1984), 240–244.

[2] H. Fernandes and E. Rechtschaffen, The football pool problem for 7 and 8 matches, *J. Combin. Theory Ser. A* 35 (1983), 109–114.

[3] W. Haas, Lower bounds for the football pool problem for 7 and 8 matches, *Electron. J. Combin.* 14 (2007), #R27.

[4] W. Haas, J.-C. Schlage-Puchta and J. Quistorff, Lower bounds on covering codes via partition matrices, *J. Combin. Theory Ser. A* 116 (2009), 478–484.

[5] H. Hämäläinen and S. Rankinen, Upper bounds for football pool problems and mixed covering code, *J. Combin. Theory Ser. A* 56 (1991), 84–95.

[6] J. Kalbfleisch and R. Stanton, A combinatorial problem in matching, *J. London Math. Soc.* 1 (1969), 60–64.

[7] K.-U. Koschnick, A new upper bound for the football pool problem for nine matches, *J. Combin. Theory Ser. A* 62 (1993), 162–167.

[8] P. R. Östergård and M. K. Kaikkonen, New upper bounds for binary covering codes, *Discrete Math.* 178 (1998), 165–179.

[9] E. R. Rodemich, Coverings by rook domains, *J. Combin. Theory* 9 (1970), 117–128.

[10] R. Singleton, Maximum distance q-nary codes, *IEEE Trans. Inf. Theory* 10 (1964), 116–118.

[11] G. J. M. van Wee, Bounds on Packings and Coverings by Spheres in q-ary and Mixed Hamming spaces, *J. Combin. Theory Ser. A* 57 (1991), 117–129.

(Received 16 Apr 2019; revised 3 Mar 2020, 18 Apr 2020)