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A dominating pair condition for a balanced bipartite digraph to be hamiltonian*

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Abstract

Let x, y be two distinct vertices in a digraph. We call the pair $\{x, y\}$ dominating if x and y have a common out-neighbour. In this paper we prove that a strong balanced bipartite digraph of order $2a$ contains a hamiltonian cycle if for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - k$, $d(y) \geq a + k$ or $d(x) \geq a + k$, $d(y) \geq 2a - k$, where $\max\{1, \frac{a}{4}\} < k \leq \frac{a}{2}$.

1 Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. The cycle problems for digraphs are among the central problems in graph theory and its applications [6]. A digraph D is called hamiltonian if it contains a hamiltonian cycle, i.e., a cycle that includes every vertex of D. There are many degree or degree sum conditions for hamiltonicity [1, 3, 4, 5, 7, 8, 9]. The following result of Meyniel on the existence of hamiltonian cycles in digraphs is basic and famous.

Theorem 1.1. *[8] Let D be a strong digraph on n vertices, where $n \geq 3$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices x, y in D, then D is hamiltonian.*

In [7], Bang-Jensen, Gutin and Li described a type of sufficient condition for a digraph to be hamiltonian. Conditions of this type combine local structure of the digraph with conditions on the degrees of non-adjacent vertices. Let D be a digraph and x, y be distinct vertices in D. If there is an arc from x to y, then we say that x dominates y and write $x \to y$. Now, $\{x, y\}$ is dominated by a vertex z if $z \to x$ and $z \to y$; in this case, we call the pair $\{x, y\}$ dominated. Likewise, $\{x, y\}$ dominates a vertex z if $x \to z$ and $y \to z$; in this case, we call the pair $\{x, y\}$ dominating.

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Theorem 1.2. *[7] Let D be a strong digraph of order $n \geq 2$. Suppose that, for every dominated pair of non-adjacent vertices $\{x, y\}$, either $d(x) \geq n$, $d(y) \geq n-1$ or $d(x) \geq n-1$, $d(y) \geq n$. Then D is hamiltonian.*

In [7], Bang-Jensen, Gutin and Li raised the following two conjectures.

Conjecture 1.3. *[7] Let D be a strong digraph of order $n \geq 2$. Suppose that $d(x) + d(y) \geq 2n-1$ for every dominating pair of non-adjacent and every dominated pair of non-adjacent vertices $\{x, y\}$. Then D is hamiltonian.*

Conjecture 1.4. *[7] Let D be a strong digraph of order $n \geq 2$. Suppose that $d(x) + d(y) \geq 2n-1$ for every dominated pair of non-adjacent vertices $\{x, y\}$. Then D is hamiltonian.*

Bang-Jensen, Guo and Yeo [5] proved that, if we replaced the degree condition $d(x)+d(y) \geq 2n-1$ with $d(x)+d(y) \geq \frac{5}{2}n-4$ in Conjecture 1.3, then D is hamiltonian. They also proved additional support for Conjecture 1.3 by showing that every digraph satisfying the condition of Conjecture 1.3 has a cycle factor. Conjecture 1.3 is still open and seems quite difficult. Conjecture 1.4 has been disproved recently in [10].

In [4], Adamus, Adamus and Yeo gave a Meyniel-type sufficient condition for hamiltonicity of a balanced bipartite digraph.

Theorem 1.5. *[4] Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(x) + d(y) \geq 3a$ for every pair of non-adjacent vertices x, y in D, then D is hamiltonian.*

In [1], Adamus proved a bipartite analogue of Conjecture 1.3.

Theorem 1.6. *[1] Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(x)+d(y) \geq 3a$ for every dominating pair of vertices and every dominated pair of vertices $\{x, y\}$ in D, then D is hamiltonian.*

In [9], the first author of this paper gave a dominating pair sufficient condition for hamiltonicity of a balanced bipartite digraph, which is a bipartite analogue of Theorem 1.2 (Theorem 1.7 below).

Theorem 1.7. *[9] Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 2$. If, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$, $d(y) \geq a+1$ or $d(x) \geq a+1$, $d(y) \geq 2a-1$, then D is hamiltonian.*

Definition 1.8. Consider a balanced bipartite digraph D of order $2a$, where $a \geq 2$. For an integer $k \geq 0$, we will say that D satisfies the condition B_k when

$$d(x) \geq 2a-k, d(y) \geq a+k \text{ or } d(x) \geq a+k, d(y) \geq 2a-k,$$

for any dominating pair of vertices $\{x, y\}$ in D.

It is natural to propose the following problem.

Problem 1.9. *[9] Consider a strong balanced bipartite digraph D of order $2a$. Suppose that D satisfies the condition B_k with $2 \leq k \leq \frac{a}{2}$. Is D hamiltonian?*

In Section 3, we will prove that if D satisfies the condition of Problem 1.9, then D contains a cycle factor. In Section 4, we will prove that if $\max\{1, \frac{a}{4}\} < k \leq \frac{a}{2}$ in Problem 1.9, then D is hamiltonian (Theorem 1.10 below).

Theorem 1.10. *Let D be a strong balanced bipartite digraph of order $2a$. If D satisfies the condition B_k with $max\{1, \frac{a}{4}\} < k \leq \frac{a}{2}$, then D is hamiltonian.*

The proof of Theorem 1.10 is based on the arguments of [1, 4].

2 Terminology

We will assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [6] for terminology not defined here. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$.

For a vertex set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ dominated by the vertices of S; i.e. $N^+(S) = \{u \in V(D) : vu \in A(D)$ for some $v \in S\}$. Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ dominating vertices of S; i.e. $N^-(S) = \{u \in V(D) : uv \in A(D)$ for some $v \in S\}$. If $S = \{v\}$ is a single vertex, the cardinality of $N^+(v)$ (resp. $N^-(v)$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the out-degree (resp. in-degree) of v in D. The degree of v is $d(v) = d^+(v) + d^-(v)$. For a pair of vertex sets X, Y of D, define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S.

A digraph D is said to be strongly connected or just strong, if for every pair of vertices x, y of D, there is an (x, y)-path. A cycle factor in D is a collection of vertex-disjoint cycles $C_1, C_2, \ldots, C_t$ such that $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_t) = V(D)$.

A digraph D is bipartite when $V(D)$ is a disjoint union of independent sets V_1 and V_2. It is called balanced if $|V_1| = |V_2|$. A matching from V_1 to V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (u_1u_2 and v_1v_2 are independent arcs when $u_1 \neq v_1$ and $u_2 \neq v_2$). If D is balanced, one says that such a matching is perfect if it consists of precisely $|V_1|$ arcs. If for every pair of vertices x, y from distinct partite sets, xy and yx are in D, then D is called complete bipartite.

3 Lemmas

The proof of Theorem 1.10 will be based on the following four lemmas.

Lemma 3.1. *Let D be a strong balanced bipartite digraph of order $2a$. Suppose that D satisfies the condition B_k for some $2 \leq k \leq \frac{a}{2}$ and D is non-hamiltonian. Then for every vertex $u \in V(D)$ there exists $v \in V(D) \setminus \{u\}$ such that $\{u, v\}$ is a dominating pair of vertices. Moreover, the degree of every vertex in D is larger than or equal to $a + k$ in D.*

Proof. Suppose that there exists a vertex $u' \in V(D)$ such that u' has no common out-neighbour with any other vertex in D.

We claim that no vertex has a common out-neighbour with any other vertex. In fact, since D is strong, for any $z \in V(D) \setminus \{u'\}$, there exists a (u', z)-path. Denote it by $P = u_1u_2 \dots u_p$, where $u_1 = u', u_p = z$. Since u_1 has no common out-neighbour with any other vertex in D, we have $d^-(u_2) = 1$. Then $d(u_2) \leq a+1$. As $2 \leq k \leq \frac{a}{2}$, $2a - k \geq a + k \geq a + 2$. By assumption, for any dominating pair of vertices $\{x, y\}$, $d(x) \geq a + 2$ and $d(y) \geq a + 2$. Hence u_2 has no common out-neighbour with any other vertex in D. By repeating the above argument, one can show that every vertex of $\{u_3, \dots, u_p\}$ has no common out-neighbour with any other vertex. By the arbitrariness of z, no vertex has a common out-neighbour with any other vertex.

Thus, for any $w \in V(D)$, $d^-(w) \leq 1$. Since D is strong, $d^-(w) \geq 1$ and so $d^-(w) = 1$. Clearly, D is a directed cycle of length $2a$, a contradiction. Therefore, for every vertex $u \in V(D)$ there exists $v \in V(D) \setminus \{u\}$ such that $\{u, v\}$ is a dominating pair of vertices. As $k \leq \frac{a}{2}$, $2a - k \geq a + k$. By the hypothesis of this lemma, $d(u) \geq a + k$. Since u is arbitrary, the degree of every vertex is larger than or equal to $a + k$. □

Lemma 3.2. *Let D be a strong balanced bipartite digraph of order $2a$. If D satisfies the condition B_k with $2 \leq k \leq \frac{a}{2}$, then D contains a cycle factor.*

Proof. If D is hamiltonian, there is nothing to prove. Now assume that D is non-hamiltonian. By Lemma 3.1, the degree of every vertex is larger than or equal to $a + k$. Let V_1 and V_2 denote the two partite sets of D. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a prefect matching from V_2 to V_1. In order to prove that D contains a perfect matching from V_1 to V_2 and a prefect matching from V_2 to V_1, by the König-Hall theorem, it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

Suppose that a non-empty set $S \subset V_1$ is such that $|N^+(S)| < |S|$. Then $V_2 \setminus N^+(S) \neq \emptyset$. If $|S| = 1$, write $S = \{x\}$, then $|N^+(S)| < |S|$ implies that $d^+(x) = 0$. It is impossible in a strong digraph. Thus $|S| \geq 2$. Then $|N^+(S)| < |S|$ implies that there exist $x_1, x_2 \in S$ and $y \in N^+(S)$ such that $\{x_1, x_2\} \to y$. Thus, $\{x_1, x_2\}$ is a dominating pair of vertices. By the hypothesis of this lemma, we assume, without loss of generality, that $d(x_1) \geq 2a - k$ and $d(x_2) \geq a + k$.

If $|S| \leq a - k$, then $|N^+(S)| < |S|$ implies $|N^+(S)| \leq a - k - 1$. Then $d(x_1) \leq a + |N^+(S)| \leq 2a - k - 1$, which contradicts $d(x_1) \geq 2a - k$. Thus, $|S| \geq a - k + 1$. Since there is no arc from S to $V_2 \setminus N^+(S)$, for any $w \in V_2 \setminus N^+(S)$, $d(w) \leq 2a - |S| \leq 2a - (a - k + 1) = a + k - 1$, which contradicts the fact that the degree of every vertex is larger than or equal to $a + k$.

This completes the proof of the existence of a perfect matching from V_1 to V_2. The proof for a perfect matching in the opposite direction is analogous. Hence D contains a cycle factor. □

Lemma 3.3. *[4] Let D be a balanced bipartite digraph. Suppose that D contains a cycle factor and D is non-hamiltonian. Let $C_1, C_2, \ldots, C_s$ be a cycle factor with minimal number of elements. Then $|(V(C_i), V(C_j))| + |(V(C_j), V(C_i))| \leq \frac{|V(C_i)|\cdot|V(C_j)|}{2}$, for all $i \neq j$.*

The next lemma shows two simple results. For convenience, we give the proof.

Lemma 3.4. *Let $a_1, a_2, \ldots, a_t$ be non-negative integers with $a_1 \leq a_2 \leq \cdots \leq a_t$ and let A be a positive integer. If $a_1 + a_2 + \cdots + a_t \leq A$, then the following hold.*

(a) For any $l \in \{1, 2, \ldots, t\}$, $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$;

(b) If $t \geq 3$ and $a_1 + a_2 = \frac{2A}{t}$, then for any $i \in \{1, 2, \ldots, t\}$, $a_i = \frac{A}{t}$ and $a_1 + a_2 + \cdots + a_t = A$.

Proof. (a) Suppose that $a_1 + a_2 + \cdots + a_l > \frac{lA}{t}$. Then $a_l > \frac{A}{t}$, as otherwise $a_1 \leq a_2 \leq \cdots \leq a_l \leq \frac{A}{t}$ implies $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$, a contradiction. Then $\frac{A}{t} < a_l \leq a_{l+1} \leq \cdots \leq a_t$ implies $\frac{(t-l)A}{t} < a_{l+1} + \cdots + a_t \leq A - (a_1 + a_2 + \cdots + a_l) < A - \frac{lA}{t} = \frac{(t-l)A}{t}$, a contradiction. Hence $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$.

(b) If $a_1 + a_2 = \frac{2A}{t}$, then $a_1 = a_2 = \frac{A}{t}$, otherwise $a_2 > \frac{A}{t}$ implies $a_i > \frac{A}{t}$, for all $i \geq 3$. Then $\frac{(t-2)A}{t} < a_3 + \cdots + a_t \leq A - (a_1 + a_2) = \frac{(t-2)A}{t}$, a contradiction. So $a_i \geq \frac{A}{t}$, for all $i \geq 3$. Then $\frac{(t-2)A}{t} \leq a_3 + \cdots + a_t \leq A - (a_1 + a_2) = \frac{(t-2)A}{t}$. It follows that all equalities hold, that is to say, $a_i = \frac{A}{t}$ and $a_1 + a_2 + \cdots + a_t = A$. ☐

4 Proof of the main result

The proof of Theorem 1.10.

Proof. Let V_1, V_2 denote the two partite sets of D. By Lemma 3.2, D contains a cycle factor $C_1, C_2, \ldots, C_s$. Assume that s is minimum possible and $s \geq 2$. Without loss of generality, assume that $|V(C_1)| \leq |V(C_2)| \leq \cdots \leq |V(C_s)|$. Clearly, $|V(C_1)| \leq a$. Denote $|V(C_1)| = 2t$, $V(C_1) \cap V_1 = \{x_1, x_2, \ldots, x_t\}$, $V(C_1) \cap V_2 = \{y_1, y_2, \ldots, y_t\}$ and $\overline{C}_1 = D - V(C_1)$.

By Lemma 3.3, the following holds:

$$\sum_{i=1}^{t} d_{\overline{C}_1}(x_i) + \sum_{i=1}^{t} d_{\overline{C}_1}(y_i) \leq \frac{|V(C_1)|(2a - |V(C_1)|)}{2} = 2t(a - t). \tag{1}$$

Without loss of generality, we may assume

$$d_{\overline{C}_1}(x_1) + \cdots + d_{\overline{C}_1}(x_t) \leq t(a - t), \tag{2}$$

as otherwise

$$d_{\overline{C}_1}(y_1) + \cdots + d_{\overline{C}_1}(y_t) \leq t(a - t). \tag{3}$$

By renaming the vertices if necessary, we may assume that $d_{\overline{C}_1}(x_1) \leq d_{\overline{C}_1}(x_2) \leq \cdots \leq d_{\overline{C}_1}(x_t)$. By Lemma 3.4(a), $d_{\overline{C}_1}(x_1) \leq a-t$. From this and Lemma 3.1, we get that

$$a+2 \leq a+k \leq d(x_1) = d_{C_1}(x_1) + d_{\overline{C}_1}(x_1) \leq 2t + a - t = a + t.$$

It follows that $t \geq 2$. According to (2) and Lemma 3.4(a), we have $d_{\overline{C}_1}(x_1) + d_{\overline{C}_1}(x_2) \leq 2(a-t)$. To complete the proof, we first proof a claim.

Claim A. Let $u, v \in V(C_1)$ such that $\{u, v\}$ is a dominating pair of vertices. If $d_{\overline{C}_1}(u) + d_{\overline{C}_1}(v) \leq 2(a-t)$, then $d_{C_1}(u) = d_{C_1}(v) = 2t$, $d_{\overline{C}_1}(u) + d_{\overline{C}_1}(v) = 2(a-t)$ and $2t = |V(C_1)| = a$.

Proof. Since $\{u, v\}$ is a dominating pair of vertices, by assumption and $2t \leq a$,

$$\begin{aligned} 3a &\leq d(u) + d(v) \\ &= d_{\overline{C}_1}(u) + d_{\overline{C}_1}(v) + d_{C_1}(u) + d_{C_1}(v) \\ &\leq 2(a-t) + 4t = 2a + 2t \leq 3a. \end{aligned}$$

It follows that all equalities hold, that is, $d_{C_1}(u) = d_{C_1}(v) = 2t$, $d_{\overline{C}_1}(u) + d_{\overline{C}_1}(v) = 2(a-t)$ and $|V(C_1)| = a$. The proof of the claim is complete. □

Now we show that $\{x_1, x_2\}$ is a dominating pair of vertices. By Lemma 3.1, $d(x_1) \geq a+k$ and $d(x_2) \geq a+k$. Thus,

$$\begin{aligned} 2(a+k) &\leq d(x_1) + d(x_2) \\ &\leq 2(a-t) + d_{C_1}(x_1) + d_{C_1}(x_2) \\ &\leq 2(a-t) + 2t + d^+_{C_1}(x_1) + d^+_{C_1}(x_2). \end{aligned}$$

From this and $k > \frac{a}{4}$, we get $d^+_{C_1}(x_1) + d^+_{C_1}(x_2) \geq 2k > \frac{a}{2} \geq t$. Note that $d^+_{C_1}(x_1) + d^+_{C_1}(x_2) > t$ means that x_1, x_2 have a common out-neighbour in C_1. So $\{x_1, x_2\}$ is a dominating pair of vertices. By Claim A,

$$d_{\overline{C}_1}(x_1) + d_{\overline{C}_1}(x_2) = 2(a-t), \tag{4}$$

$$d_{C_1}(x_1) = d_{C_1}(x_2) = 2t \text{ and } |V(C_1)| = a. \tag{5}$$

It follows from (4) and Lemma 3.4(b) that $d_{\overline{C}_1}(x_1) + \cdots + d_{\overline{C}_1}(x_t) = t(a-t)$. From this and (1), we get (3). Moreover, by (5), for any $y_i, y_j \in V(C_1) \cap V_2$, $\{y_i, y_j\}$ is a dominating pair of vertices. Assume that $d_{\overline{C}_1}(y_1) \leq d_{\overline{C}_1}(y_2) \leq \cdots \leq d_{\overline{C}_1}(y_t)$. By (3) and Lemma 3.4(a), $d_{\overline{C}_1}(y_1) + d_{\overline{C}_1}(y_2) \leq 2(a-t)$. Using Claim A to y_1 and y_2, we get that $d_{\overline{C}_1}(y_1) + d_{\overline{C}_1}(y_2) = 2(a-t)$. This together with Lemma 3.4(b) and (3) implies that, for any distinct $y_i, y_j \in V(C_1) \cap V_2$, $d_{\overline{C}_1}(y_i) + d_{\overline{C}_1}(y_j) = 2(a-t)$. Since $\{y_i, y_j\}$ is a dominating pair of vertices, by Claim A, $d_{C_1}(y_i) = d_{C_1}(y_j) = 2t$. Hence, $D[V(C_1)]$ is a complete bipartite digraph.

Now observe that, by the minimality of $|V(C_1)|$ and $|V(C_1)| = a$, we have $s = 2$ and $|V(C_2)| = a$ as well. Consequently, we can swap C_1 and C_2 and repeat the argument to get that $D[V(C_2)]$ is also a complete bipartite digraph.

Since D is strong, it is not difficult to obtain that $(V(C_1) \cap V_1, V(C_2)) \neq \emptyset$ and $(V(C_2), V(C_1) \cap V_2) \neq \emptyset$, or $(V(C_1) \cap V_2, V(C_2)) \neq \emptyset$ and $(V(C_2), V(C_1) \cap V_1) \neq \emptyset$. Note that in a complete bipartite digraph there exists a hamiltonian path between x and y, where x and y belong to different partite sets. So D must be hamiltonian. □

5 Related problems

To conclude the paper, we mention four related problems.

Remark 1. According to Theorem 1.10, the remaining case of Problem 1.9 is $2 \leq k \leq \frac{a}{4}$.

Remark 2. A balanced bipartite digraph containing cycles of all even length is called bipancyclic.

In [2], Adamus proved that the hypothesis of Theorem 1.6 implies bipancyclicity of D, except for a single exceptional digraph (Theorem 5.1 below).

Theorem 5.1. *[2] Let D be a strong balanced bipartite digraph of order $2a$ with $a \geq 3$. If $d(x)+d(y) \geq 3a$ for every dominating pair of vertices and every dominated pair of vertices $\{x, y\}$ in D, then D either is bipancyclic or is a directed cycle of length $2a$.*

It is natural to ask whether the hypothesis of Theorem 1.10 also implies bipancyclicity of D, except for some exceptional digraphs.

Remark 3. As another related problem, perhaps Theorem 1.6 can even be generalized to the following.

Conjecture 5.2. *There is an integer $k \geq 0$ such that every strong balanced bipartite digraph of order $2a$ satisfying $d(x) + d(y) \geq 3a + k$ for every dominating pair of vertices $\{x, y\}$ is hamiltonian.*

Remark 4. Perhaps we can also consider the following ordinary digraph analogue of Theorem 1.10.

Conjecture 5.3. *There is an integer $k \geq 1$ such that every strong digraph of order n satisfying either $d(x) \geq n + k$, $d(y) \geq n - 1 - k$ or $d(x) \geq n - 1 - k$, $d(y) \geq n + k$ for every dominated pair of non-adjacent vertices $\{x, y\}$ is hamiltonian.*

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