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# Graphs whose crossing numbers are the same as their line graphs*

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**Abstract**

In this paper we give a necessary and sufficient condition for a graph to have its crossing number equal to the crossing number of its line graph.

## 1 Introduction

By a drawing of a graph $G$, we mean a drawing in the plane such that every edge is represented by an arc. The arcs are allowed to cross, but they may not pass through vertices (except for their endpoints) and no point is an internal point of three or more arcs. A *crossing* is a common internal point of two arcs.

The *crossing number* $cr(G)$ of a graph is the smallest crossing number of any drawing of $G$ in the plane, where the crossing number $cr_\phi(G)$ of a drawing $\phi$ is the number of pairs of nonadjacent edges that intersect in the drawing. It is implicit that the edges in a drawing are Jordan arcs and thus are non-self-intersecting. It is easy to see that a drawing with minimum crossing number must be a *good* drawing; that is, two edges have at most one point in common, which is either a common end vertex or a crossing point. A good drawing $\phi$ of a graph $G$ is called *optimal* if $cr_\phi(G) = cr(G)$.

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The investigation of the crossing number of graphs is a classical but however very difficult problem (see [5] or [8]). Garey and Johnson [6] have proved that the problem of determining the crossing number of graphs is NP-complete. At present the classes of graphs whose crossing numbers are determined are very scarce, and we only know the crossing number of some classes of special graphs, for example: the complete graphs with a small number of vertices [19]; the complete bipartite graphs with fewer vertices in one bipartite partition [10, 19]; certain generalized Petersen graphs [14]; and some Cartesian products of two cycles [4, 7, 12, 15, 16] and of paths and stars [11].

The *line graph* of a graph $G$, denoted by $L(G)$, is the graph with vertex set $E(G)$ in which two vertices are joined if and only if they are adjacent edges in $G$. In particular, the study of the crossing number of the line graph of a graph has drawn the attention of many authors; for example, see [1, 2, 9, 13, 17]. An earlier result [17] characterizing the planarity of the line graph of a planar graph is the following theorem.

**Theorem 1.1**. *The line graph of a planar graph $G$ is planar if and only if the maximum degree $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut-vertex of $G$.*

It is natural to consider the crossing number of the line graph for a graph whose crossing number is at least one. Jendrol and Klešč in [9] have given the following main result.

**Theorem 1.2**. *Let $G$ be a nonplanar graph with $cr(G) = 1$. Then $cr(L(G)) = 1$ if and only if the following conditions hold:*

(1) *the maximum degree $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of $G$;*

(2) *there exists a drawing of $G$ in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.*

In many cases, a line graph has a larger crossing number than that of its primal graph: for example, the star graph with $n + 1$ vertices is planar and its crossing number is 0, whereas its line graph is the complete graph on $n$ vertices and its crossing number is $\Omega(n^4)$ if $n \geq 8$ (see [18]).

In fact Theorem 1.2 above gives a necessary and sufficient condition for both the line graph and its primal graph to have the same crossing number 1. In this paper we consider the analogous problem, and extend the crossing number value "1" to any integer $k \geq 1$. That is, for any integer $k \geq 1$, we give a necessary and sufficient condition for the line graph to have the same crossing number $k$ as that of its primal graph. Surprisingly, the required conditions are analogous. Specifically, we obtain the following theorem.

**Theorem 1.3**. *Let $G$ be a graph with $cr(G) = k \geq 1$. Then $cr(L(G)) = k$ if and only if the following conditions hold:*

(1) *$\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of $G$;*

(2) *there exists a drawing of $G$ in the plane with exactly $k$ crossings in which each crossed edge is incident with a vertex of degree* 2.

From Theorem 1.3 above, we can easily obtain the following corollaries; although a little trivial, they are still interesting.

**Corollary 1.4**. *If a nonplanar graph $G$ has no vertices of degree* 2*, then the crossing numbers of $G$ and $L(G)$ are different.*

**Corollary 1.5**. *Let $G$ be a graph with $\Delta(G) \leq 4$. If every vertex of degree* 4 *is a cut-vertex of $G$, then $G$ has a subdivision $H$ such that $cr(H) = cr(L(H))$.*

Therefore Theorem 1.3 is a generalization of Theorem 1.2. Our method is different from that used in [9]. Our paper is organized as follows. In Section 2 we give some properties showing that the line graph contains some special subgraphs homeomorphic to its primal graph. In Section 3 we give some elementary properties of a graph and its line graph. Sections 4 and 5 contain, respectively, the proofs of the necessity and the sufficiency of Theorem 1.3.

All graphs considered here are simple; that is, neither loops nor multiple edges are permitted. Furthermore, all the graphs are connected, unless we know otherwise from the context. For any $A \subseteq E(G) \cup V(G)$, the graph $G \backslash A$ is that obtained from $G$ by deleting all elements in $A$ (note that when we delete a vertex, we must also delete all edges incident with this vertex). If $A$ contains a single element $x$, we simply write $G \backslash x$ instead of $G \backslash A$. If $A$ is a vertex subset (an edge subset, respectively) of a graph $G$, then $G[A]$ denotes the vertex-induced subgraph (the edge-induced subgraph, respectively) of $G$. The *degree* of a vertex $u$, denoted by $\deg_G(u)$, is the number of edges of $G$ incident with $u$, and $\Delta(G)$ is the maximum value among all the degrees of vertices of $G$. A *path* is a connected graph with exactly two vertices of degree 1 and all the rest of the vertices of degree 2 (we usually assume that it has at least two vertices), while a *cycle* is a connected graph with each vertex of degree 2. A path (cycle, respectively) of $k$ vertices is also referred as a $k$*-path* ($k$*-cycle*, respectively). If a graph $H$ is isomorphic to a subdivision of a graph $G$, we then say that $H$ is *homeomorphic* to $G$.

Let $G$ be a graph and $L(G)$ be its line graph. We see from the definition that each edge $e$ of $G$ naturally corresponds to a vertex of $L(G)$. Throughout this paper, we usually use $\overline{v}_e$ to denote the vertex of $L(G)$ that *naturally corresponds to* $e$ of $G$. Moreover, let $v$ be a vertex of $G$ of degree $m$, whose incident edges are precisely $e_i$ $(1 \leq i \leq m)$. Then the vertex-induced subgraph $L(G)[\{\overline{v}_{e_i} \mid 1 \leq i \leq m\}]$ of $L(G)$ is isomorphic to the complete graph $K_m$. Also we usually use $\mathcal{K}_v$ to denote the induced subgraph $L(G)[\{\overline{v}_{e_i} | 1 \leq i \leq m\}]$, and for convenience we shall say that $\mathcal{K}_v$ is a subgraph of $L(G)$ that *naturally corresponds to* the vertex $v$ of $G$. For any two distinct vertices $v$ and $u$ of a graph $G$, it is obvious that the two subgraphs $\mathcal{K}_v$ and $\mathcal{K}_u$ of $L(G)$ are either vertex-disjoint, or else have exactly one vertex $\overline{v}_e$ in common (the latter case only occurs when $u$ and $v$ are joined by an edge $e$ in $G$).

Let $\phi$ be a good drawing of a graph $G$. If $A$ and $B$ are two edge subsets of $G$, then $cr_\phi(A, B)$ denotes the number of such crossings involving an edge of $A$ and another of $B$.

For other terminology and definitions without explanation here, we follow [8].

## 2 Special subgraphs in line graphs

**Definition 2.1**. Let $G$ be a graph and $L(G)$ be its line graph, and let $\pi$: $V(G) \longrightarrow V(L(G))$ be a mapping from $V(G)$ to $V(L(G))$. We say that $\pi$ is an *induced homoeomorphic mapping* from $G$ to $L(G)$ if it is an injection such that, for each vertex $u \in V(G)$, $\pi(u) = \overline{v}_e$, where $e$ is an edge of $G$ incident with $u$.

From the above definition, we easily see that if $G$ is a tree, then there does not exist an induced homoeomorphic mapping from $G$ to its line graph $L(G)$.

A vertex $x$ of $L(G)$ is called *principal* under $\pi$ if there exists a vertex $u$ of $G$ such that $\pi(u) = x$; otherwise, $x$ is called *secondary*.

Let $\pi$ be an induced homoeomorphic mapping from a graph $G$ to its line graph $L(G)$. We see from Definition 2.1 that, for each edge $e = uv$ of $G$, the line graph $L(G)$ has a unique path of length 1 or 2 (namely, an edge or a 2-path) which corresponds to the edge $e$ of $G$. Thus, we can define an extension of $\pi$ for each edge $e$ of $G$ as follows.

**Definition 2.2**. Let $\pi$ be an induced homoeomorphic mapping from a graph $G$ to its line graph $L(G)$. For any edge $e = uv$ of $G$ , we define:

$$\pi(e) = \begin{cases} \pi(u)\pi(v), \text{ if } \pi(u) = \overline{v}_e \text{ or } \pi(v) = \overline{v}_e \ \big(\text{an edge of } L(G)\big); \\ \\ \pi(u)\overline{v}_e\pi(v), \ \text{ if } \pi(u) \neq \overline{v}_e \text{ and } \pi(v) \neq \overline{v}_e \ \big(\text{a path of length 2 of } L(G)\big). \end{cases}$$

From the definition, we know that $\pi(e)$ is well defined and that if $\pi(e) = \pi(u)\overline{v}_e\pi(v)$, namely a 2-path of $L(G)$, then $\overline{v}_e$ is the common vertex of the two subgraphs $\mathcal{K}_u$ and $\mathcal{K}_v$ of $L(G)$.

Take $E^* = \bigcup\limits_{e \in E(G)} \{\text{all edges in } \pi(e)\}$ as an edge subset of $L(G)$. We now consider the edge-induced subgraph $L(G)[E^*]$ of $L(G)$. For given $\pi$, $E^*$ is uniquely determined, and so is $L(G)[E^*]$. In this sense, we also say that $L(G)[E^*]$ is *induced by* $\pi$. Since each principal vertex of $L(G)$ must be incident with an edge in $E^*$, we see that $L(G)[E^*]$ contains all principal vertices of $L(G)$. Moreover, we have the following lemma.

**Lemma 2.1** *Suppose that $\pi$ is an induced homoeomorphic mapping from a graph $G$ to its line graph $L(G)$. If $L(G)[E^*]$ is the subgraph of $L(G)$ induced by $\pi$, then $L(G)[E^*]$ is homeomorphic to $G$.*

**Proof**. First, as stated above, $L(G)[E^*]$ contains all principal vertices of $L(G)$. Again, we observe that, for a vertex $x$ of $L(G)[E^*]$, if $x$ is a secondary vertex of $L(G)$ under $\pi$, then $x$ must have degree 2 in $L(G)[E^*]$. Note that secondary vertices are subdivision vertices. Thus it is not difficult to check from the definitions that $\pi$ induces also a homeomorphic mapping between $G$ and $L(G)[E^*]$. $\square$

The above Lemma 2.1 validates the choice of terminology "induced homoeomorphic mapping from $G$ to $L(G)$" in Definition 2.1.

The rest of this section will be devoted to finding conditions under which not only $L(G)$ contains a subgraph homeomorphic to $G$, but also $L(G)$ has an edge whose removal contains a subgraph homeomorphic to $G$. We first give some lemmas.

**Lemma 2.2** *Let $G$ be a tree and $u$ be any vertex of $G$. Then there exists a mapping $\pi$ from $V(G)\backslash\{u\}$ to $V(L(G))$ such that $\pi$ is an injection and, for each vertex $x \in V(G)\backslash\{u\}$, we have $\pi(x) = \overline{v}_e$, where $e$ is an edge of $G$ incident with $x$.*

**Proof**. We first transfer $G$ into a rooted tree $\overrightarrow{G}$ by choosing $u$ as the root. For any vertex $x \in V(G)\backslash\{u\}$, since there is a unique in-degree edge $e$ incident with $x$, we define the mapping $\pi$ from $V(G)\backslash\{u\}$ to $V(L(G))$ by $\pi(x) = \overline{v}_e$. It is easily verified that $\pi$ is as desired. $\square$

In [9], Jendrol and Kleščč proved that if $G$ is a graph containing at least one cycle then the line graph $L(G)$ has a subgraph homeomorphic to $G$. Nevertheless, in order to keep this paper as self-contained as possible, with the help of Lemma 2.2 we will give a proof of this result by using the method of "induced homoeomorphic mapping".

**Lemma 2.3** *Let $G$ be a graph with at least one cycle. Then there exists an induced homeomorphic mapping $\pi$ from $G$ to $L(G)$, and thus the subgraph of $L(G)$ induced by $\pi$ is homeomorphic to $G$.*

**Proof**. We only prove the former, for the latter is direct from Lemma 2.1. Our method is by induction on the number of edges of $G$. If $G$ is a unicycle graph, then there exists an edge $f$ in $G$ such that $G' = G\backslash f$ is a tree. Let $u$ be an end vertex of $f$. By Lemma 2.2, there exists a mapping $\pi'$ from $V(G')\backslash\{u\}$ to $V(L(G'))$, as stated in Lemma 2.2. Now, we extend $\pi'$ into a desired mapping $\pi$ as follows: for any vertex $x \in V(G)$, we define

$$\pi(x) = \begin{cases} \pi'(x), & \text{if } x \neq u; \\ \overline{v}_f, & \text{if } x = u. \end{cases}$$

It is easily checked from Definition 2.1 that $\pi$ is an induced homeomorphic mapping from $G$ to $L(G)$. Assume now that the conclusion holds for a graph with at least one cycle and with fewer edges than $G$. Let $G$ be a graph that has at least two cycles. Then we can choose an edge $h$ of $G$ such that $G\backslash h$ is connected and has at least one cycle. Thus, by the inductive hypothesis there exists an induced homeomorphic mapping $\pi''$ from $G\backslash h$ to $L(G\backslash h)$. Since both $G$ and $G\backslash h$ have the same vertex set and $L(G\backslash h)$ is a subgraph of $L(G)$, we can see that $\pi''$ is also an induced homeomorphic mapping from $G$ to $L(G)$. The proof is thus finished by the inductive hypothesis. $\square$

**Lemma 2.4** *Suppose that there exists an induced homeomorphic mapping $\pi$ from a graph $G$ to its line graph $L(G)$, and let $\mathcal{G}^*$ be the subgraph of $L(G)$ induced by $\pi$. Assume that $e_1$ and $e_2$ are a pair of adjacent edges of $G$ with a common end vertex $v$. If $\mathcal{G}^*$ contains the edge $\overline{v}_{e_1}\overline{v}_{e_2}$, then either $\pi(v) = \overline{v}_{e_1}$ or $\pi(v) = \overline{v}_{e_2}$.*

**Proof**. Let $v_i$ be the other end vertex of $e_i$ for $i = 1, 2$. Recall that as in the paragraph before Lemma 2.1, $\mathcal{G}^*$ contains all principal vertices of $L(G)$. Since $\overline{v}_{e_1}\overline{v}_{e_2}$ is an edge of $\mathcal{G}^*$, it follows from the definition of $\mathcal{G}^*$ that there exists an edge $f = xy \in E(G)$, where $x$ and $y$ are two end vertices of $f$, such that one of the following two cases happens:

**Case 1**: $\pi(f)$ is a 2-path of $L(G)$, and $\overline{v}_{e_1}\overline{v}_{e_2}$ is an edge of $\pi(f)$;

**Case 2**: $\pi(f)$ is an edge of $L(G)$, and $\overline{v}_{e_1}\overline{v}_{e_2}$ is $\pi(h)$.

Suppose Case 1 happens. In this case we can write $\pi(f) = \pi(x)\overline{v}_f\pi(y)$. Since $\overline{v}_{e_1}\overline{v}_{e_2}$ is an edge of $\pi(f)$, by the definition, either $\overline{v}_{e_1} = \overline{v}_f$ and $\overline{v}_{e_2} \in \{\pi(x), \pi(y)\}$, or $\overline{v}_{e_2} = \overline{v}_f$ and $\overline{v}_{e_1} \in \{\pi(x), \pi(y)\}$. If the former holds, clearly $e_1 = f$. Since $G$ is simple, we have $\{v, v_1\} = \{x, y\}$, and so $\overline{v}_{e_2} \in \{\pi(x), \pi(y)\} = \{\pi(v), \pi(v_1)\}$. On the other hand, we can conclude that $\overline{v}_{e_2} \neq \pi(v_1)$, or otherwise, $e_2$ must be incident with $v_1$, which contradicts the simplicity of $G$. It therefore follows that $\overline{v}_{e_2} = \pi(v)$. If the latter holds, by the same arguments we find that $\overline{v}_{e_1} = \pi(v)$.

Suppose Case 2 happens. In this case we can write $\pi(f) = \pi(x)\pi(y)$. Since $\pi(f)$ is an edge of $\mathcal{G}^*$, $\overline{v}_f \in \{\pi(x), \pi(y)\}$ by the definition $\pi(f)$. Again, since the edge $\overline{v}_{e_1}\overline{v}_{e_2}$ is $\pi(h)$, we have $\{\pi(x), \pi(y)\} = \{\overline{v}_{e_1}, \overline{v}_{e_2}\}$. Therefore, $\overline{v}_f \in \{\overline{v}_{e_1}, \overline{v}_{e_2}\}$. If $\overline{v}_f = \overline{v}_{e_1}$, then this implies that $f = e_1$, and therefore $\{x, y\} = \{v, v_1\}$. So we obtain $\pi(v) \in \{\pi(v), \pi(v_1)\} = \{\pi(x), \pi(y)\} = \{\overline{v}_{e_1}, \overline{v}_{e_2}\}$. If $\overline{v}_f = \overline{v}_{e_2}$, the arguments are analogous.

The two cases covered above complete the proof of the lemma. □

**Lemma 2.5** *Let $e$ be an edge of a graph $G$ with an end vertex $u$. If $e$ is not a cut-edge of $G$, then there exists an induced homeomorphic mapping $\pi$ from $G$ to $L(G)$ such that $\pi(u) \neq \overline{v}_e$.*

**Proof**. Let $G' = G \backslash e$. Since $e$ is not a cut-edge of $G$, either $G'$ is a connected graph containing at least one cycle or $G'$ is a tree. If $G'$ is the former, by Lemma 2.3 there exists an induced homeomorphic mapping $\pi'$ from $G'$ to $L(G')$. Because $\overline{v}_e \notin V(L(G'))$, obviously $\pi'(u) \neq \overline{v}_e$. On the other hand, since both $G$ and $G'$ have the same vertex set and $L(G')$ is a subgraph of $L(G)$, by Definition 2.1, $\pi'$ is also an induced homeomorphic mapping from $G$ to $L(G)$, and so $\pi'$ is as desired. If $G'$ is a tree, then $G$ has a cycle $C$ containing $e$. By the simplicity of $G$, we can choose another edge $e''$ from $C$ such that $e''$ is also incident with $u$ but $e'' \neq e$, and moreover such that $G'' = G \backslash e''$ is still a tree. By Lemma 2.2, there exists a mapping $\pi''$ from $V(G'')\backslash\{v\}$ to $V(L(G''))$, stated as in Lemma 2.2. Now we extend $\pi''$ into a desired mapping $\pi$ from $G$ to $L(G)$ as follows: for any vertex $x \in V(G)$, we define

$$\pi(x) = \begin{cases} \pi''(x), & \text{if } x \neq u; \\ \\ \overline{v}_{e''}, & \text{if } x = u. \end{cases}$$

It is easily checked that $\pi$ is an induced homeomorphic mapping from $G$ to $L(G)$. Clearly, $\pi(u) = \overline{v}_{e''} \neq \overline{v}_e$. This proves the lemma. □

With the above lemmas, we conclude this section with the following main result, which indicates that under certain conditions the line graph has an edge whose removal contains a subgraph homeomorphic to its primal graph.

**Lemma 2.6** *Assume that $G$ is a graph containing at least one cycle, $v$ is a vertex of $G$ with the degree $\deg_G(v) \geq 3$, and $e_1$ and $e_2$ are two edges of $G$ incident with $v$. Let $\mathcal{K}_v$ be the subgraph of $L(G)$ naturally corresponding to $v$, and let $\alpha = \overline{v}_{e_1}\overline{v}_{e_2}$ be an edge of $\mathcal{K}_v$. Then we have*

(1) *If $v$ is not a cut-vertex of $G$, then $L(G)\backslash\alpha$ has a subgraph homeomorphic to $G$.*

(2) *Let $\beta$ be an edge of $\mathcal{K}_v$, and $\beta \neq \alpha$. If $\beta$ and $\alpha$ are not adjacent, then either $L(G)\backslash\alpha$ or $L(G)\backslash\beta$ has a subgraph homeomorphic to $G$.*

(3) *Let $\beta$ be an edge of $\mathcal{K}_v$, and $\alpha \neq \beta$. If at least one of $e_1$ and $e_2$ is not a cut-edge of $G$, then either $L(G)\backslash\alpha$ or $L(G)\backslash\beta$ has a subgraph homeomorphic to $G$.*

**Proof**. We first prove (1). Set $G' = G\backslash e_1$. Since $v$ is not a cut-vertex of $G$ and has degree at least 3, we know that $G'$ is connected and moreover $e_2$ is not a cut-edge of $G'$. By Lemma 2.5, there exists an induced homeomorphic mapping $\pi'$ from $G'$ to $L(G')$ such that $\pi'(v) \neq \overline{v}_{e_2}$. Because $e_1 \notin E(G')$, obviously $\pi'(v) \neq \overline{v}_{e_1}$. Since both $G$ and $G'$ have the same vertex set and $L(G')$ is a subgraph of $L(G)$, by Definition 2.1, $\pi'$ is also an induced homeomorphic mapping from $G$ to $L(G)$, and thus by Lemma 2.3, $\pi'$ induces a subgraph $\mathcal{G}^*$ of $L(G)$ that is homeomorphic to $G$. Because $\pi'(v) \notin \{\overline{v}_{e_1}, \overline{v}_{e_2}\}$, it follows from Lemma 2.4 that $\alpha$ does not belong to $\mathcal{G}^*$. This proves (1).

We next prove (2). Let $\beta = \overline{v}_f\overline{v}_h$ be an edge $\beta$ of $\mathcal{K}_v$, where $f$ and $h$ are two edges of $G$ incident with $v$. Because $\alpha$ and $\beta$ are not adjacent, we have $\{f,h\} \cap \{e_1,e_2\} = \emptyset$, and thus $\{\overline{v}_f, \overline{v}_h\} \cap \{\overline{v}_{e_1}, \overline{v}_{e_2}\} = \emptyset$. Since $G$ contains at least one cycle, by Lemma 2.3 there exists an induced homeomorphic mapping $\pi$ from $G$ to $L(G)$ that induces a subgraph $\mathcal{G}^*$ of $L(G)$ homeomorphic to $G$. Assume to the contrary that both $\alpha$ and $\beta$ belong to $\mathcal{G}^*$. Then, by Lemma 2.4, $\pi(v) \in \{\overline{v}_{e_1}, \overline{v}_{e_2}\} \cap \{\overline{v}_f, \overline{v}_h\}$. This contradicts the fact that $\{\overline{v}_f, \overline{v}_h\} \cap \{\overline{v}_{e_1}, \overline{v}_{e_2}\} = \emptyset$. This proves (2).

We finally prove (3). Without loss of generality, assume that $e_1$ is not a cut-edge of $G$. Because of the truth of (2), we restrict our consideration to the case that $\alpha$ and $\beta$ are adjacent. Therefore, we consider the following two cases, according to whether $\overline{v}_{e_1}$ or $\overline{v}_{e_2}$ is the common end vertex of $\alpha$ and $\beta$.

**Case 3.1**. Suppose $\overline{v}_{e_1}$ is the common end vertex of $\alpha$ and $\beta$. Then we can write $\beta = \overline{v}_{e_1}\overline{v}_f$, where $f$ is an edge of $G$ incident with $v$, and $f \neq e_1$, $f \neq e_2$. Obviously, $\overline{v}_f \neq \overline{v}_{e_1}$ and $\overline{v}_f \neq \overline{v}_{e_2}$. By our assumption that $e_1$ is not a cut-edge of $G$, we know from Lemma 2.5 that there exists an induced homeomorphic mapping $\pi$ from $G$ to $L(G)$ such that $\pi(v) \neq \overline{v}_{e_1}$. Again, by the latter part of Lemma 2.3, $\pi$ induces a subgraph $\mathcal{G}^*$ of $L(G)$ that is homeomorphic to $G$. Assume to the contrary that both $\alpha$ and $\beta$ belong to $\mathcal{G}^*$. We then know from Lemma 2.4 that $\pi(v) \in \{\overline{v}_{e_1}, \overline{v}_{e_2}\}$ and $\pi(v) \in \{\overline{v}_{e_1}, \overline{v}_{e_f}\}$. It follows that $\pi(v) \in \{\overline{v}_{e_1}, \overline{v}_{e_2}\} \bigcap \{\overline{v}_{e_1}, \overline{v}_f\} = \{\overline{v}_{e_1}\}$, namely $\pi(v) = \overline{v}_{e_1}$, a contradiction.

**Case 3.2**. Suppose $\overline{v}_{e_2}$ is the common end vertex of $\alpha$ and $\beta$. If $e_2$ is not a cut-edge of $G$, the arguments are analogous as in Case 3.1 above. Therefore, assume that $e_2$ is a cut-edge of $G$. Let $G_1$ and $G_2$ be the two components of $G\backslash e_2$. Without loss of generality, assume that $e_1$ is an edge of $G_1$, and let the other end vertex of $\beta$ be $\overline{v}_h$, where $h$ is an edge of $G$ incident with $v$. Certainly, $h$ is an edge of $G_1$. Since $e_1$ is not a cut-edge of $G$ by our assumption, it is known that $e_1$ is not a cut-edge of $G_1$. By Lemma 2.5, there exists an induced homeomorphic mapping $\pi'$ from $G_1$ to $L(G_1)$ such that $\pi'(v) \neq \overline{v}_{e_1}$. Now we again deal with the following two subcases according to whether $G_2$ has cycles or not.

**Subcase 3.2.1**. Suppose $G_2$ has cycles. By Lemma 2.3, there exists an induced homeomorphic mapping $\pi''$ from $G_2$ to $L(G_2)$. Now define a mapping $\pi$: $V(G) \longrightarrow V(L(G))$ as follows: for any vertex $x \in V(G)$,

$$\pi(x) = \begin{cases} \pi'(x), \text{ if } x \in V(G_1); \\ \\ \pi''(x), \text{ if } x \in V(G_2). \end{cases}$$

We can verify that $\pi$ is an induced homeomorphic mapping from $G$ to $L(G)$. Thus, by Lemma 2.3, $\pi$ induces a subgraph $\mathcal{G}^*$ of $L(G)$ that is homeomorphic to $G$. Clearly, $\pi(v) \neq \overline{v}_{e_1}$ and $\pi(v) \neq \overline{v}_{e_2}$. Therefore it follows from Lemma 2.4 that $\alpha$ does not belong to $\mathcal{G}^*$.

**Subcase 3.2.2**. Suppose $G_2$ has no cycles, namely $G_2$ is a tree. Let $v_2$ be the other end vertex of $e_2$ that belongs to $G_2$. First, by Lemma 2.2 there exists a mapping $\pi'''$ from $V(G_2)\backslash\{v_2\}$ to $V(L(G_2))$, as stated in Lemma 2.2. We now define a mapping $\theta$: $V(G) \longrightarrow V(L(G))$ as follows: for any vertex $x \in V(G)$,

$$\theta(x) = \begin{cases} \pi'(x), & \text{if } x \in V(G_1); \\ \\ \pi'''(x), & \text{if } x \in V(G_2) \text{ and } x \neq v_2; \\ \\ \overline{v}_{e_2}, & \text{if } x = v_2. \end{cases}$$

Similarly we can check that $\theta$ is an induced homeomorphic mapping from $G$ to $L(G)$, and thus by Lemma 2.3, $\theta$ induces a subgraph $\mathcal{G}^{**}$ of $L(G)$ that is homeomorphic to $G$. Note that $\theta(v) \neq \overline{v}(e_1)$ and $\theta(v) \neq \overline{v}(e_2)$. It thus follows from Lemma 2.4 that $\alpha$ does not belong to $\mathcal{G}^{**}$.

Therefore the proof of the lemma is finished. □

## 3 Elementary properties on the crossing number of a graph and its line graph

Before proving the necessity of Theorem 1.3 in the next section, we first give some lemmas in this section.

Let $\phi$ be a good drawing of a graph $G$. Then $\phi$ partitions the plane into some parts. For convenience, each part is called a *region* of $\phi$, and exactly one region that is not bounded is called an *infinite region* while the rest of the regions are called *finite regions*. We note that the boundary of each region is not necessarily a closed walk of the drawn $G$.

With the help of the stereographic project of the plane graph, the following lemma is easily obtained from standard graph drawing techniques.

**Lemma 3.1** *Let $\phi$ be a good drawing of a graph $G$. Then we have*

(1) *for a vertex $v$ of $G$, there exists a good drawing $\phi_1$ of $G$ such that $v$ lies on the boundary of the infinite region of $\phi_1$, and $cr_{\phi_1}(G) = cr_{\phi}(G)$;*

(2) *for an edge $e$ of $G$ that is not crossed under $\phi$, there exists a good drawing $\phi_2$ of $G$ such that $e$ is not crossed under $\phi_2$, $e$ lies on the boundary of the infinite region of $\phi_2$, and $cr_{\phi_2}(G) = cr_{\phi}(G)$;*

(3) *for a cycle $C$ of $G$ that is the boundary of a region of $\phi$, there exists a good drawing $\phi_3$ of $G$ such that $C$ is the boundary of the infinite region of $\phi_3$, and $cr_{\phi_3}(G) = cr_{\phi}(G)$. This conclusion is also true if we replace "infinite region" by "finite region".*

Let $v$ be a vertex of a graph $G$. If $G$ can be decomposed into two subgraphs $G_1$ and $G_2$ such that $G_1 \cup G_2 = G$, $V(G_1) \cap v(G_2) = \{v\}$, and $E(G_1) \cap E(G_2) = \emptyset$, then we say that $G$ is obtained by *amalgamating $G_1$ and $G_2$ at the vertex $v$*, and we use $G_1 \odot_v G_2$ to denote such $G$.

**Lemma 3.2** *Let $G_1 \odot_v G_2$ be a graph obtained by amalgamating two graphs $G_1$ and $G_2$ at a vertex $v$. If $\phi$ is an optimal drawing of $G_1 \odot_v G_2$, then we have*

(1) *both of these two restricted drawings $\phi|G_1$ and $\phi|G_2$ are optimal drawings of $G_1$ and $G_1$, respectively;*

(2) $cr_{\phi}\big(E(G_1), E(G_2)\big) = 0$.

**Proof**. Combined the optimality of $\phi$ with Lemma 3.1 (1), the claim is easily obtained by a routing analysis, and we omit the details. □

Analogous to the vertex-amalgamating graph above, we define the edge-amalgamating graph. Let $e = xy$ be an edge of a graph $G$, where $x$ and $y$ are two end vertices of $e$. If $G$ can be decomposed into two subgraphs $G_1$ and $G_2$ such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{x, y\}$, and $E(G_1) \cap E(G_2) = \{e\}$, then we say that $G$ is obtained by *amalgamating $G_1$ and $G_2$ along the edge $e$*, and we use $G_1 \ominus_e G_2$ to denote such $G$.

**Lemma 3.3** *Let $G_1 \ominus_e G_2$ be a graph obtained by amalgamating two graphs $G_1$ and $G_2$ along an edge $e$. If $\phi$ is an optimal drawing of $G_1 \ominus_e G_2$ such that $e$ is not crossed, then we have*

(1) $cr_\phi\big(E(G_1), E(G_2)\big) = 0$;

(2) *for each* $i = 1, 2$, *there does not exist a good drawing* $\omega_i$ *of* $G_i$ *such that* $e$ *is not crossed and* $cr_{\omega_i}(G_i) < cr_{\phi|G_i}(G_i)$.

**Proof**. This is straightforward, by applying Lemma 3.1 (2) and the assumption that $e$ is not crossed under the optimal drawing $\phi$. □

Let $C$ be a cycle of a graph $G$. If $G\backslash V(C)$ is connected, then $C$ is called *nonseparating*, and if the vertex-induced subgraph $G[V(C)]$ of $G$ is just $C$ itself, then $C$ is called *induced*. A cycle is called an *induced nonseparating* cycle, if it is both induced and nonseparating.

Let $C$ be a cycle of a graph $G$. If $G$ can be decomposed into two subgraphs $G_1$ and $G_2$ such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = V(C)$, and $E(G_1) \cap E(G_2) = E(C)$, then we say that $G$ is obtained by *amalgamating* $G_1$ *and* $G_2$ *along the cycle* $C$, and we use $G_1 \oplus_C G_2$ to denote such $G$.

**Lemma 3.4** *Let* $G_1 \oplus_C G_2$ *be a graph obtained by amalgamating two graphs* $G_1$ *and* $G_2$ *along a cycle* $C$. *Suppose that* $C$ *is an induced nonseparating cycle in both* $G_1$ *and* $G_2$. *If* $\phi$ *is an optimal drawing of* $G_1 \oplus_C G_2$ *such that no edges of* $C$ *are crossed, then* $cr_\phi\big(E(G_1), E(G_2)\big) = 0$.

**Proof**. We note the fact that if a cycle $C$ of a graph $G$ is both induced and nonseparating, and moreover there are no crossings on $C$ under the optimal drawing $\phi$, then $C$ must bound a region of $\phi$. Combined with Lemma 3.1(3), the conclusion of the lemma is easy to obtain by a routine analysis; we omit the details. □

With the help of the lemmas above, we conclude this section with a key lemma that will be used in the proof of the necessity of Theorem 1.3.

**Lemma 3.5** *Let* $G$ *be a graph containing a cut-vertex* $v$ *with* $\deg_G(v) = 3$ *or* $4$, *and let* $\mathcal{K}_v$ *be the subgraph of* $L(G)$ *naturally corresponding to* $v$. *If* $\phi_L$ *is an optimal drawing of* $L(G)$ *such that* $\mathcal{K}_v$ *has a crossed edge, then* $\mathcal{K}_v$ *must have another crossed edge.*

**Proof**. We first consider the case $\deg_G(v) = 3$. Let $e_1$, $e_2$ and $e_3$ be three edges incident to $v$. Since $v$ is a cut-vertex of $G$, at least one edge among $e_1$, $e_2$ and $e_3$, without loss of generality, say $e_1$, is a cut-edge of $G$ (not excluding that $e_2$ or $e_3$ is also a cut-edge of $G$). Let $G_1'$ and $G_2$ be two components of $G\backslash\{e_1\}$, where $G_1'$ contains $v$ and $G_1 = G_1'\backslash\{v\}$ (see Figure 1 (A)). Then we can sketch the outline of the structure of $L(G)$, as displayed in Figure 1 (B), where $\mathcal{G}_1$ is the line graph of $G_1'$, and $\mathcal{G}_2$ is the line graph of the induced subgraph $G[V(G_2) \cup \{v\}]$, and $\mathcal{K}_v$ is the complete graph $K_3$ naturally corresponding to $v$ and containing the three edges $\alpha = \overline{v}_{e_1}\overline{v}_{e_2}$, $\beta = \overline{v}_{e_1}\overline{v}_{e_3}$ and $\gamma = \overline{v}_{e_2}\overline{v}_{e_3}$.

From the structure we can write $L(G)$ as follows:

$$L(G) = \big(\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v\big) \odot_{\overline{v}_{e_1}} \mathcal{G}_2.$$

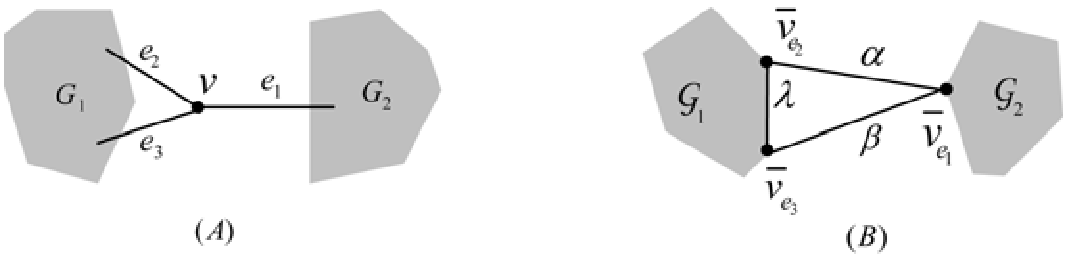


*Figure 1: The structure of $G$ (left) and its line graph $L(G)$ (right).*

Assume to the contrary that $\mathcal{K}_v$ has a unique crossed edge $x \in \{\alpha, \beta, \gamma\}$ under $\phi_L$. Let $y$ ($\neq x$) be another edge of $L(G)$ that makes a crossing with $x$. Because $\phi_L$ is a good drawing of $L(G)$, obviously $y \notin \{\alpha, \beta, \gamma\}$. Combining the optimality of $\phi_L$ with Lemma 3.2(1), it follows that these two restricted drawings $\omega_1 = \phi_L|(\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v)$ and $\omega_2 = \phi_L|\mathcal{G}_2$ are two optimal drawings of $\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v$ and $\mathcal{G}_2$, respectively. Since $y \notin \{\alpha, \beta, \gamma\}$, $y$ must be an edge of $\mathcal{G}_1$ by Lemma 3.2 (2). We now distinguish two cases as follows.

**Case 1**: if $x = \alpha$, or $x = \beta$: We only consider the case $x = \alpha$, for it is analogous for $x = \beta$. At this time we notice that $\gamma$ is not a crossed edge under $\omega_1$. Again because $\omega_1$ is an optimal drawing of $\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v$, by Lemma 3.3 (1), $y \in \{\alpha, \beta, \gamma\}$, contradicting $y \notin \{\alpha, \beta, \gamma\}$.

**Case 2**: if $x = \gamma$: We first see that $\overline{v}_{e_1}$ is a vertex of $\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v$ of degree 2, and that both $\alpha$ and $\beta$ are not crossed edges under $\omega_1$. In this case we can obtain a good drawing $\omega_1'$ of $\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v$ by redrawing the edge $\gamma$ closely along one of two sides of the path $\overline{v}_{e_2}\alpha\overline{v}_{e_1}\beta\overline{v}_{e_3}$. We see that this drawing $\omega_1'$ has at least one crossing fewer than $\omega_1$, contradicting the fact that $\omega_1$ is an optimal drawing of $\mathcal{G}_1 \ominus_\gamma \mathcal{K}_v$.

The above two cases conclude the proof for $\deg_G(v) = 3$. We then consider the case $\deg_G(v) = 4$. Let $e_i$ ($i = 1, 2, 3, 4$) be the four edges incident to $v$. Since $v$ is a cut-vertex of $G$, without loss of generality there are four cases as follows:

**Case A**: Each $e_i$ ($1 \leq i \leq 4$) is incident to a different component of $G\backslash\{v\}$ (see Figure 2 (A)), where each $G_i$ is a component containing the other end vertex of $e_i$.

**Case B**: Two edges, say $e_1$ and $e_2$, are incident to the same component of $G\backslash\{v\}$, and $e_3$ and $e_4$ are, respectively, incident to two difficult components of $G\backslash\{v\}$ (see Figure 2 (B)), where $G_1$ is a component containing the other end vertices of $e_1$ and $e_2$, while $G_i$ ($i = 2, 3$) is the one containing the other end vertex of $e_i$.

**Case C**: Two edges, say $e_1$ and $e_2$, are incident to the same component of $G\backslash\{v\}$, and $e_3$ and $e_4$ are incident to another component of $G\backslash\{v\}$ (see Figure 2 (C)), where $G_1$ is a component containing the other end vertices of $e_1$ and $e_2$, while $G_2$ is the one containing the other end vertices of $e_3$ and $e_4$.

**Case D**: Three edges, say $e_1$, $e_2$ and $e_3$, are incident to the same component of $G\backslash\{v\}$, and $e_4$ is incident to another component of $G\backslash\{v\}$ (see Figure 2 (D)), where $G_1$ is a component containing the other end vertices of $e_1$, $e_2$ and $e_3$, while $G_2$ is the

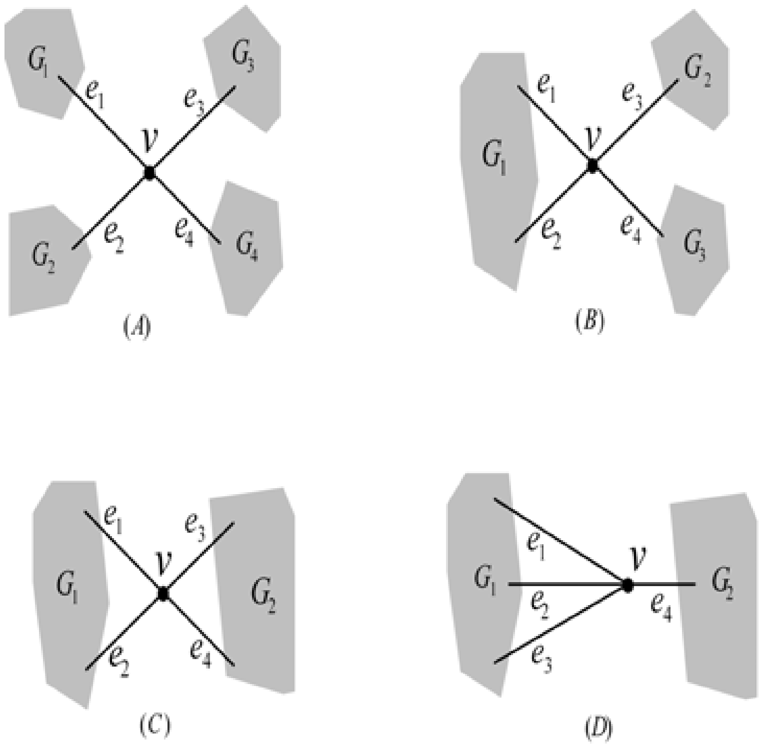


*Figure 2: The possible structures of the graph $G$ associated with the cut-vertex $v$.*

one containing the other end vertex of $e_4$.

Note that $\mathcal{K}_v$ is a subgraph of $L(G)$ isomorphic to the complete graph $K_4$. Let $x$ be a crossed edge of $\mathcal{K}_v$, and $y$ be an edge of $L(G)$ that makes a crossing with $x$ under $\phi_L$. If $y$ is also an edge of $\mathcal{K}_v$, then the desired conclusion of the lemma is obtained. Therefore, in the following we always assume that $y \notin E(\mathcal{K}_v)$, and keep in mind that all edges of $\mathcal{K}_v$, except for $x$, are not crossed under $\phi_L$. We now deal with the above four cases A–D.

**If Case A occurs**. We can see that the general structure of $L(G)$ can be sketched as in Figure 3, where $\mathcal{G}_i = L(G[V(G_i) \cup \{v\}])$ for $1 \leq i \leq 4$.

Therefore we can denote $L(G)$ by

$$L(G) = \Big(\big((\mathcal{G}_1 \odot_{\overline{v}_{e_1}} \mathcal{K}_v) \odot_{\overline{v}_{e_2}} \mathcal{G}_2\big) \odot_{\overline{v}_{e_3}} \mathcal{G}_3\Big) \odot_{\overline{v}_{e_4}} \mathcal{G}_4.$$

Since $\phi_L$ is an optimal drawing of $L(G)$, a recursive application of Lemma 3.2(1) and (2) yields that $y$ must be an edge of $\mathcal{K}_v$, contradicting the assumption that $y \notin E(\mathcal{K}_v)$.

**If Case B occurs**. Then $L(G)$ has the following structure as shown in Figure 4, where $\mathcal{G}_{12} = L(G[V(G_1) \cup \{v\}])$, $\mathcal{G}_3 = L(G[V(G_2) \cup \{v\}])$, and $\mathcal{G}_4 = L(G[V(G_3) \cup \{v\}])$. Similarly, we can denote $L(G)$ by

$$L(G) = \big((\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v) \odot_{\overline{v}_{e_3}} \mathcal{G}_3\big) \odot_{\overline{v}_{e_4}} \mathcal{G}_4, \quad \text{where } \gamma \text{ is the edge } \overline{v}_{e_1}\overline{v}_{e_2}.$$

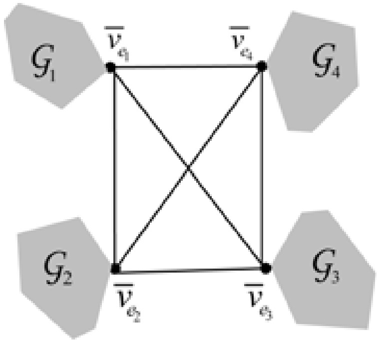


*Figure 3: The general structure of $L(G)$ corresponding to Case A.*

Since $\phi_L$ is an optimal drawing of $L(G)$, by a recursive application of Lemma 3.2 (1) and (2), we can obtain that the restricted $\omega = \phi_L|(\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v)$ is an optimal drawing of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$, and moreover that $y$ must be an edge of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$. Note that $x$ is the unique crossed edge of $\mathcal{K}_v$ under $\omega$. We consider two subcases.

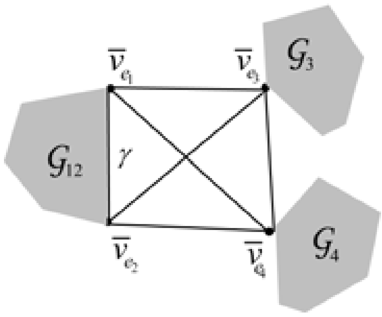


*Figure 4: The general structure of $L(G)$ corresponding to Case B.*

**Subcase (B1)**: If $x \neq \gamma$. Then $\gamma$ is not crossed under $\omega$. Since $\omega$ is an optimal drawing of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$, with the help of Lemma 3.3 (1) we obtain that $y$ must be an edge of $\mathcal{K}_v$, also contradicting the assumption that $y \notin E(\mathcal{K}_v)$.

**Subcase (B2)**: If $x = \gamma$. Note that all edges in $\mathcal{K}_v$ except for $x$ are not crossed under $\omega$. Since $\overline{v}_{e_3}$ is a vertex of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$ of degree 3, we can obtain a good drawing $\omega'$ of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$ by choosing one of the two sides of the 2-path $\overline{v}_{e_1}\overline{v}_{e_3}\overline{v}_{e_2}$ and redrawing the edge $x$ closely along the chosen side of this 2-path. We can observe that $\omega'$ has at least one crossing fewer than $\omega$, contradicting the fact that $\omega$ is an optimal drawing of $\mathcal{G}_{12} \ominus_\gamma \mathcal{K}_v$.

**If Case C occurs**. Then $L(G)$ has the structure as shown in Figure 5, where $\mathcal{G}_{12} = L(G[V(G_1) \cup \{v\}])$, $\mathcal{G}_{34} = L(G[V(G_2) \cup \{v\}])$, $\alpha$ and $\beta$ denote the edges $\overline{v}_{e_1}\overline{v}_{e_2}$ and $\overline{v}_{e_3}\overline{v}_{e_4}$, respectively. We now consider two subcases.

**Subcase (C1)**: If $x \neq \alpha$ and $x \neq \beta$. From the structure of $L(G)$ we can denote $L(G)$ by

$$L(G) = \big(\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v\big) \ominus_\beta \mathcal{G}_{34}.$$

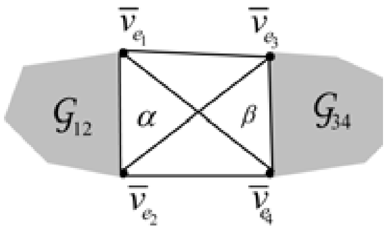


*Figure 5: The general structure of $L(G)$ corresponding to Case C.*

Since $\phi_L$ is an optimal drawing of $L(G)$ and the edge $\beta = \overline{v}_{e_3}\overline{v}_{e_4}$ is not crossed under $\phi_L$, it thus follows from Lemma 3.3 (1) that $y$ must be an edge of $\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v$. On the other hand we can also denote $L(G)$ by

$$L(G) = \big(\mathcal{G}_{34} \ominus_\beta \mathcal{K}_v\big) \ominus_\alpha \mathcal{G}_{12}.$$

By the same reason we deduce that $y$ is an edge of $\mathcal{G}_{34} \ominus_\beta \mathcal{K}_v$. Since $\mathcal{G}_{12}$ and $\mathcal{G}_{34}$ are edge-disjoint, it implies that $y$ must be an edge of $\mathcal{K}_v$, contradicting the assumption that $y \notin E(\mathcal{K}_v)$.

**Subcase (C2)**: If $x = \alpha$, or $x = \beta$. We only consider the case $x = \alpha$, for it is analogous for the case $x = \beta$. As stated in the above subcase (C1), $L(G)$ can be written as

$$L(G) = \big(\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v\big) \ominus_\beta \mathcal{G}_{34}.$$

Since $\phi_L$ is an optimal drawing of $L(G)$ and the edge $\beta = \overline{v}_{e_3}\overline{v}_{e_4}$ is not crossed under $\phi_L$, by Lemma 3.3 (1) we know that $y$ must be an edge of $\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v$. Let $\omega = \phi_L|(\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v)$ be the restricted drawing of $\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v$. Note that all other edges of $\mathcal{K}_v$ except for $x$ are not crossed under $\omega$. Since $\overline{v}_{e_3}$ is a vertex of $\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v$ of degree 3, and the edges $\overline{v}_{e_1}\overline{v}_{e_3}$ and $\overline{v}_{e_3}\overline{v}_{e_2}$ are not crossed under $\omega$, we can obtain a good drawing $\omega'$ of $\mathcal{G}_{12} \ominus_\alpha \mathcal{K}_v$ by redrawing the edge $x$ closely along one of the two sides of the 2-path $\overline{v}_{e_1}\overline{v}_{e_3}\overline{v}_{e_2}$, such that $x$ is not crossed. At the same time we easily see that $\omega'$ has at least one crossings fewer than $\omega$ and that the edge $\beta$ is not crossed under $\omega'$. But this contradicts Lemma 3.3 (2).

**If Case D occurs**. Then $L(G)$ has the general structure as shown in Figure 6, where $\mathcal{G}_{123} = L(G[V(G_1) \cup \{v\}])$, and $\mathcal{G}_4 = L(G[V(G_2) \cup \{v\}])$. Therefore we can denote $L(G)$ by

$$L(G) = \big(\mathcal{G}_{123} \oplus_C \mathcal{K}_v\big) \odot_{\overline{v}_{e_4}} \mathcal{G}_4, \text{ where } C \text{ denotes the 3-cycle } \overline{v}_{e_1}\overline{v}_{e_2}\overline{v}_{e_3}\overline{v}_{e_1}.$$

Let $\omega_1 = \phi_L|(\mathcal{G}_{123} \oplus_C \mathcal{K}_v)$ and $\omega_2 = \phi_L|\mathcal{G}_4$ be two restricted drawings of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$ and $\mathcal{G}_4$, respectively. Since $\phi_L$ is an optimal drawing of $L(G)$, by Lemma 3.2, both $\omega_1$ and $\omega_2$ are, respectively, the optimal drawings of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$ and $\mathcal{G}_4$, and moreover $y$ must be an edge of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$. Since $x$ is an edge of $\mathcal{K}_v$, we now consider two subcases.

**Subcase (D1)**: If the edge $x \in \{\overline{v}_{e_1}\overline{v}_{e_2}, \overline{v}_{e_1}\overline{v}_{e_3}, \overline{v}_{e_2}\overline{v}_{e_3}\}$. Assume that $x = \overline{v}_{e_1}\overline{v}_{e_2}$, for the arguments of the rest cases are analogous. Note that all other edges in $\mathcal{K}_v$

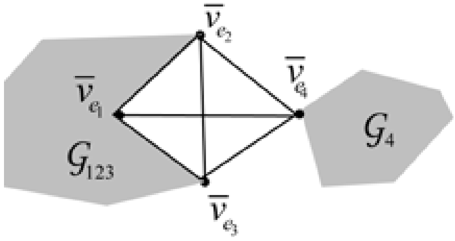


*Figure 6: The general structure of $L(G)$ corresponding to Case D.*

except for $x$ are not crossed under $\omega_1$. Since $\overline{v}_{e_4}$ is a vertex of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$ of degree 3, we can obtain a good drawing $\omega_1'$ of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$ by redrawing the edge $x$ closely along one of the two sides of the 2-path $\overline{v}_{e_1}\overline{v}_{e_4}\overline{v}_{e_2}$. We see that $\omega_1'$ has at least one crossing fewer than $\omega_1$, contradicting the fact that $\omega_1$ is an optimal drawing of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$.

**Subcase (D2)**: If the edge $x \in \{\overline{v}_{e_1}\overline{v}_{e_4}, \overline{v}_{e_2}\overline{v}_{e_4}, \overline{v}_{e_3}\overline{v}_{e_4}\}$. We easily verify that the 3-cycle $C = \overline{v}_{e_1}\overline{v}_{e_2}\overline{v}_{e_3}\overline{v}_{e_1}$ is an induced nonseparating cycle of both $\mathcal{G}_{123}$ and $\mathcal{K}_v$. Also note that in this subcase no edges of $C$ are crossed under $\omega_1$. Since $\omega_1$ is an optimal drawing of $\mathcal{G}_{123} \oplus_C \mathcal{K}_v$, it follows from Lemma 3.4 that $y$ must be an edge of $\mathcal{K}_v$. This contradicts our assumption that $y \notin E(\mathcal{K}_v)$.

The above cases now complete the proof for the case $\deg_G(v) = 4$, and the proof of the lemma is obtained. $\square$

## 4 The proof of the necessity of Theorem 1.3

We first give the following lemma, whose proof is straightforward from standard graph drawing techniques, and is easily obtained by using Lemma 3.2.

**Lemma 4.1** *Let $\phi$ be an optimal drawing of a graph $G$. If $e$ is a crossed edge, then $e$ is not a cut-edge of $G$.*

**The proof of the necessity of Theorem 1.3**. We first prove the necessity (1) of Theorem 1.3. Clearly, $G$ must have cycles, otherwise $G$ is a tree and thus $cr(G) = 0$, contradicting $cr(G) = k \geq 1$. Let $\phi_L$ be an optimal drawing of $L(G)$, that is, $cr_{\phi_L}(L(G)) = k$. Assume to the contrary that $G$ has a vertex $v$ with $\deg_G(v) \geq 5$. Note that the subgraph $\mathcal{K}_v$ of $L(G)$, which naturally corresponds to the vertex $v$ of $G$, is a complete graph on at least five vertices. Therefore we can see that there exist two non-adjacent edges of $\mathcal{K}_v$; let them be $\alpha$ and $\beta$, such that $\alpha$ and $\beta$ are crossed with each other under $\phi_L$. This thus implies that $cr\big(L(G)\backslash\alpha\big) \leq k-1$ and $cr\big(L(G)\backslash\beta\big) \leq k-1$. On the other hand, since $\alpha$ and $\beta$ are not adjacent in $\mathcal{K}_v$, by Lemma 2.6 (2) either $L(G)\backslash\alpha$ or $L(G)\backslash\beta$ has a subgraph homeomorphic to $G$. So it follows that either $cr\big(L(G)\backslash\alpha\big) \geq cr(G) = k$ or $cr\big(L(G)\backslash\beta\big) \geq cr(G) = k$. This contradiction implies $\Delta(G) \leq 4$. Again, assume to the contrary that $G$ has a non-cut-vertex $v$ of degree 4. Note also that the subgraph $\mathcal{K}_v$ of $L(G)$ that naturally corresponds to the vertex $v$ of $G$ is the complete graph $K_4$. Since $v$ is not a cut-vertex of $G$, we contract the edges of $L(G)$ that are not incident with the vertices of $\mathcal{K}_v$, and then obtain a graph that is isomorphic to the complete graph $K_5$. It is thus known that $\mathcal{K}_v$ has an edge $\alpha'$ that is crossed under $\phi_L$. This shows that $cr\big(L(G)\backslash\alpha'\big) \leq k-1$. Again since $v$ is not a cut-vertex of $G$, by Lemma 2.6 (1), $L(G)\backslash\alpha'$ has a subgraph homeomorphic to $G$, and thus $cr\big(L(G)\backslash\alpha'\big) \geq cr(G) = k$. This contradiction shows that each vertex of degree 4 must be a cut-vertex of $G$. So the necessity (1) of Theorem 1.3 is proved.

We now prove the necessity (2) of Theorem 1.3. As just mentioned above, since $G$ has cycles, it follows from Lemma 2.3 that there exists an induced homeomorphic mapping $\pi$ from $G$ to $L(G)$ and moreover the subgraph $\mathcal{G}^*$ induced by $\pi$ is homeomorphic to $G$. Let $\psi_L$ be an optimal drawing of $L(G)$ and let $\psi^* = \phi_L|\mathcal{G}^*$ be the restricted drawing of $\mathcal{G}^*$. Obviously, $cr_{\psi^*}(\mathcal{G}^*) \leq cr_{\psi_L}\big(L(G)\big) = k$. On the other hand, since $\mathcal{G}^*$ is homeomorphic to $G$, we have that $cr(\mathcal{G}^*) = cr(G) = k$. It thus implies that $cr_{\psi^*}(\mathcal{G}^*) = cr(\mathcal{G}^*) = k$, namely, $\psi^*$ is an optimal drawing of $\mathcal{G}^*$. Noting that $\mathcal{G}^*$ is homeomorphic to $G$, and moreover, by the definition of $\mathcal{G}^*$ and the process of the proof of Lemma 2.1, we see that this homeomorphism between $G$ and $\mathcal{G}^*$ preserves each edge $e$ of $G$ corresponding to $\pi(e)$ (an edge or a 2-path of $\mathcal{G}^*$). Associated with the optimal drawing $\psi^*$ of $\mathcal{G}^*$, naturally we can obtain an optimal drawing $\psi$ of $G$, namely $cr_\psi(G) = k$, such that, for any edge $e$ of $G$, $\psi$ has a crossing appearing on $e$ if and only if $\psi^*$ has a crossing appearing on $\pi(e)$ (roughly speaking, we can superimpose the drawing $\psi^*$ of $\mathcal{G}^*$ on $G$ to obtain the drawing $\psi$, by drawing the vertices of $G$ very close to the corresponding vertices of $\mathcal{G}^*$, and letting each edge $e$ of $G$ be drawn very close to $\pi(e)$, an edge or a 2-path of $\mathcal{G}^*$). Since $\psi^*$ $(= \psi_L|\mathcal{G}^*)$ is a restricted drawing, this again indicates that $\psi$ has a crossing appearing on an edge $e$ of $G$ if and only if $\psi_L$ has a crossing appearing on $\pi(e)$ of $L(G)$. Assume that $e_1$ is any crossed edge of $G$ under $\psi$, whose two end vertices are $u$ and $v$. Since $cr_\psi(G) = k$, in order to prove the necessity (2) of Theorem 1.3, it suffices to prove that either $\deg_G(v) = 2$ or $\deg_G(u) = 2$. First, since $e_1$ is a crossed edge under the optimal drawing $\psi$, by Lemma 4.1, $e_1$ is not a cut-edge of $G$, and thus $\deg_G(u) \neq 1$ and $\deg_G(v) \neq 1$. Combining with the truth of the necessity (1) of Theorem 1.3, we now assume to the contrary that $3 \leq \deg_G(u) \leq 4$ and $3 \leq \deg_G(v) \leq 4$. Let $e_1, e_2, e_3, \ldots, e_t$ be all edges of $G$ incident with $v$, and $e_1, e_2', e_3', \ldots, e_s'$ be all edges of $G$ incident with $u$, where $t = \deg_G(v)$ and $s = \deg_G(u)$. Since $e_1$ is a crossed edge of $G$ under $\psi$, just as stated above, $\psi_L$ has a crossing appearing on $\pi(e_1)$. We now distinguish two cases according to whether $\pi(e_1)$ is an edge or a 2-path of $L(G)$.

**Case 1**. If $\pi(e_1) = \pi(u)\pi(v)$ is an edge of $L(G)$. In this case we see from the induced homeomorphic mapping $\pi$ that either $\pi(u) = \overline{v}_{e_1}$ and $\pi(v) \in \{\overline{v}_{e_2}, \ldots, \overline{v}_{e_t}\}$, or $\pi(v) = \overline{v}_{e_1}$ and $\pi(u) \in \{\overline{v}_{e_2'}, \ldots, \overline{v}_{e_s'}\}$. Without loss of generality we only deal with the former, for the arguments are completely analogous for the latter. Let $\pi(v) = \overline{v}_{e_i}$ $(2 \leq i \leq t)$. That is to say, $\pi(e_1) = \overline{v}_{e_1}\overline{v}_{e_i}$. Note that $\pi(e_1)$ (let it be $\alpha$) is an edge of the subgraph $\mathcal{K}_v$ of $L(G)$, which naturally corresponds to the vertex $v$ of $G$. We consider two subcases according to whether $v$ is a cut-vertex of $G$ or not.

**Case 1.1**. If $v$ is a cut-vertex of $G$. Since $\deg_G(v) = 3$ or 4 by our assumption and

$\psi_L$ is an optimal drawing of $L(G)$ with a crossed edge $\alpha$ in $\mathcal{K}_v$, by Lemma 3.5, $\mathcal{K}_v$ has another crossed edge $\beta$. This therefore implies that $cr(L(G)\backslash\alpha) \le cr(L(G)) - 1 = k-1$ and $cr(L(G)\backslash\beta) \le cr(L(G))-1 = k-1$. On the other hand, noting that $e_1$ is not a cut edge of $G$, we apply Lemma 2.6 (3) to obtain that either $L(G)\backslash\alpha$ or $L(G)\backslash\beta$ has a subgraph homeomorphic to $G$. This shows that either $cr(L(G)\backslash\alpha) \ge cr(G) = k$, or $cr(L(G)\backslash\beta) \ge cr(G) = k$. A contradiction appears.

**Case 1.2**. If $v$ is not a cut-vertex of $G$. Since $\psi_L$ is an optimal drawing of $L(G)$ with the crossed edge $\alpha$, $cr(L(G)\backslash\alpha) \le cr(L(G)) - 1 = k - 1$. On the other hand, since $\deg_G(v) = 3$ or 4 by our assumption, it follows from Lemma 2.6 (1) that $L(G)\backslash\alpha$ has a subgraph homeomorphic to $G$. Therefore, $cr(L(G)\backslash\alpha) \ge cr(G) = k$. A contradiction appears too.

**Case 2**. If $\pi(e_1)$ is a 2-path of $L(G)$. In this case we know from the induced homeomorphic mapping $\pi$ that we can let $\pi(e_1) = \pi(u)\overline{v}_{e_1}\pi(v)$ and know that $\pi(v) \in \{\overline{v}_{e_2}, \dots, \overline{v}_{e_t}\}$ and $\pi(u) \in \{\overline{v}_{e'_2}, \dots, \overline{v}_{e'_s}\}$. Without loss of generality, we can let $\pi(v) = \overline{v}_{e_2}$ and $\pi(u) = \overline{v}_{e'_2}$. Since $\psi$ has a crossing appearing on $e$, just as stated above, $\psi_L$ has a crossing appearing on $\pi(e_1)$ $(= \overline{v}_{e'_2}\overline{v}_{e_1}\overline{v}_{e_2})$. Therefore, $\psi_L$ has a crossing appearing on the edge $\overline{v}_{e'_2}\overline{v}_{e_1}$ or on the edge $\overline{v}_{e_1}\overline{v}_{e_2}$. Without loss of generality, assume that $\psi_L$ has a crossing appearing on the edge $\overline{v}_{e_1}\overline{v}_{e_2}$. For the remainder of the proof, we considered the two cases that $v$ is a cut-vertex of $G$ or not. We omit the details, for the argument is the same as in Case 1.

Now the proof of the necessity (2) of Theorem 1.3 is obtained, and so the proof is complete. □

## 5 The proof of the sufficiency of Theorem 1.3

Before proving the sufficiency of Theorem 1.3, we first give some lemmas. Let $\phi$ be a good drawing of a graph $G$, and let $e_i$ $(i = 1, 2)$ be an edge of $G$ with two end vertices $u_i$ and $v_i$. Suppose that $\alpha$ is a crossing point involving these two edges $e_1$ and $e_2$. Then we can produce a new graph $G'$ from the drawn graph $G$, whose vertex set and edge set are respectively $V(G) \cup \{\alpha\}$ and $\big(E(G)\backslash\{e_1, e_2\}\big) \cup \{\alpha u_1, \alpha v_1, \alpha u_2, \alpha v_2\}$. At this time, we shall say that $G'$ is obtained from the drawn graph $G$ by *viewing $\alpha$ as a new vertex.*

With the help of Lemma 3.1 (1) and conventional graph-drawing techniques we can easily get the following lemma.

**Lemma 5.1** *Let $\phi$ be an optimal drawing of a graph $G$ with a crossing $\alpha$ involving two edges $e_1$ and $e_2$. Suppose that $G'$ is a graph obtained by viewing $\alpha$ as a new vertex. Then we have:*
(1) $cr(G') = cr(G) - 1$;
(2) *$\alpha$ is not a cut-vertex of $G'$;*
(3) *if $x$ is a cut-vertex of $G$, then $x$ is also a cut-vertex of $G'$.*

Let $G$ be a graph with a vertex $v$ of degree 4, and let $e_i$ $(1 \le i \le 4)$ be four edges incident with $v$. Assume that $\phi$ is an optimal drawing of $G$, where the four edges $e_i$ are drawn so that the clock-wise order around $v$ is, for example: $e_1 \to e_2 \to e_3 \to e_4 \to e_1$ (see the left side of Figure 7). Then, in a very small neighborhood of $v$, we delete $v$ and add a 4-cycle $C = v_1v_2v_3v_4v_1$ by a suitable drawing, put $e_i$ incident with $v_i$ $(1 \le i \le 4)$, and eventually obtain a graph $G'$ (see the right side of Figure 7). For convenience, we say that the graph $G'$ is obtained from the drawn graph $G$ by *blocking up the vertex $v$ into a 4-cycle.*

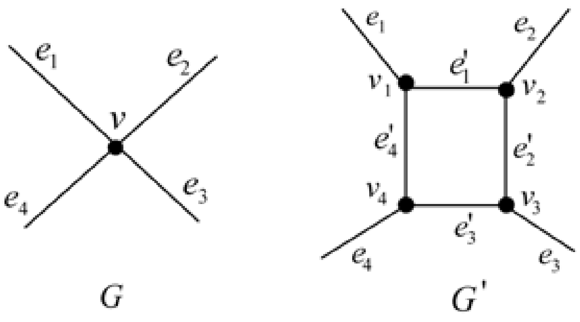


*Figure 7: Block up $v$ into a 4-cycle $v_1v_2v_3v_4v_1$.*

By the way that $G'$ is constructed, the optimal drawing $\phi$ of $G$ induces a good drawing $\phi'$ of $G'$ that keeps the number of crossings of $\phi$ unchanged. This thus implies that $cr(G') \le cr(G)$. In fact the equality holds, and we have the following result.

**Lemma 5.2** (1) *$cr(G) = cr(G')$;*
(2) *$G' \backslash \{v_1, v_2, v_3, v_4\}$ is connected;*
(3) *if $x$ is a cut-vertex of $G$, then $x$ is also a cut-vertex of $G'$.*

**Proof**. The conclusions (2) and (3) are direct from Lemma 5.1 (2) and (3), respectively. We only prove the conclusion (1) by verifying that $cr(G') \ge cr(G)$. For convenience, write $e_i'$ to denote the edge $v_iv_{i+1}$, where the indices are read modulo 4 for $1 \le i \le 4$. By the way that we construct $G'$, we first observe the fact that a new graph $G^*$ that is homeomorphic to $G$ can be obtained by first deleting a pair of adjacent edges from the 4-cycle $C$, for example, $e_1'$ and $e_2'$, and then adding a new edge $e^*$ connecting these two vertices $v_4$ and $v_2$ (see Figure 7). Now, let $\phi'$ be an optimal drawing of $G'$, and let $\mathcal{C}(e_i')$ denote the number of crossings appearing on $e_i'$ for each $1 \le i \le 4$. Without loss of generality, let $\mathcal{C}(e_1')$ be maximal in $\{\mathcal{C}(e_1'), \mathcal{C}(e_2'), \mathcal{C}(e_3'), \mathcal{C}(e_4')\}$. Since $v_3$ is a vertex of degree 3 in $G'$, we can choose one side of the two sides of the 2-path $v_4e_3'v_3e_2'v_2$, and add a new edge $e^*$ connecting $v_4$ and $v_2$ by drawing $e^*$ closely nearer this side. This drawing increases at most $\mathcal{C}(e_3') + \mathcal{C}(e_2')$ crossings. Again, deleting these two edges $e_1'$ and $e_2'$ decreases at least $\mathcal{C}(e_1') + \mathcal{C}(e_2')$ crossings. Since $\mathcal{C}(e_3') \le \mathcal{C}(e_1')$, the above arguments show that we can obtain a new graph $G^*$ that is homeomorphic to $G$, together with its a good drawing $\phi^*$, such that $cr_{\phi^*}(G^*) \le cr_{\phi'}(G') = cr(G')$. Hence, $cr(G) = cr(G^*) \le cr(G')$, and the lemma is proved. $\square$

We now explain some notation used in the latter part of this section. Let $G$ be a graph and $x$ be a 2-degree vertex of $G$, whose two end vertices are $u$ and $v$. We use $P_x^2$ to denote the 2-path $uxv$ in $G$. Also, because the subgraph $\mathcal{K}_x$ in $L(G)$ naturally corresponds to the vertex $x$, $\mathcal{K}_x$ in $L(G)$ is isomorphic to the complete graph $K_2$; for convenience, we sometimes view $\mathcal{K}_x$ as an edge of $L(G)$. If a crossing $\alpha$ involves two edges $e$ and $f$, we write $\alpha = (e, f)$.

The last two lemmas in this section are closely related to the drawing of line graphs, and we first provide with an example to illustrate how to draw the line graph in terms of a drawing of its primal graph.

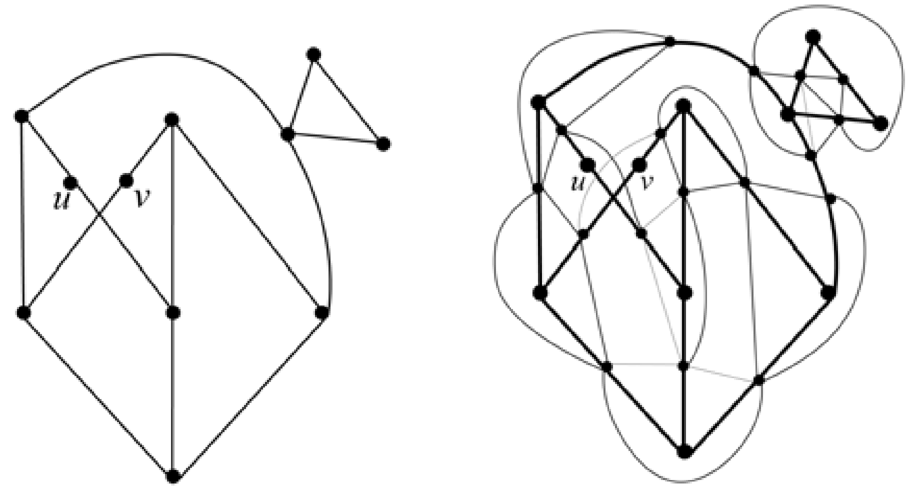


*Figure 8: Drawings of a specific graph and its line graph.*

**Example 5.1**. Let $G$ be a graph shown in the left of Figure 8. In fact the left of Figure 8 gives an optimal drawing $\phi$ of $G$ with exactly one crossing, where these two crossed edges are respectively incident with two 2-degree vertices $u$ and $v$. Associated with the drawing $\phi$ of $G$, the right side of Figure 8 depicts a drawing of its line graph $L(G)$ by inserting each vertex into its corresponding edge of $G$ and drawing all edges of $L(G)$ in the plane (see the thin vertices and edges). We observe that this drawing of $L(G)$ produces exactly one crossing that involves the two edges $\mathcal{K}_u$ and $\mathcal{K}_v$ of $L(G)$, and moreover that the edge $\mathcal{K}_u$ ($\mathcal{K}_v$, respectively) is drawn closely along (arbitrary) one side of the path $P_u^2$ ($P_v^2$, respectively) and through the path $P_v^2$ ($P_u^2$, respectively).

**Definition 5.1**. Let $G$ be a graph with $cr(G) = k$ ($k \geq 1$), and suppose that the following conditions are satisfied:

(1) $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of $G$;

(2) there exits a drawing of $G$ with exactly $k$ crossings $\alpha_i = (e_i, f_i)$ ($1 \leq i \leq k$);

(3) the edges $e_i$ and $f_i$ are respectively incident with two 2-degree vertices $u_i$ and $v_i$ of $G$.

Associated with the drawing $\phi$ of $G$, we draw $L(G)$ as follows: *insert each vertex of $L(G)$ on the corresponding edge of $G$, and draw all the edges of $L(G)$ in the plane such that there are exactly $k$ crossing $\beta_i = (\mathcal{K}_{u_i}, \mathcal{K}_{v_i})$ for $1 \le i \le k$, and moreover such that each edge $\mathcal{K}_{u_i}$ (respectively, $\mathcal{K}_{v_i}$) of $L(G)$ is drawn closely along one side of the path $P^2_{u_i}$ (respectively, $P^2_{v_i}$) and through the path $P^2_{v_i}$ (respectively, $P^2_{u_i}$).* If such a drawing $\omega$ of $L(G)$ exists, then we say that $\omega$ is a *conjugate drawing of $L(G)$ on the drawing $\phi$ of $G$*, or simply, a *conjugate drawing on $\phi$*.

The right side of Figure 8 gives a conjugate drawing of the line graph on the drawing of its primal graph. As for other general graphs $G$ with $cr(G) = 1$, the correctness of the following lemma can be directly seen from the proof of Theorem 1.2 in [9].

**Lemma 5.3** *Let $G$ be a graph with $cr(G) = 1$. If the following conditions hold:*

(1) *$\Delta(G) \le 4$, and every vertex of degree 4 is a cut-vertex of $G$,*

(2) *there exists a drawing $\phi$ of $G$ in the plane with exactly one crossing in which the two crossed edges are respectively incident with two 2-degree vertices $u$ and $v$,*

*then $L(G)$ has a conjugate drawing $\omega$ on $\phi$.*

With the help of Lemma 5.3, under the similar conditions we now give a generalization of Lemma 5.3 for a graph with arbitrary crossing number value.

**Lemma 5.4** *Let $G$ be a graph with $cr(G) = k \ge 1$. If the following conditions hold:*

(1) *$\Delta(G) \le 4$, and every vertex of degree 4 is a cut-vertex of $G$,*

(2) *there exists a drawing $\phi$ of $G$ in the plane with exactly $k$ crossings $\alpha_i = (e_i, f_i)$ for $1 \le i \le k$, in which the two crossed edges $e_i$ and $f_i$ are respectively incident with two 2-degree vertices $u_i$ and $v_i$,*

*then $L(G)$ has a conjugate drawing $\omega$ on $\phi$.*

**Proof**. Assume that a graph $G$, together with its a drawing $\phi$, satisfies the conditions of the lemma. Our proof is by induction on $k$. If $k = 1$, it is direct from Lemma 5.3. Now assume that $k \ge 2$ and that the conclusion holds for the case $\ell < k$. Since $k \ge 2$, for the drawn graph $G$ we first choose a crossed edge, say $e_1 = sv$. By condition (2) of the lemma, without loss of generality, let $v$ be the vertex of degree 2. Then pick up a crossing $\alpha$ appearing on $e_1$ so as to be nearest to $s$. Let $\alpha = \alpha_1 = (e_1, f_1)$, where $f_1 = tu$. Similarly, without loss of generality, let $u$ be the vertex of degree 2. Assume that $r$ and $l$ are, respectively, the other adjacent vertices of $u$ and $v$ (see the left side of Figure 9).

Associated with the graph $G$ and its drawing $\phi$, according to the following steps we will produce a new graph $G^*$, together with its drawing $\phi^*$, so as to apply the inductive hypothesis to the drawn graph $G^*$.

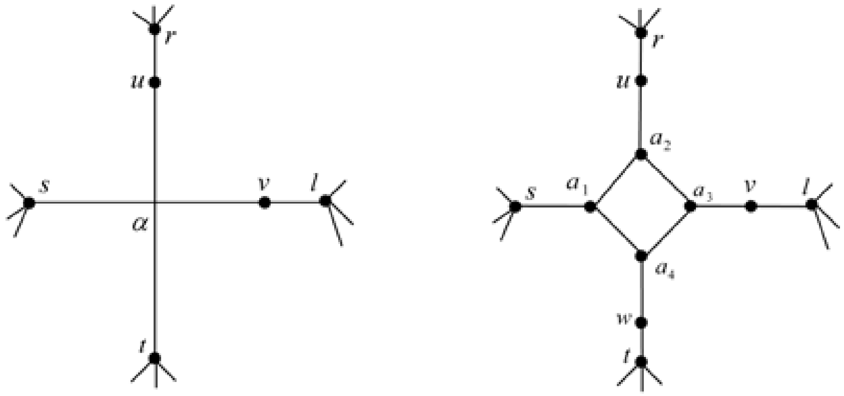


*Figure 9: The local situation of graphs $G$ (left) and $G^*$ (right).*

**Step 1**: *Construct a graph $G'$ from the drawn graph $\phi(G)$ by viewing the crossing $\alpha$ as a new vertex* (see the left side of Figure 9). By Lemma 5.1 (1), $cr(G') = k - 1$. Moreover, by the way that $G'$ is constructed, $\phi$ naturally induces a good drawing $\phi'$ of $G'$ satisfying $cr_{\phi'}(G') = k - 1$, that is to say, $\phi'$ is an optimal drawing of $G'$.

**Step 2**: *Construct a graph $G''$ from the drawn graph $\phi'(G')$ by blocking up the vertex $\alpha$ into a 4-cycle $C = a_1a_2a_3a_4a_1$* (see the right side of Figure 9). Similarly, by the construction of $G''$, $\phi'$ naturally induces a good drawing $\phi''$ of $G''$ so that $cr_{\phi''}(G'') = cr_{\phi'}(G') = k-1$. Since $\phi'$ is an optimal drawing of $G'$, by Lemma 5.2 (1), $\phi''$ is also an optimal drawing of $G''$. Also, by Lemma 5.2 (2) and (3), we have that $G''\backslash\{a_1, a_2, a_3, a_4\}$ is connected and each vertex of degree 4 is a cut-vertex of $G''$. Clearly, $\Delta(G'') \leq 4$ as $\Delta(G) \leq 4$.

**Step 3**: *Construct a graph $G^*$ from the drawn graph $\phi''(G'')$ by inserting a new 2-degree vertex $w$ on the edge $a_4w$ so that there are no crossings appearing on the section $tw$* (see the right side of Figure 9). Similarly, $\phi''$ naturally induces a good drawing $\phi^*$ of $G^*$ satisfying $cr_{\phi^*}(G^*) = k - 1$. Because $G^*$ is homeomorphic to $G''$, we have that $cr(G^*) = cr(G'') = k - 1$ by Lemma 5.2 (1), implying that the drawing $\phi^*$ is an optimal drawing of $G^*$. Clearly, $\Delta(G^*) \leq 4$, each vertex of degree 4 is a cut-vertex of $G^*$, and $G^*\backslash\{a_1, a_2, a_3, a_4\}$ is connected. On the other hand, we also observe that, except for the crossing $\alpha$, the drawing $\phi^*$ does not change the crossings of $\phi$. Moreover, by the way of constructing $G^*$, we note that every crossed edge under the drawing $\phi^*$ is still incident with a 2-degree vertex of $G^*$ (the motivation for inserting the 2-degree vertex $w$ in $G''$ is to ensure that if the original drawing $\phi$ has a crossing appearing on the section $t\alpha$, then the corresponding crossed edge of $G^*$ must be incident with the 2-degree vertex $w$).

In summary, we have the following two claims according to Steps 1–3 above.

**Claim 1**. *$G\backslash\{e_1, f_1\} = G^*\backslash\{w, a_1, a_2, a_3, a_4\}$, and the drawing $\phi$ restricted on $G\backslash\{e_1, f_1\}$ is consistent with the drawing $\phi^*$ restricted on $G^*\backslash\{w, a_1, a_2, a_3, a_4\}$.*

**Claim 2**. *For the constructed graph $G^*$ with $cr(G^*) = k-1$, and its drawing $\phi^*$, we have the following:*

(1) $\Delta(G^*) \leq 4$, *and every vertex of degree* 4 *is a cut-vertex of* $G^*$*;*

(2) *the drawing* $\phi^*$ *has exactly* $k-1$ *crossings, in which the two crossed edges corresponding to a crossing are incident with two* 2*-degree vertices of* $G^*$*; and*

(3) $G^*\backslash\{a_1, a_2, a_3, a_4\}$ *is connected.*

By Claim 2 and the inductive hypothesis, the line graph $L(G^*)$ has a conjugate drawing $\omega^*$ on $\phi^*$ that has $k-1$ crossings. For convenience, denote the $k-1$ crossings of $\phi^*$ by $\alpha_i^* = (e_i^*, f_i^*)$ for $1 \leq i \leq k-1$, where $e_i^*$ and $f_i^*$ are edges of $G^*$ that are respectively incident with 2-degree vertices $u_i^*$ and $v_i^*$, and denote the $k-1$ crossings of $\omega^*$ by $\beta_i = (\mathcal{K}_{u_i^*}, \mathcal{K}_{v_i^*})$ $(1 \leq i \leq k-1)$. In Figure 10, we use the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 to respectively denote the vertices of $L(G^*)$ that naturally correspond to the edges $a_1a_2$, $a_2a_3$, $a_3a_4$, $a_4a_1$, $a_1s$, $a_2u$, $a_3v$, $a_4w$, $wt$, $vl$, and $ur$ of $G^*$ (see Figure 9, for these thin vertices).

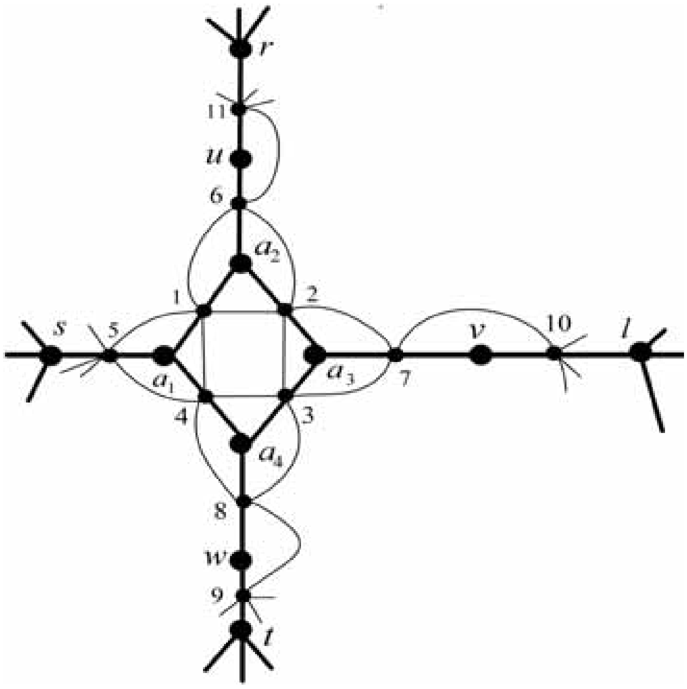


*Figure 10: Two local drawings of* $\phi^*$ *of* $G^*$ (*the thick vertices and edges*) *and its conjugate drawing* $\omega^*$ *of* $L(G^*)$ (*the thin vertices and edges*).

We first see that the 4-cycle $\mathcal{C}$ = 1-2-3-4-1 is an induced nonseparating cycle of $L(G^*)$, because $G^*\backslash\{a_1, a_2, a_3, a_4\}$ is connected by Claim 2(3). On the other hand, by the inductive hypothesis we see that each crossing $\beta_i$ of $\omega^*$ involves those two edges $\mathcal{K}_{u_i^*}$ and $\mathcal{K}_{v_i^*}$ of $L(G^*)$ $(1 \leq i \leq k-1)$, where both $u^*$ and $v^*$ are the 2-degree vertices of $G^*$. Because each edge of $\mathcal{C}$ belongs to only such a subgraph $\mathcal{K}_{a_j}$ $(1 \leq j \leq 4)$, where $a_j$ is a 3-degree vertex of $G^*$, it follows that there is no crossing of $\omega^*$ involving any edge of $\mathcal{C}$. Thus, $\mathcal{C}$ bounds a region $F$ under the drawing $\omega^*$. Without loss of generality, let $F$ be a finite region. Similarly, we easily check that these four 3-cycles $\mathcal{C}_1 = $ 1-2-6-1, $\mathcal{C}_2 = $ 2-7-3-2, $\mathcal{C}_3 = $ 3-8-4-3, and $\mathcal{C}_4 = $ 1-4-5-1 are all induced nonseparating cycles of $L(G^*)$, moreover that there is no crossing of $\omega^*$ involving any edge of each $\mathcal{C}_j$ $(1 \leq j \leq 4)$. Therefore, each cycle $\mathcal{C}_j$ $(1 \leq j \leq 4)$ bounds a region $F_j$ under $\omega^*$. By the assumption that $F$ is a finite region, these four regions $F_j$ $(1 \leq j \leq 4)$ must lie outside $F$. These two local situations of the drawings $\phi^*$ and $\omega^*$ are displayed in Figure 10, where the thick vertices and edges correspond to $\phi^*$, and the thin to $\omega^*$.

Now, combining Claim 1, together with the construction of $G^*$ and $\phi^*$, we can do the following so as to recover the original graph $G$ and its drawing $\phi$:

(1) Delete from $G^*$ the four edges of the 4-cycle $C = a_1a_2a_3a_4a_1$.

(2) Join $a_1$ and $a_3$ (respectively, $a_2$ and $a_4$) and draw the edge $a_1a_3$ (respectively, $a_2a_4$) along the original section of $\phi(e_1)$ ($\phi(f_1)$, respectively) that lies in the region bounded by the 4-cycle $C = a_1a_2a_3a_4a_1$. Note that this also recovers the original crossing $\alpha$.

(3) "Ignore" all 2-degree vertices $a_1$, $a_2$, $a_3$, $a_4$, and $w$.

In order to complete the proof of the lemma, we shall prove that $L(G)$ has a conjugate drawing $\omega$ on $\phi$. We first have the following claim.

**Claim 3.** *The graph* $L(G^*) \setminus \{1,2,3,4,5,6,7,8\}$ *is isomorphic to the graph* $L(G)\setminus\{\mathcal{K}_u, \mathcal{K}_v\}$ *(where* $\mathcal{K}_u$ *and* $\mathcal{K}_v$ *are viewed as two edges).*

**Proof**. Let the vertex 5 (respectively, the vertex 9) of $L(G^*)$ be the vertex of $L(G)$ that corresponds to the edge $e_1$ (respectively, $f_1$) of $G$ (see Figure 10). Then, by the constructions of $G$ and $G^*$, the truth of the claim follows. □

By Claim 3, we can suppose $L(G)\setminus\{\mathcal{K}_u, \mathcal{K}_v\}$ is just $L(G^*)\setminus\{1,2,3,4,5,6,7,8\}$, and furthermore that $\omega$ is such a drawing of $L(G)$ whose restricted drawing on $L(G)\setminus\{\mathcal{K}_u, \mathcal{K}_v\}$ is consistent with the restricted drawing $\omega^*$ on $L(G^*)\setminus\{1,2,3,4,5,6,7,8\}$. Just as said in the proof of Claim 3, let the vertex 5 and the vertex 9 be the vertex of $L(G)$ that corresponds to the edge $e_1$ and $f_1$ of $G$, respectively. Thus, in order to obtain the graph $L(G)$ and the desired drawing $\omega$, we only care how to draw the two extra edges $\mathcal{K}_u$ and $\mathcal{K}_v$. We can proceed as follows:

(1) delete from $L(G^*)$ all the vertices in $\{1,2,3,4,5,6,7,8\}$ (including all the edges incident with them);

(2) add the edge $\mathcal{K}_u$ joining the vertices 11 and 9, and the edge $\mathcal{K}_v$ joining the vertices 5 and 10;

(3) draw the edge $\mathcal{K}_u$ (respectively, $\mathcal{K}_v$) along the path $P_1 = $ 11-6-2-3-8-9 (respectively, the path $P_2 = $ 10-7-3-4-5), where the edges $\mathcal{K}_u$ (respectively, $\mathcal{K}_v$) lies one side of the path $P_u^2$ (respectively, $P_v^2$) of $G$ and passes through the edge $f_1$ (respectively, $e_1$) of $G$. Observe that the two drawn edges $\mathcal{K}_u$ and $\mathcal{K}_v$ intersect at the point that corresponds to the vertex 3. Denote this crossing by $\alpha^* = (\mathcal{K}_u, \mathcal{K}_u)$.

We now consider the drawing $\omega$. As we mentioned above, the drawing $\omega^*$ of $L(G^*)$ has $k-1$ crossings $\beta_i = (\mathcal{K}_{u_i^*}, \mathcal{K}_{v_i^*})$ $(1 \leq i \leq k-1)$, each of which involves no edges in {1-5, 1-6, 1-2, 4-8, 2-7}. Hence, drawing the edges $\mathcal{K}_u$ and $\mathcal{K}_v$ results in only increasing the crossing $\alpha^*$. So, $\omega$ has exactly $k$ crossings in all, including

$\alpha^*$ and $k-1$ crossings $\beta_i = (\mathcal{K}_{u_i^*}, \mathcal{K}_{v_i^*})$ $(1 \leq i \leq k-1)$. At this time, we note the following fact: if a crossing $\beta_i = (\mathcal{K}_{u_i^*}, \mathcal{K}_{v_i^*})$ involves the edge 11-6 or the edge 9-8 (that corresponds to $\mathcal{K}_u$ or $\mathcal{K}_w$ of $L(G^*)$, respectively), then this edge is replaced by the edge 11-9 that corresponds to $\mathcal{K}_u$ of $L(G)$; and if a crossing $\beta_i = (\mathcal{K}_{u_i^*}, \mathcal{K}_{v_i^*})$ involves the edge 10-7 (that corresponds to $\mathcal{K}_v$ of $L(G^*)$), then this edge is replaced by the edge 10-5 that corresponds to $\mathcal{K}_v$ of $L(G)$. Since $\omega^*$ is a conjugate drawing on $\phi^*$, combining with the definition and the arguments above, we easily check that $\omega$ is also a conjugate drawing on $\phi$. Hence, the proof of the lemma is finished by the induction hypotheses. $\square$

**The proof of the sufficiency of Theorem 1.3**. Let $G$ be a graph together with its a good drawing $\phi$ satisfying the conditions (1) and (2) of Theorem 1.3. First, by Lemma 5.4, we have that $cr(L(G)) \leq k$. On the other hand, by Lemma 2.3, $L(G)$ has a subgraph homeomorphic to $G$, and thus $cr(L(G)) \geq cr(G) = k$. It follows that $cr(G) = L(G) = k$, proving the sufficiency of Theorem 1.3.

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## References

[1] D. G. Akka, S. Jendrol, M. Kleščč and S. V.Panshetty, On line graphs with crossing number 2, *Nuniv. Beograd. Publ. Elektrotechn. Fak. Ser. Mat.* **8** (1997), 3–8.

[2] D. G. Akka and S. V. Panshetty, Forbidden subgraphs for graphs with line graphs of crossing number $\leq 1$, *Period. Hungar.* **26** (1993), 175–185.

[3] L. W. Beineke and R. D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, *J. Graph Theory* **4** (1980), 145–155.

[4] A. M. Dean and R. B. Richter, The crossing number of $C_4 \times C_4$, *J. Graph Theory* **19** (1995), 125–129.

[5] P. Erdős and R. K. Guy, Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52–58.

[6] M. R. Garey and D. S. Johnson, Crossing number is NP-complete, *SIAM J. Algebraic. Discrete Methods* **4** (1983), 312–316.

[7] L. Glebsky and G. Salazar, The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$, *J. Graph Theory* **47** (2004), 53–72.

[8] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA (1969).

[9] S. Jendrol and M. Klešč, On the graphs whose line graphs have crossing number one, *J. Graph Theory* **37** (2001), 181–188.

[10] D. J. Kleitman, The crossing number of $K_{5,n}$, *J. Combin. Theory* **9** (1970), 315–323.

[11] M. Klešč, The crossing numbers of products of path and stars with 4-vertex graphs, *J. Graph Theory* **18** (6) (1994), 605–614.

[12] M. Klešč, R. B. Richter and I. Stobert, The crossing number of $C_5 \times C_n$, *J. Graph Theory* **22** (3) (1996), 239–243.

[13] V. R. Kulli, D. G. Akka and L. W. Beineke, On the line graphs with crossing number 1, *J. Graph Theory* **3** (1979), 87–90.

[14] D. McQuillan and R. B. Richter, On the crossing numbers of certain generalized Petersen graphs, *Discrete Math.* **104** (1992), 311–320.

[15] R. B. Richter and C. Thomassen, Intersections of curve systems and the crossing number of $C_5 \times C_5$, *Discrete Comput. Geom.* **13** (1995), 149–159.

[16] R. B. Richter and G. Salazar, The crossing number of $C_6 \times C_n$, *Australas. J. Combin.* **23** (2001), 135–143.

[17] J. Sedlaáček, "Some properties of interchange graphs," *Theory of Graphs and Its Aplications*, Academic Press, New York, 1964, pp. 145–150.

[18] A. T. White and L. W. Beineke, Toplogical graph theory, in "Selected Topics in Graph theory" (L. W. Beineke and R. J. Wilson, Eds.), Academic Press, 1978, 15–50.

[19] D. R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing number conjecture, *J. Graph Theory* **17** (1993), 657–671.


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