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Permutations in which pairs of numbers are not simultaneously close in position and close in size

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Abstract

Let $\|i,j\|_n$ be the minimum of $(i-j) \bmod n$ and $(j-i) \bmod n$. Given integers n and k, we seek a sequence $a_0, \ldots, a_{n-1}$ which is a permutation of $0, 1, \ldots, n-1$ and such that whenever $\|i,j\|_n < s$ we have $\|a_i, a_j\|_n \geq k$, with s as large as possible given k and n. We solve the problem completely when k divides n or k and n are relatively prime and in some other cases, but the problem remains open in general. We also consider the related problem in which $\|i,j\|_n < s$ is replaced with $|i-j| < s$ and determine the maximum possible s for all cases of n and k. We also prove similar results for several extensions and variations of these problems.

1 Introduction

In this paper we consider permutations of $\mathbb{Z}_n$ for positive integers n. For convenience, we write such a permutation $\mathcal{A}$ as the sequence $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$, where $\mathcal{A}(i) = a_i$ for all $i \in \mathbb{Z}_n$. We are interested in certain conditions on permutations $\mathcal{A}$ based on the following definition. For an integer n and elements $i, j \in \mathbb{Z}_n$, we define $\|i,j\|_n$ to be the shortest distance between i and j in $\mathbb{Z}_n$, that is,

$$\|i,j\|_n = \min\{(i-j) \bmod n,\ (j-i) \bmod n\}$$

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and note that

$$\|i, j\|_n = \|i - j, 0\|_n \tag{1.1}$$

for all i, j. We omit the subscript n when the modulus is clear from the context.

We say that an (s, k)-*clash* (or simply a *clash* when the parameters are clear) occurs between (distinct) elements a_i and a_j in $\mathcal{A}$ if

$$\|i, j\|_n < s \text{ and } \|a_i, a_j\|_n < k.$$

If there is no such pair we say that $\mathcal{A}$ is *clash-free*. For example, the permutation $0, 2, 4, 6, 8, 1, 3, 5, 7$ is (3,2)-clash-free but $\mathcal{A} = 0, 3, 5, 1, 4, 7, 2, 8, 6$ is not since a clash occurs between $a_5 = 7$ and $a_7 = 8$. Clearly, if $a_0, a_1, \ldots, a_{n-1}$ is clash-free then so is any of its rotations $a_i, a_{i+1}, \ldots, a_{n-1}, a_0, \ldots, a_{i-1}$ for $1 \le i \le n-1$, as is its reverse $a_{n-1}, a_{n-2}, \ldots, a_0$. We let $\sigma(n, k)$ be the maximum possible value of s for which an (s, k)-clash-free permutation of $\mathbb{Z}_n$ exists. Similarly, $\kappa(n, s)$ is the maximum possible value of k for which an (s, k)-clash-free permutation of $\mathbb{Z}_n$ exists. We also write $\nu(s, k)$ for the minimum value of n such that there exists an (s, k)-clash-free permutation of $\mathbb{Z}_n$. The aim of this paper is to determine $\sigma(n, k)$ and $\kappa(n, s)$ (and to a lesser extent $\nu(s, k)$) for various values of n, k and s. We point out here that $\sigma(n, k)$ (and $\kappa(n, s)$) is *not* non-decreasing in n. For example, we will see later that $\sigma(17, 4) > \sigma(18, 4)$.

Before stating the results of this paper, we give a brief overview of previous work related to clash-free permutations. Brualdi, Kiernan, Meyer and Schroeder [4], page 254, considered a very special case of clash-free permutations and proved the following theorem.

Theorem 1.1 *For all $n \ge 3$, $\sigma(n, 2) = \lfloor (n-1)/2 \rfloor$.*

However, they formulated and proved Theorem 1.1 in a totally graph theoretical setting. We give a brief description. Let G be a graph with vertices $0, 1, \ldots, n-1$ and edges $e_0, \ldots, e_{m-1}$. The *cyclic matching sequencibility* of G, denoted $\mathrm{cms}(G)$, is the largest integer s for which there exists an ordering $e_{a_0}, \ldots, e_{a_{m-1}}$ of the edges of G so that for all i, j with $\|i, j\|_m < s$, the edges e_{a_i} and e_{a_j} are disjoint or equivalently edges e_{a_i} and e_{a_j} form a matching. If we let G be C_n, the cycle on n vertices, with edges $e_i = \{i, i+1 \bmod n\}$ for $i = 0, \ldots, n-1$, then the edges e_{a_i} and e_{a_j} are adjacent if and only if $\|a_i, a_j\|_n < 2$. Thus, $\mathrm{cms}(C_n) = s$ if and only if $\sigma(n, 2) = s$. The question of the value of $\sigma(n, k)$ can be seen as determining cyclic matching sequencibility of the tight k-cycle on n vertices, but we do not explore this point of view further.

Clash-free permutations can also be formulated as so-called *permuted packings*. Roughly speaking, a permuted packing of some shape S into the plane $[0, n] \times [0, n]$ is the collection of disjoint copies of S centred at points $(i, \pi(i))$ for each $i \in \mathbb{Z}_n$, for some permutation $\pi \in S_n$. In [2] and [3], the authors considered permuted packings of diamonds (spheres with respect to the L_1 norm) for the study of so-called d-prolific permutations. Clash-free permutations can be consider as permuted packings of rectangles on the $n \times n$ torus, i.e., on $[0, n] \times [0, n]$, where the left and right edges are identified as are the top and bottom edges, in the following way. For a permutation $\mathcal{A} = a_0, \ldots, a_{n-1}$, and an element $i \in \mathbb{Z}_n$, let R_i, be the $k \times s$ open rectangle in $[0, n] \times [0, n]$ centred at (i, a_i). That is

$$R_i = \left(i - \frac{s}{2}, i + \frac{s}{2}\right) \times \left(a_i - \frac{k}{2}, a_i + \frac{k}{2}\right), \tag{1.2}$$

where entries are considered modulo n. Then the permutation $\mathcal{A}$ is (s, k)-clash-free if and only if the rectangles $R_0, \ldots, R_{n-1}$ are disjoint.

We now give a summary of the results and structure of this paper. In Section 2 we prove Theorem 2.1 which shows that there is a duality between the parameters s and k. In fact, we show that σ and κ are the same function in Corollary 2.2. Section 3 is devoted to the main results about $\sigma(n, k)$ and is divided into three subsections. In Subsection 3.2 we prove Theorem 3.2 which shows that if k and n are relatively prime then $\sigma(n, k) = \lfloor n/k \rfloor$ and, as a corollary, we obtain the value of $\nu(s, k)$ for all s, k. In Subsection 3.3 we prove Theorem 3.6 which shows that if k divides n then $\sigma(n, k) = n/k - 1$. Combined with Theorem 3.2, this leaves open cases when $\gcd(k, n) > 1$ but k does not divide n. Subsection 3.4 is devoted to obtaining bounds and, in some cases, exact results in such cases using Theorems 3.2 and 3.6. However, in general the exact value of $\sigma(n, k)$ remains open and a conjecture of our expectations is presented at the end of Subsection 3.4.

In Section 4, we consider an extension of clash-free permutations arising from [6], by introducing a new parameter r. This is more naturally explained as permuted packings where we instead only require that no set of $r+1$ rectangles have a common intersection, for an integer $r \geq 1$. We extend the results on clash-free permutations (corresponding to the case when $r = 1$) to general values of r. We also prove two results that extend arbitrary clash-free permutations with $r = 1$ to those with $r > 1$; see Propositions 4.6 and 4.7.

In Section 5 we consider two variations of clash-free permutations. For the first in Subsection 5.2, we instead only require that in a permutation $a_0, \ldots, a_{n-1}$, there is no pair i, j with $|i - j| < s$ and $\|a_i, a_j\|_n < k$. Note that the analogue of this for cyclic matching sequencibility has also been considered before in [1], [4] and [5]. When considered as a permuted packing of rectangles, this variation is equivalent to changing the topology from an $n \times n$ torus to an $n \times n$ cylinder, i.e., $[0, n] \times [0, n]$ where only one pair of the edges is identified. The weakened condition above simplifies the problem significantly and the main result of this subsection is Theorem 5.2, which determines the natural analogue of $\sigma(n, k)$ for all values of n and k. In fact, we do this for the natural analogue of the extension mentioned previously. In Subsection 5.3 we also briefly consider a variation of $\sigma(n, k)$ where we consider the permuted packing of rectangles on the $n \times n$ plane, i.e., on $[0, n] \times [0, n]$ with no pairs of edges identified. The paper ends with several conjectures and open questions.

2 Preliminary results

In this section we show that σ and κ are equivalent and obtain necessary condition on the parameters, s, k and n, for the existence of an (s,k)-clash-free permutations of $\mathbb{Z}_n$. By definition, there exists an (s,k)-clash-free permutation of $\mathbb{Z}_n$ only if $\sigma(n,k) \geq s$ and $\kappa(n,s) \geq k$ and if either of these inequalities holds then a clash-free permutation with these parameters exists. So $\sigma(n,k)$ and $\kappa(n,s)$ are related by

$$\sigma(n,k) \geq s \text{ if and only if } \kappa(n,s) \geq k. \tag{2.1}$$

Theorem 2.1 *The permutation $\mathcal{A}$ is (s,k)-clash-free if and only if $\mathcal{A}^{-1}$, the inverse permutation of $\mathcal{A}$, is (k,s)-clash-free.*

Proof: Let $\mathcal{A} = a_0, \ldots, a_{n-1}$. Then $\mathcal{A}$ is (s,k)-clash-free if and only if there is no pair i, j such that

$$\begin{array}{rll} & \|a_i, a_j\|_n & < k \text{ and } \|i,j\|_n < s \\ \text{i.e., no } i,j \text{ such that} & \|a_i, a_j\|_n & < k \text{ and } \|\mathcal{A}^{-1}(a_i), \mathcal{A}^{-1}(a_j)\|_n < s \\ \text{i.e., no } i,j \text{ such that} & \|i,j\|_n & < k \text{ and } \|\mathcal{A}^{-1}(i), \mathcal{A}^{-1}(j)\|_n < s. \end{array}$$

As the final line is exactly the condition for $\mathcal{A}^{-1}$ to be (k,s)-clash-free, this completes the proof. □

The following is immediate from the theorem.

Corollary 2.2 *For any positive integer n and any m in $[1,n]$,*

$$\sigma(n,m) = \kappa(n,m).$$

Proof: Let $\mathcal{A}$ be a permutation of $\mathbb{Z}_n$ that is $(\sigma(n,m), m)$-clash-free. Then $\mathcal{A}^{-1}$ is an $(m, \sigma(n,m))$-clash-free permutation of $\mathbb{Z}_n$. This means $\kappa(n,m) \geq \sigma(n,m)$. Similarly $\sigma(n,m) \geq \kappa(n,m)$ and the corollary follows. □

We now obtain a necessary condition of the parameters s, k and n for (s,k)-clash-free permutations of $\mathbb{Z}_n$ to exist and an upper bound on $\sigma(n,k)$.

Lemma 2.3 *If $1 < k < n$ and there exists an (s,k)-clash-free permutation of $\mathbb{Z}_n$, then $sk \leq n-1$, and hence $\sigma(n,k) \leq \lfloor (n-1)/k \rfloor$ and $\kappa(n,s) \leq \lfloor (n-1)/s \rfloor$.*

Proof: Suppose that $a_0, a_1, \ldots, a_{n-1}$ is an (s,k)-clash-free permutation. On an $n \times n$ torus construct n open rectangles $R_0, R_1, \ldots, R_{n-1}$ such that R_i has centre at (i, a_i) and dimensions $k \times s$. As described after (1.2), each of the $k \times s$ rectangles $R_0, R_1, \ldots, R_{n-1}$ are pairwise disjoint. Thus the sum of their areas cannot exceed the area of the torus and so $n \cdot sk \leq n^2$.

Suppose that $sk = n$. Then the rectangles tile the torus. Since the x-coordinate of the centres of the rectangles are distinct there must be two contiguous rectangles R_i and R_j as shown in Figure 1. The spot marked by X can only be covered by a rectangle whose centre has the same y-coordinate as the centre of R_i, which is impossible, so we conclude that $sk < n$. This means we have $s \leq (n-1)/k$ and so $\sigma(n,k) \leq \lfloor (n-1)/k \rfloor$. The bound on $\kappa(n,s)$ is obtained in a similar fashion. □

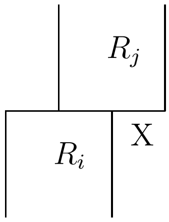

Figure 1: Contiguous rectangles: what rectangle can cover the spot marked X?

3 Results on $\sigma(n,k)$

3.1 Outline

In this section we determine the value of $\sigma(n,k)$ in various cases of n and k, as described in the Introduction. In the subsection to follow, we determine $\sigma(n,k)$ when $\gcd(k,n) = 1$. As a corollary, we obtain the value of $\nu(s,k)$ in all cases. The proof of Theorem 3.6, which gives the value of $\sigma(n,k)$ when k divides n, is presented in Subsection 3.3. In the final subsection, Subsection 3.4, a summary and discussion of the implications of the results of the section is presented.

3.2 Case when $\gcd(k,n) = 1$

In this subsection we prove Theorem 3.2 and Corollary 3.3. First we need the following result, which is Theorem 3 of [8].

Theorem 3.1 *If n and q are relatively prime integers with $0 < q < n$ then the set $\{\lfloor in/q \rfloor \,:\, i = 0,\ldots,q-1\}$ equals the set $\{i\overline{q} \bmod n,\ i = 0,\ldots,q-1\}$ where $q\overline{q} = -1 \pmod{n}$.*

Since

$$0 < \left\lfloor \frac{n}{q} \right\rfloor < \left\lfloor \frac{2n}{q} \right\rfloor < \cdots < \left\lfloor \frac{(q-1)n}{q} \right\rfloor < n$$

and

$$n - \left\lfloor \frac{(q-1)n}{q} \right\rfloor = \left\lceil \frac{n}{q} \right\rceil$$

it follows that

$$\|0, i\overline{q}\|_n \geq \left\lfloor \frac{n}{q} \right\rfloor \tag{3.1}$$

for $0 < i < q$.

Theorem 3.2 *If $\gcd(s,n) = 1$ then $\kappa(n,s) = \lfloor (n-1)/s \rfloor$ and if $\gcd(k,n) = 1$ then $\sigma(n,k) = \lfloor (n-1)/k \rfloor$.*

Proof: By Corollary 2.2, the two parts of the theorem's statement are equivalent. We prove the first, so suppose that $\gcd(s,n)=1$. By Lemma 2.3, $\kappa(n,s)$ cannot be greater than $\lfloor (n-1)/s \rfloor$ so it is sufficient to present an (s,k)-clash-free permutation achieving $k=\lfloor (n-1)/s \rfloor$. Let $\overline{s}$ be the integer in the interval $[1,n-1]$ satisfying $s\overline{s}=-1 \pmod n$. Then define the permutation $\mathcal{A}=a_0,\ldots,a_{n-1}$ by

$$a_i = i\overline{s} \bmod n \text{ for } i=0,\ldots,n-1.$$

We claim that no clash occurs in this permutation. Suppose, for the sake of contradiction, that a clash does occur between a_i and a_j where, without loss of generality, $j>i$. Then,

$$\|i,j\|_n < s \text{ and } \|a_i,a_j\|_n < \left\lfloor \frac{n}{s} \right\rfloor .$$

So by the definition of $\mathcal{A}$,

$$\left\lfloor \frac{n}{s} \right\rfloor > \|a_i,a_j\|_n = \|i\overline{s}, j\overline{s}\|_n = \|0,\overline{s}(j-i)\|_n = \|\overline{s}(n+i-j),0\|_n .$$

As $0<\|i,j\|_n<s$, and $\|i,j\|_n$ is either $\|i,j\|=j-i$ or $\|i,j\|=n+i-j$, the above inequality contradicts (3.1). This completes the proof. □

We now obtain the value of $\nu(s,k)$, as a corollary.

Corollary 3.3 *Let s and k be integers greater than* 1. *Then* $\nu(s,k)=sk+1$.

Proof: By Lemma 2.3, $\nu(s,k)\geq sk+1$. For $n=sk+1$, there is an (s,k)-clash-free permutation of length n by Theorem 3.2, since $\gcd(k,n)=1$ and $s=\lfloor (n-1)/k \rfloor$. Thus, $\nu(s,k)\leq sk+1$, completing the proof. □

3.3 Case when k divides n

In this section we show that if k divides n then $\sigma(n,k)=\lfloor (n-1)/k\rfloor = n/k-1$. That is, we prove Theorem 3.6, which is mentioned in the Introduction. By Lemma 2.3, we know that $\sigma(n,k)\leq n/k-1$, so it is sufficient to present an (s,k)-clash-free permutation of $\mathbb{Z}_n$ where k divides n and $s=n/k-1$. We give an algorithm for constructing such a permutation, then prove its correctness.

Algorithm: Let $n=mk$ so that $s=m-1$. We construct a $(k+1)\times m$ array A with rows indexed $0,\ldots,k$ and columns indexed $0,\ldots,m-1$ as follows. For $i=0,\ldots,k$ we define

$$A[i,0]=\begin{cases} 0 & \text{if } i < (m-1 \bmod k) \\ m-1 & \text{if } i \geq (m-1 \bmod k) . \end{cases}$$

For $i=0,\ldots,k$ and $j=1,\ldots,m-1$ we define

$$A[i,j]=\begin{cases} m-\left\lceil \frac{m}{k} \right\rceil + \frac{m-i-j-1}{k} & \text{if } m-i-j-1 \equiv 0 \pmod k \\ j-\left\lceil \frac{m}{k} \right\rceil + \left\lceil \frac{m-i-j-1}{k} \right\rceil & \text{otherwise.} \end{cases}$$

We then produce another array T defined for all $i = 0, \ldots, k$ and $j = 0, \ldots, m-1$ by

$$T[i,j] = A[i,j]k + i \bmod n\,.$$

Finally, we obtain the permutation

$$\mathcal{A} = T[0,0], T[0,1], \ldots, T[0,m-1], T[1,0], T[1,1] \ldots, T[k-1,m-1]\,. \tag{3.2}$$

Note that the last row of T is not used in constructing the permutation $\mathcal{A}$. It is used, however, in the proof of correctness.

EXAMPLE. With $k = 4$ and $m = 7$ the formulas above become

$$A[i,0] = \begin{cases} 0 & \text{for } i < (6 \bmod 4) \\ 6 & \text{for } i \geq (6 \bmod 4)\,, \end{cases}$$

and

$$A[i,j] = \begin{cases} 5 + \frac{6-i-j}{4} & \text{if } 6-i-j \equiv 0 \pmod 4 \\ j - 2 + \left\lceil \frac{6-i-j}{4} \right\rceil & \text{otherwise,} \end{cases} \tag{3.3}$$

for $i = 0, \ldots, 5$ and $j = 1, \ldots, 6$. This produces the array:

$$A = \begin{bmatrix} 0 & 1 & 6 & 2 & 3 & 4 & 5 \\ 0 & 6 & 1 & 2 & 3 & 5 & 4 \\ 6 & 0 & 1 & 2 & 5 & 3 & 4 \\ 6 & 0 & 1 & 5 & 2 & 3 & 4 \\ 6 & 0 & 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

The array T is then

$$\begin{bmatrix} 0 & 4 & 24 & 8 & 12 & 16 & 20 \\ 1 & 25 & 5 & 9 & 13 & 21 & 17 \\ 26 & 2 & 6 & 10 & 22 & 14 & 18 \\ 27 & 3 & 7 & 23 & 11 & 15 & 19 \\ 0 & 4 & 24 & 8 & 12 & 16 & 20 \end{bmatrix}$$

and the required permutation is

$$0, 4, 24, \ldots, 11, 15, 19.$$

Before showing that the algorithm produces a clash-free permutation, we require the following notation and simple observations. Fix $i \in [1, k-1]$ and define B_i to be the set of values of $A[i,j]$, with $j \geq 1$, for which k divides $m - i - j - 1$. These are the values of $A[i,j]$ defined by the first part of the definition of $A[i,j]$. Define F_i to be the set of values of $A[i,j]$ for which k does not divide $m - i - j - 1$. It is easy but tedious to derive the following facts. Their derivation is left as an exercise for the reader.

(1) The set B_i forms a decreasing arithmetic sequence with common difference 1, and the set F_i forms an increasing arithmetic sequence with common difference 1.

(2) If $i < (m-1 \bmod k)$, then

$$\begin{aligned} \max(B_i) &= m-1 \\ \min(B_i) &= m - \lceil m/k \rceil \\ \max(F_i) &= m-1-\lceil m/k \rceil \\ \min(F_i) &= 1. \end{aligned}$$

(3) If $i \geq (m-1 \bmod k)$, then

$$\begin{aligned} \max(B_i) &= m-2 \\ \min(B_i) &= m - \lceil m/k \rceil \\ \max(F_i) &= m-1-\lceil m/k \rceil \\ \min(F_i) &= 0. \end{aligned}$$

Recall that $A[i,0] = 0$ if $i < (m-1 \bmod k)$ and $A[i,0] = m-1$ if $i \geq (m-1 \bmod k)$. Set $F = F_i \cup \{A[i,0]\} = F_i \cup \{0\}$ and $B = B_i$ if $i < (m-1 \bmod k)$ and $B = B_i \cup \{A[i,0]\} = B_i \cup \{m-1\}$ and $F = F_i$ if $i \geq (m-1 \bmod k)$. We now find that the sets F and B are independent of i and

$$\begin{aligned} F &= \{0, 1, \ldots, m - \lceil m/k \rceil - 1\} \\ B &= \{m - \lceil m/k \rceil, \ldots, m-1\}. \end{aligned}$$

From this we see that the rows of A are permutations of $0, 1, \ldots, m-1$.

Now we show that the permutation produced by the algorithm is clash-free, by proving the following two lemmas. The first lemma determines the necessary conditions for a clash to occur and the second establishes that these necessary conditions cannot occur; thus the permutation produced by the algorithm is clash-free.

Lemma 3.4 *Let i_1 and i_2 be elements of $\mathbb{Z}_k$ such that $\|i_1, i_2\| = i_2 - i_1 \bmod k$. Then the element $T[i_1, j_1]$ of $\mathcal{A}$ clashes with the element $T[i_2, j_2]$ only if each of the following conditions holds*
(a) Either $i_2 = i_1 + 1 \leq k-1$, or $i_1 = k-1$ and $i_2 = 0$.
(b) $j_2 \leq j_1 - 2$.
(c) Either $A[i_1, j_1] = A[i_1+1, j_2]$ or $A[i_1, j_1] = (A[i_1+1, j_2]+1) \bmod m$. Note that if $i_1 = k-1$ then $A[i_1+1, j_1]$ will be in the last row of array A.

Proof: We first assume that $i_1 \leq i_2$ and $\{i_1, i_2\} \neq \{0, k-1\}$. The member of $\mathcal{A}$ arising from entry $A[i,j]$ has index $mi + j$ and value $T[i,j] = kA[i,j] + i$. Suppose a clash occurs between $T[i_1, j_1] = kA[i_1, j_1] + i_1$ and $T[i_2, j_2] = kA[i_2, j_2] + i_2$,

For the indices to clash we must have

$$\|mi_1 + j_1, mi_2 + j_2\|_n < m - 1. \tag{3.4}$$

Since j_1 and j_2 are in the interval $[0, m-1]$ and since we are not considering the case when $\{i_1, i_2\} = \{0, k-1\}$ this implies that $\|i_1, i_2\|_k \le 1$ so that

$$i_2 = i_1 \text{ or } i_2 = i_1 + 1. \tag{3.5}$$

For the values to clash we must have

$$\|kA[i_1, j_1] + i_1, kA[i_2, j_2] + i_2\|_n < k. \tag{3.6}$$

If $i_1 = i_2$ this becomes $k\|A[i_1, j_1], A[i_1, j_2]\|_n < k$ which is impossible for $j_1 \ne j_2$. This observation together with (3.5) implies that $i_2 = i_1 + 1$. This means (3.4) becomes

$$\|j_1, m + j_2\|_n < m - 1 \Rightarrow j_2 \le j_1 - 2 \tag{3.7}$$

and (3.6) becomes

$$\|kA[i_1, j_1], kA[i_1 + 1, j_2] + 1\|_n < k$$

which implies that

$$A[i_1, j_1] = A[i_1 + 1, j_2] \text{ or } A[i_1, j_1] = A[i_1 + 1, j_2] + 1 \bmod m$$

and the statement of the lemma is satisfied.

Now suppose that $\{i_1, i_2\} = \{0, k-1\}$. We consider the k-th row of the array A. It is easy to check that $A[k, j] = A[0, j] - 1 \bmod m$. Therefore,

$$T[k, j] = (A[0, j] - 1)k + k = A[0, j]k = T[0, j]$$

for all j. It follows that a clash occurs between $T[k-1, j_1]$ and $T[0, j_2]$ if and only if a clash occurs between $T[k-1, j_1]$ and $T[k, j_2]$. By the same reasoning as above this occurs only if

$$A[k-1, j_1] = A[k, j_2] \bmod m \text{ or } A[k-1, j_1] = A[k, j_2] + 1 \bmod m$$

and $j_2 \le j_1 - 2$. □

The next lemma shows that these conditions for a clash do not occur.

Lemma 3.5 *For all $i = 1, \ldots, k-1$ and $j = 2, \ldots, m-1$, if $A[i, j] = A[i+1, j']$ or $A[i, j] - 1 = A[i+1, j']$ then $j' \ge j - 1$.*

Proof: We consider five cases.

Case 1. If $A[i, j] = A[i+1, j']$ and both are in B then

$$\begin{aligned} & m - \left\lceil \frac{m}{k} \right\rceil + \frac{m - i - j - 1}{k} = m - \left\lceil \frac{m}{k} \right\rceil + \frac{m - (i+1) - j' - 1}{k} \\ \Rightarrow\ & m - i - j - 1 = m - i - j' - 2 \\ \Rightarrow\ & j' = j - 1. \end{aligned}$$

Case 2. If $A[i,j] = A[i+1,j']$ and both are in F, then

$$\begin{aligned} j - \left\lceil \frac{m}{k} \right\rceil + \left\lceil \frac{m-i-j-1}{k} \right\rceil &= j' - \left\lceil \frac{m}{k} \right\rceil + \left\lceil \frac{m-i-j'-2}{k} \right\rceil \\ \Rightarrow \quad \left\lceil \frac{m-i-j-1}{k} \right\rceil - \left\lceil \frac{m-i-j'-2}{k} \right\rceil &= j' - j. \end{aligned}$$

If $m-i-j-1 \not\equiv 1 \pmod{k}$, then both sides of the above equal 0 when $j' = j$. If $m-i-j-1 \equiv 1 \pmod{k}$, then both sides of the above are 1 when $j' = j+1$. By (1), there is at most one entry in row $i+1$ equal to $A[i,j]$ so that in either case we must have $j' \geq j$.

Case 3. If $A[i,j] - 1 = A[i+1,j']$ and both are in B then

$$\begin{aligned} m - \left\lceil \frac{m}{k} \right\rceil + \frac{m-i-j-1}{k} - 1 &= m - \left\lceil \frac{m}{k} \right\rceil + \frac{m-i-j'-2}{k} \\ \Rightarrow \quad -j-k &= -j'-1 \end{aligned}$$

so that $j' = j+k-1 \geq j-1$.

Case 4. If $A[i,j] - 1 = A[i+1,j']$ and both are in F, then

$$\begin{aligned} j - \left\lceil \frac{m}{k} \right\rceil + \left\lceil \frac{m-i-j-1}{k} \right\rceil - 1 &= j' - \left\lceil \frac{m}{k} \right\rceil + \left\lceil \frac{m-i-j'-2}{k} \right\rceil \\ \Rightarrow \quad \left\lceil \frac{m-i-j-1}{k} \right\rceil - \left\lceil \frac{m-i-j'-2}{k} \right\rceil &= j' - (j-1). \end{aligned}$$

Both sides equal 0 if $j' = j-1$. Since there exists only one entry in row $i+1$ equal to $A[i,j] - 1$ we must have $j' = j-1$.

Case 5. If $A[i,j] - 1 = A[i+1,j']$ and one is in B and the other in F we must have $A[i,j]$ being the least element in B and $A[i+1,j']$ being the greatest in F. By (3), we have that the greatest element of F equals $m - \lceil \frac{m}{k} \rceil - 1$. Using the formula for $A[i,j]$ we see that $A[i+1,m-1] = m - \lceil \frac{m}{k} \rceil - 1$ so $j' = m-1$ which is greater than or equal to $j-1$ for any $j \leq m-1$.

In each case we have shown that $j' \geq j-1$. This completes the proof. □

We can now determine $\sigma(n,k)$, when k divides n.

Theorem 3.6 *If k divides n, then $\sigma(n,k) = \lfloor (n-1)/k \rfloor = n/k - 1$. Equivalently, if s divides n, then $\kappa(n,s) = \lfloor (n-1)/s \rfloor = n/s - 1$*

Proof: By Corollary 2.2, the two statements of the theorem are equivalent, so we only prove the first. Let $\mathcal{A}$ be the permutation of $\mathbb{Z}_n$ formed by the algorithm given in the first part of this subsection for k dividing n. By Lemma 3.4, an $(n/k-1,k)$-clash can only occur between $T[i,j]$ and $T[i',j']$ for $0 \leq i \leq i' \leq k-1$ in $\mathcal{A}$ if conditions (a) and (c) are satisfied and $j' \leq j-2$. Lemma 3.5 shows that in fact $j' \geq j-1$ whenever conditions (a) and (c) are satisfied. Thus, $\mathcal{A}$ is $(n/k-1,k)$-clash-free and so $\sigma(n,k) \geq \lfloor (n-1)/k \rfloor$. By Lemma 2.3, $\sigma(n,k) \leq \lfloor (n-1)/k \rfloor$ and the theorem follows. □

3.4 Consequences and results in other cases

In this subsection we bring together the results of the previous two subsections; see Theorem 3.7. We also consider the value of $\sigma(n,k)$ in cases not covered by the previous results, by proving Corollary 3.8, which demonstrates that in some instances that exact results can still be obtained. The subsection ends with a brief discussion about cases that remain open and a conjecture of what we expect.

We begin by proving the following consequence of Theorems 3.2 and 3.6.

Theorem 3.7 *Let n and k be integers and $s = \lfloor (n-1)/k \rfloor$. If at least one of $k \mid n$, $s \mid n$, $\gcd(n,k) = 1$ or $\gcd(n,s) = 1$ holds, then*

$$\sigma(n,k) = s\,.$$

Proof: The cases when $\gcd(n,k) = 1$ and $k \mid n$ follow immediately from Theorems 3.2 and 3.6, respectively. So let either $\gcd(n,s) = 1$ or $s \mid n$ hold. By assumption, $sk \leq n-1$ from which it follows that $k \leq \lfloor (n-1)/s \rfloor$. So by Theorems 3.2 and 3.6, $\kappa(n,s) = \lfloor (n-1)/s \rfloor \geq k$, when $\gcd(n,s) = 1$ and $s \mid n$, respectively. Thus (2.1), implies that $\sigma(n,k) \geq s$. As Lemma 2.3 says $\sigma(n,k) \leq \lfloor (n-1)/k \rfloor = s$, we conclude that $\sigma(n,k) = s$. □

We can also determine the value of $\sigma(n,k)$ for some of the values of n and k that do not satisfy the condition of Theorem 3.7, using the following result.

Corollary 3.8 *If k_1, k_2 and n are integers with $k_1 < k_2 < n$, and*

$$\sigma(n,k_2) = \left\lfloor \frac{n-1}{k_2} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{n-1}{k_1} \right\rfloor = \left\lfloor \frac{n-1}{k_2} \right\rfloor$$

then

$$\sigma(n,k_1) = \left\lfloor \frac{n-1}{k_1} \right\rfloor .$$

Proof: First we note that if a, b and n are positive integers with $n \geq b \geq a$ then

$$\sigma(n,a) \geq \sigma(n,b)\,, \tag{3.8}$$

which is immediate from definitions. Since $k_2 > k_1$, the inequality (3.8) implies that $\sigma(n,k_1) \geq \sigma(n,k_2)$. Lemma 2.3 says that $\sigma(n,k_1) \leq \lfloor (n-1)/k_1 \rfloor$. With the hypothesis that $\sigma(n,k_2) = \lfloor (n-1)/k_2 \rfloor$, we obtain the result. □

As an example of the use of Corollary 3.8 consider the case $n = 46$, $k_1 = 10$ and $k_2 = 11$. Here $\sigma(46,11) = 4$ by Theorem 3.2 and $\lfloor \frac{46-1}{10} \rfloor = \lfloor \frac{46-1}{11} \rfloor$, so $\sigma(46,10) = 4$. From this example we see that the conditions in Theorem 3.7 are not necessary for $\sigma(n,k) = \lfloor (n-1)/k \rfloor$ to hold.

We summarise the values of $\sigma(n,k)$ for $n \leq 30$. in Table 1. Each of the values given is obtained from Theorem 3.7, Corollary 3.8 and the fact that $\sigma(n,k) = 1$

n	k: 4	6	8	9	10	12	14
9	2	1	1	1	1	1	1
10	2	1	1	1	1	1	1
12	2	1	1	1	1	1	1
14	3	2	1	1	1	1	1
15	3	2	1	1	1	1	1
16	3	2	1	1	1	1	1
18	≥ 3	2	2	1	1	1	1
20	4	3	2	2	1	1	1
21	5	3	2	2	2	1	1
22	5	3	2	2	2	1	1
24	5	3	2	2	2	1	1
25	6	4	3	2	2	2	1
26	≥ 5	≥ 3	3	2	2	2	1
27	6	4	3	2	2	2	1
28	6	4	3	3	2	2	1
30	7	4	3	3	2	2	2
$\sigma(n,k) = \lfloor (n-1)/k \rfloor$ for each prime n and k				$\sigma(n,k) = 1$ for $2k \geq n$			

Table 1: Values of $\sigma(n, k)$ for selected small values of n and k.

when $2k \geq n$, which follows easily from the definition of $\sigma(n, k)$. For simplicity, we have excluded particularly simple cases, each of which the value of $\sigma(n, k)$ can be determined by the fact that $\sigma(n, k) = 1$ when $2k \geq n$ or the immediate consequence of Theorem 3.7 that $\sigma(n, k) = \lfloor (n-1)/k \rfloor$ if either n or k is prime. For the three cases when $\sigma(n, k)$ cannot be determined using any of the results given thus far, we provide the lower bound determined by the fact that $\sigma(n, k) \geq \sigma(n, k+1)$.

By the table, the only cases when $\sigma(n, k)$ is not determined for $n \leq 30$ are $\sigma(18, 4)$, $\sigma(26, 4)$ and $\sigma(26, 6)$. By a computer search, we determined that no $(4, 4)$-clash-free permutations of $\mathbb{Z}_{18}$ or $(6, 4)$-clash-free permutations of $\mathbb{Z}_{26}$ exist (and therefore $(4, 6)$-clash-free permutations of $\mathbb{Z}_{26}$ do not exist). As summarised in Table 1, equation (3.8) and Theorem 3.7 imply that $\sigma(18, 4) \geq \sigma(18, 5) = 3$, $\sigma(26, 4) \geq \sigma(26, 5) = 5$ and $\sigma(26, 6) \geq \sigma(26, 7) = 3$. Thus, $\sigma(18, 4) = 3$, $\sigma(26, 4) = 5$, $\sigma(26, 6) = 3$, while $\sigma(n, k) = \lfloor (n-1)/k \rfloor$ in all other cases for $n \leq 30$.

We determined that $\nu(s, k) = sk + 1$ for all $s, k > 1$ in Subsection 3.2. But, perhaps surprisingly, it is possible that $n > \nu(s, k)$, yet no (s, k)-clash-free permutation of $\mathbb{Z}_n$ exists. For example, with $s = 4$ and $k = 4$ we have $\nu(4, 4) = 17$ but $\sigma(18, 4) = 3$. That is, there is a $(4, 4)$-clash-free permutation of $\mathbb{Z}_{17}$, but no $(4, 4)$-clash-free permutation of $\mathbb{Z}_{18}$.

There can be a wide gap between the highest theoretically possible value of $\sigma(n, k)$ determined by Lemma 2.3 and the lower bound we can obtain. For example

$\sigma(420, 47) = 8$ by Theorem 3.2, so (2.1) and Theorem 2.1 imply $\sigma(420, 8) \geq 47$ (note that (3.8) only yields $\sigma(420, 8) \geq \sigma(420, 9) = 46$). On the other hand by Lemma 2.3, $\sigma(420, 8) \leq 52$. However, we do not expect the actual value $\sigma(n, k)$ to be much different than $\lfloor (n-1)/k \rfloor$ for any values of n and k. In particular, we conjecture the following.

Conjecture 3.9 *Let n and k be integers and $s = \lfloor \frac{n-1}{k} \rfloor$. Then*

$$s - 1 \leq \sigma(n, k) \leq s\,.$$

4 Extension

In this section we consider an extension of clash-free permutations, which we call (s, k, r)-*clash-free permutations*, that arises from introducing a new parameter r. As stated in the Introduction from the viewpoint of permuted packings of $k \times s$ rectangles, this extension is to consider packing rectangles (as given in (1.2)), so that no set of $r + 1$ rectangles have a non-empty intersection for a fixed integer $r \geq 1$. When $r = 1$, we recover the condition for (s, k)-clash-free permutations described earlier. The main purpose of this section is to extend Theorems 3.2 and 3.6 to (s, k, r)-clash-free permutations; see Theorem 4.2 and Corollary 4.5, respectively. The section ends with two methods for constructing (s, k, r)-clash-free permutations from (s, k)-clash-free permutations. We start by explicitly describing the extension, using the following notation. For $1 \leq r < n$ and $0 \leq i_0 < \cdots < i_r \leq n$, let

$$\|i_0, \ldots, i_r\|_n = \min\{i_l - i_{l+1} \bmod n \ : \ l = 0, \ldots, r\}\,,$$

where we use the convention $i_{r+1} = i_0$. That is, $\|i_0, \ldots, i_r\|_n$ is the minimum length l of a cyclic interval $\{i + j \bmod n \ : \ j = 0, \ldots, l\}$ that contains all of $i_0, \ldots, i_r$. So clearly $\|i_{j_0}, \ldots, i_{j_{r'}}\|_n \leq \|i_0, \ldots, i_r\|_n$ for any subset $\{i_{j_0}, \ldots, i_{j_{r'}}\}$ of $\{i_0, \ldots, i_r\}$. For a permutation $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$, an (s, k, r)-*clash* occurs between the (distinct) elements $a_{i_0}, \ldots, a_{i_r}$ if

$$\|i_0, \ldots, i_r\|_n < s \text{ and } \|a_{i_0}, \ldots, a_{i_r}\|_n < k\,.$$

If the permutation $\mathcal{A}$ does not contain any such elements, then it is said to be (s, k, r)-*clash-free* (or just clash-free if the parameters are clear). For example when $k = 3$, $s = 3$, $n = 7$ and $r = 2$ the permutation $\mathcal{A} = 0, 3, 6, 2, 5, 1, 4$ is $(3, 3, 2)$-clash-free, but $\mathcal{A} = 0, 2, 5, 6, 4, 1, 3$ is not as the elements $a_2 = 5$, $a_3 = 6$ and $a_4 = 4$ clash. The maximum value of s for which an (s, k, r)-clash-free permutation exists for a given n, k and r is denoted by $\sigma(n, k, r)$. Similarly, the maximum value of k for which a clash-free permutation exists for a given n, s and r is denoted by $\kappa(n, s, r)$. Note that then natural analogues of (2.1), Lemma 2.1 and Corollary 2.2 hold for σ and κ when $r \geq 1$ hold.

Before proving the main results of this section we first show that Lemma 2.3 can be extended to the following.

Lemma 4.1 *Let k, n and r be integers with k and r both less than n. If $r \geq k$ then*

$$\sigma(n, k, r) = n$$

and if $r < k$ then

$$\sigma(n, k, r) \leq \lfloor (rn - 1)/k \rfloor.$$

Proof: It is easy to check that no (s, k, r)-clash can occur if $r \geq k$, so in this case $\sigma(n, k, r) = n$, which is the first part of the lemma. Now suppose that $r < k$ and that $a_0, a_1, \ldots, a_{n-1}$ is an (s, k, r)-clash-free permutation. On an $n \times n$ torus construct n open rectangles $R_0, R_1, \ldots, R_{n-1}$. Each R_i has centre at (i, a_i) and dimensions $k \times s$. If R_i and R_j are different rectangles then $i \neq j$ and $a_i \neq a_j$. Since no point on the torus can belong to more than r rectangles we have

$$nsk \leq rn^2. \tag{4.1}$$

Suppose we have equality in (4.1). Then every point on the torus that is not on the boundary of a rectangle belongs to exactly r rectangles, which means each point on the boundary of one rectangle must also be on the boundary of another rectangle. Suppose R_i is on the left and R_j on the right of a section of a common vertical boundary. This section cannot be the whole of the common vertical boundary for then we'd have $a_i = a_j$ which is impossible. So part of the right hand vertical boundary of R_i must be shared with a rectangle other than R_j, say with R_l. But this would mean $j = l$, which is impossible. We conclude that there is strict inequality in (4.1) so that $sk < rn$ and therefore $s \leq (rn - 1)/k$ and

$$\sigma(n, k, r) \leq \lfloor (rn - 1)/k \rfloor.$$

□

We can now show the following theorem.

Theorem 4.2 *If* $\gcd(s, n) = 1$ *then* $\kappa(n, s, r) = \lfloor (rn - 1)/s \rfloor$ *and if* $\gcd(k, n) = 1$ *then* $\sigma(n, k, r) = \lfloor (rn - 1)/k \rfloor$.

Proof: By the duality of σ and κ, the two parts of the Theorem's statement are equivalent. We will prove the first, so suppose that $\gcd(s, n) = 1$.

Lemma 4.1 implies that $\sigma(n, k, r) \leq \lfloor (rn - 1)/k \rfloor$ and the duality between $\sigma(n, k, r)$ and $\kappa(n, k, r)$ implies that $\kappa(n, s, r)$ cannot be greater than $\lfloor (rn - 1)/s \rfloor = \lfloor rn/s \rfloor$. Therefore it is sufficient to present a permutation achieving $k = \lfloor rn/s \rfloor$. Let $\bar{s}$ be the integer in $[2, n - 1]$ such that $s\bar{s} \equiv -1 \pmod{n}$. Then define $\mathcal{A} = a_0, \ldots, a_{n-1}$ by

$$a_i = i\bar{s} \bmod n \text{ for } i = 0, \ldots, n - 1.$$

We claim that no clash occurs in this permutation. Suppose, for the sake of contradiction, that a clash does occur between $a_{i_0}, \ldots, a_{i_r}$, i.e,

$$\|i_0, \ldots, i_r\| < s \text{ and } \|a_{i_0}, \ldots, a_{i_r}\| < k$$

where, without loss of generality, $i_0 < \cdots < i_r$. If $\|i_0, \ldots, i_r\| = i_r - i_0$, then

$$\|a_{i_0}, \ldots, a_{i_r}\| = \|i_0\bar{s}, i_1\bar{s}, \ldots, i_r\bar{s}\| = \|0, (i_1 - i_0)\bar{s}, \ldots, (i_r - i_0)\bar{s}\|$$

where $0 < i_1 - i_0 < \cdots < i_r - i_0 < s$. If $\|i_0, \ldots, i_r\| = n + i_l - i_{l+1}$ for some $0 \leq l \leq r-1$, then

$$\begin{aligned}\|a_{i_0}, \ldots, a_{i_r}\| &= \|i_0\bar{s}, i_1\bar{s}, \ldots, i_r\bar{s}\| \\ &= \|0, (i_{l+2}-i_{l+1})\bar{s}, \ldots, (i_r-i_{l+1})\bar{s}, (n+i_0-i_{l+1})\bar{s}, \ldots, (n+i_l-i_{l+1})\bar{s}\|\end{aligned}$$

where $0 < i_{l+2} - i_{l+1} < \cdots < i_r - i_{l+1} < n + i_0 - i_{l+1} < \cdots < n + i_l - i_{l+1} < s$. In either case, there are integers $0 = j_0 < j_1 < \cdots < j_r < s$ such that

$$\|a_{i_0}, \ldots, a_{i_r}\| = \|0, j_1\bar{s}, \ldots, j_r\bar{s}\|\,.$$

By Theorem 3.1, $\{0, j_1\bar{s} \bmod n, \ldots, j_r\bar{s} \bmod n\} \subseteq \{\lfloor in/s \rfloor \;:\; i = 0, \ldots, s-1\}$. So let $\{0, j_1\bar{s} \bmod n, \ldots, j_r\bar{s} \bmod n\} = \{0, \lfloor h_1 n/s \rfloor, \ldots, \lfloor h_r n/s \rfloor\}$, where $0 = h_0 < h_1 < \cdots < h_r < s$. We cannot have $\|0, \lfloor h_1 n/s \rfloor, \ldots, \lfloor h_r n/s \rfloor\| = \lfloor h_r n/s \rfloor - 0$ as $h_r \geq r$ and we have assumed

$$\begin{aligned}\|0, \lfloor h_1 n/s \rfloor, \ldots, \lfloor h_r n/s \rfloor\| &= \|a_{i_0}, \ldots, a_{i_r}\| \\ &< k \\ &= \lfloor rn/s \rfloor.\end{aligned} \tag{4.2}$$

So there is some $0 \leq l \leq r-1$ such that $\|0, \lfloor h_1 n/s \rfloor, \ldots, \lfloor h_r n/s \rfloor\| = n + \lfloor h_l n/s \rfloor - \lfloor h_{l+1} n/s \rfloor$. As $h_r < s$,

$$\begin{aligned} s - 1 &\geq h_r - h_0 \\ &= \sum_{i=0}^{r-1}(h_{i+1} - h_i) \\ &= h_{l+1} - h_l + \sum_{i \neq l}(h_{i+1} - h_i) \\ &\geq h_{l+1} - h_l + r - 1 \end{aligned}$$

i.e., $h_{l+1} - h_l \leq s - r$. Therefore, using the inequality $\lfloor a \rfloor > a - 1$ implies that

$$\begin{aligned} n + \left\lfloor \frac{h_l n}{s} \right\rfloor - \left\lfloor \frac{h_{l+1} n}{s} \right\rfloor &> n + \frac{h_l n}{s} - 1 - \frac{h_{l+1} n}{s} \\ &= \frac{(s - (h_{l+1} - h_l))n}{s} - 1 \\ &\geq \frac{rn}{s} - 1 \\ &\geq \left\lfloor \frac{rn}{s} \right\rfloor - 1\,. \end{aligned}$$

On the other hand, by (4.2),

$$n + \lfloor h_l n/s \rfloor - \lfloor h_{l+1} n/s \rfloor = \|0, \lfloor h_1 n/s \rfloor, \ldots, \lfloor h_r n/s \rfloor\| < \left\lfloor \frac{rn}{s} \right\rfloor\,.$$

Therefore we have the contradiction,

$$\left\lfloor \frac{rn}{s} \right\rfloor > n + \left\lfloor \frac{h_l n}{s} \right\rfloor - \left\lfloor \frac{h_{l+1} n}{s} \right\rfloor > \left\lfloor \frac{rn}{s} \right\rfloor - 1\,.$$

This proves the theorem. □

Let $\nu(s,k,r)$ be the minimum n such that an (s,k,r)-clash-free permutation exists. We have a partial result on the value of $\nu(s,k,r)$, by the theorem above.

Corollary 4.3 *Let n, s, k and r be positive integers such that $k > r$, $s > r$ and $n = \lceil (sk+1)/r \rceil$. If $\gcd(n,k) = 1$ or $\gcd(n,s) = 1$, then $\nu(s,k,r) = n$.*

Proof: By Lemma 4.1 we have $\nu(s,k,r) \geq n$. If $\gcd(n,k) = 1$, then Theorem 4.2 implies that an (s,k,r)-clash-free permutation of $\mathbb{Z}_n$ exists and so $\nu(s,k,r) \leq n$. The case when $\gcd(n,s) = 1$ is similar. This completes the proof. □

Next we prove Corollary 4.5. First we require the following result, which is a special case of Proposition 2.5 from [7]. Note that Proposition 2.5 from [7] is formulated in terms of cyclic matching sequencibility and applies to any graph or hypergraph G; choosing G appropriately yields the result presented here. We avoid using a formulation in terms of cyclic matching sequencibility here and present a proof for completeness. In the proof, for a given r and l with $\gcd(r,l) = 1$, we let $\alpha(i) = ir^{-1} \bmod l$ for all $i \in \mathbb{Z}_l$. Also for an $l \times m$ array T and $\mathcal{A} = T[0,0], T[0,1], \ldots, T[l-1, m-1]$, we let $\alpha(\mathcal{A}) = T[\alpha(0),0], T[\alpha(0),1], \ldots, T[\alpha(1),0], \ldots, T[\alpha(l-1), m-1]$.

Theorem 4.4 *Let $T[i,j]$ be an $l \times m$ array and $\mathcal{A} = T[0,0], T[0,1], \ldots, T[l-1, m-1]$ be an (s,k) clash-free permutation such that $\|T[i,j], T[i,j']\| \geq k$ for all $i = 0, \ldots, m-1$ and $j \neq j'$. If $\gcd(l,r) = 1$, then $\alpha(\mathcal{A})$ is an (s',k,r)-clash-free permutation of $\mathbb{Z}_n$ where $s' = (r-1)m + s$.*

Proof: For x and y in $\mathbb{Z}_n$ let

$$I_y := \{y, \ldots, y+k-1 \bmod n\}$$

and

$$S_x := \{T[\alpha(i),j] : im+j = x, \ldots, x+s'-1 \bmod n, i \in \mathbb{Z}_l, j \in \mathbb{Z}_m\} = \{a_x, \ldots, a_{x+s'-1}\}$$

and note that $|I_y| = k$ and $|S_x| = s'$. Let $\{a_{i_1}, \ldots, a_{i_p}\}$ be a subset of S_x for some x. Then

$$\|i_1, i_2, \ldots, i_p\| < s' \tag{4.3}$$

and if $\{a_{i_1}, \ldots, a_{i_p}\}$ is also a subset of I_y for some y then

$$\|a_{i_1}, \ldots, a_{i_p}\| < k. \tag{4.4}$$

It follows that $\alpha(\mathcal{A})$ is (s', k, r)-clash-free if and only if for all values of x and y in $\mathbb{Z}_n$ we have $|I_y \cap S_x| \leq r$. We'll now show that this condition holds.

The condition that $\|T(i,j), T[i,j']\| \geq k$ for all $i \in \mathbb{Z}_m$ and $j \neq j'$ means that at most one element of each row of T belongs to I_y. The set S_x contains $s' = (m-1)r$ consecutive elements modulo n of $\alpha(\mathcal{A})$. These must come from r or $r+1$ consecutive rows modulo l of T. If they come from exactly r rows of T then $S_x \cap I_y$ contains at most r elements and we are done. Suppose the elements of S_x come from $r+1$ rows of T, say from rows $\alpha(i'), \ldots, \alpha(i'+r)$.

Then S_x consists of the last z elements of row $\alpha(i')$ for some positive z less than s, all of rows $\alpha(i'+1)$ to $\alpha(i'+r-1)$ (each of which contains at most one element of I_y), and the first $s-z$ elements of row $\alpha(i'+r)$ of T. That is, S_x contains $\{T[\alpha(i'), j], j = m-z \ldots, m-1\}$ and $\{T[\alpha(i'+r), j], j = 0, \ldots, s-z-1\}$. But

$$\alpha(i'+r) \quad = \quad (i'+r)r^{-1} \bmod l = \alpha(i') + 1 \bmod l.$$

So these s elements are

$$T[\alpha(i'), m-z+1], \ldots, T[\alpha(i'), m-1], T[\alpha(i')+1, 0] \ldots T[\alpha(i'), s-z-1]$$

which form a sequence of s consecutive members of $\mathcal{A}$. Since $\mathcal{A}$ was assumed to be (s,k)-clash-free this sequence can contain at most one element of I_y so $|I_y \cap S_x| \leq r$ as required. □

Recall that the proof of Theorem 3.6 used an array T constructed from the algorithm from Subsection 3.3. One can easily check that the array T satisfies the condition of Lemma 4.4, since each row of T is the set of residues of k modulo n. Therefore, the following result is immediate from Theorem 4.4 and the array T constructed from the algorithm from Subsection 3.3.

Corollary 4.5 *Let k divide n and* $\gcd(r,k) = 1$. *Then*

$$\sigma(n,k,r) = \frac{rn}{k} - 1\,.$$

Proof: By Theorems 3.6 and 4.4, $\sigma(n,k,r) \geq (r-1)\frac{n}{k} + \frac{n}{k} - 1 = \frac{rn}{k} - 1$ and $\sigma(n,k,r) \leq \frac{rn}{k} - 1$, by Lemma 4.1. □

We end the section with two results about constructing clash-free permutations for general r from those with $r = 1$.

Proposition 4.6 *Let $\mathcal{A} = a_0, \ldots, a_{n-1}$ be an (s,k)-clash-free permutation. Then $\mathcal{A}' = ra_0, ra_0+1, \ldots, ra_0+r-1, ra_1, \ldots, ra_{n-1}+r-1$ is an (rs, rk, r)-clash-free permutation of $\mathbb{Z}_{rn}$.*

Proof: Let $\mathcal{A}' = a'_0, \ldots, a'_{rn-1}$, where $a'_{ir+j} = ra_i + j$ for every $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_r$. Suppose a clash occurs in $\mathcal{A}'$. That is, there exist $r+1$ members $a'_{i_0}, \ldots, a'_{i_r}$ of $\mathcal{A}'$ whose indices lie is some interval modulo rn of length $rs-1$, and whose corresponding values in the permutation lie in an interval modulo rn of length less than rk. Say the set of indices modulo rn is:

$$\begin{aligned} & \{i'r + j : j = j' + 1, \ldots, r-1\} \\ \cup \ & \{ir + j : i = i' + 1, \ldots, i' + s - 1, j = 0, \ldots, r-1\} \\ \cup \ & \{(i' + s)r + j : j = 0, \ldots, j' - 1\}. \end{aligned}$$

for integers i' and j'. Now consider our $r+1$ members of the permutation. Since each of these has index in the set above and their residue modulo r can take only r different values, there must be at least one pair of indices with with the same residue modulo r. Say these indices are $i_1r + j_1$ and $i_2r + j_1$. By assumption $\|i_1r + j_1, i_2r + j_1\| < rs$ so that $\|i_1, i_2\| < s$. Also by assumption, $\|ra_{i_1} + j_1, ra_{i_2} + j_1\| < rk$, which implies that $\|a_{i_1}, a_{i_2}\| < k$. This contradicts the hypothesis that $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$ is (s,k)-clash-free. We conclude that no clash occurs in $\mathcal{A}'$. □

The following example demonstrates that Proposition 4.6 is best possible in the sense that the conclusion for $\mathcal{A}'$ is no longer true if either rs is replaced with $s' > rs$, or rk is replaced with $k' > rk$.

EXAMPLE. Let $\mathcal{A} = 0, 3, 6, 2, 5, 1, 4$. It is easy to check that $\mathcal{A}$ is $(2,3)$-clash-free. The permutation (of $\mathbb{Z}_{14}$) constructed from $\mathcal{A}$ as in the proposition above for $r = 2$ is $\mathcal{A}' = 0, 1, 6, 7, 12, 13, 4, 5, 10, 11, 2, 3, 8, 9$ and is by the proposition $(4,6,2)$-clash-free. On the other hand $\mathcal{A}$ is not $(5,6,2)$-clash-free as $a_0 = 0, a_1 = 1$ and $a_4 = 12$ would clash and is not $(4,7,2)$-clash-free as $a_0 = 0, a_1 = 1$ and $a_2 = 6$ would clash.

Proposition 4.7 *Let $\mathcal{A}$ be an (s,k)-clash-free permutation. Then $\mathcal{A}$ is also (rs,k,r)-clash-free.*

Proof: Let $\mathcal{A} = a_0, \ldots, a_{n-1}$ and suppose for a contradiction that $a_{i_0}, \ldots, a_{i_r}$ clash. That is

$$\|i_0, \ldots, i_r\| < rs \quad \text{and} \quad \|a_{i_0}, \ldots, a_{i_r}\| < k \tag{4.5}$$

where without loss of generality $i_0 < i_1 \cdots < i_r$.

Now

$$\|i_0, \ldots, i_r\| = \sum_{j=0}^{r-1}(i_{j+1} - i_j) + (n + i_0 - i_r) - m\,,$$

where m is the largest of the $r+1$ terms in parentheses. Since the sum is less than rs at least one of the remaining r terms, say $i_{l+1} - i_l$, is less than s. That is,

$$\|i_l, i_{l+1}\| < s.$$

We also have

$$\|a_{i_l}, a_{i_{l+1}}\| \leq \|a_{i_0}, \ldots, a_{i_r}\| < k$$

so that a_{i_l} and $a_{i_{l+1}}$ (s,k)-clash in $\mathcal{A}$, contradicting the hypothesis of the proposition. □

As pointed out by a referee, one could instead prove the proposition using the fact that no $k \times s$ rectangle which is open on the north and east sides and closed on the other sides can contain more than one point (i, a_i).

5 Variants of clash-free permutations

5.1 Outline

In this section, we consider two variations of (s, k, r)-clash-free permutations. As explained in the Introduction, when considered from the viewpoint of permuted packing, these variations change the topology from a torus to a cylinder or to a plane. It turns out that the cylinder is significantly easier than the torus and we determine the analogue of $\sigma(n, k, r)$ for all n, k and r; see Theorem 5.2 below. In the final subsection, we consider (s, k)-clash-free permutations on the plane and prove an analogue to Theorem 3.6.

5.2 Variant on the cylinder

We begin by defining the variant of (s, k, r)-clash-free permutations on the cylinder, using the following notation. For $i_0, \ldots, i_r$ we let $|i_0, \ldots, i_r| = \max\{i_0, \ldots, i_r\} - \min\{i_0, \ldots, i_r\}$. When $r = 1$, we have that $|i_0, i_1| = |i_1 - i_0|$. For a permutation $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$, a *weak* (s, k, r)*-clash* occurs between the (distinct) elements $a_{i_0}, \ldots, a_{i_r}$ if

$$|i_0, \ldots, i_r| < s \text{ and } \|a_{i_0}, \ldots, a_{i_r}\|_n < k. \tag{5.1}$$

If the permutation $\mathcal{A}$ does not contain any such elements, then it is said to be *weakly* (s, k, r)*-clash-free*. For example when $k = 3$, $s = 3$, $n = 7$ and $r = 2$, the permutation $\mathcal{A} = 0, 2, 4, 6, 3, 5, 1$ is weakly clash-free but is not clash-free as $a_0 = 0, a_1 = 2$ and $a_6 = 1$ clash. Let $\hat{\sigma}(n, k, r)$ be the maximum s for which a weakly (s, k, r)-clash-free permutation exists for n, k and r. As $|i_0, \ldots, i_r| < s$ implies that $\|i_0, \ldots, i_r\| < s$, we have that $\hat{\sigma}(n, k, r) \geq \sigma(n, k, r)$ for all n, k and r. We will determine $\hat{\sigma}(n, k, r)$ for all parameters n, k and r. First we prove the natural analogue to Lemma 4.1 for weakly (s, k, r)-clash-free permutations.

Lemma 5.1 *Let k, n and r be integers with k and r both less than n. If $r \geq k$ then*

$$\hat{\sigma}(n, k, r) = n$$

and if $r < k$ then

$$\hat{\sigma}(n, k, r) \leq \lfloor (rn - 1)/k \rfloor.$$

Proof: It is immediate from definitions that any permutation of $\mathbb{Z}_n$ is weakly (n, k, r)-clash-free when $r \geq k$ and so in this case $\hat{\sigma}(n, k, r) = n$. Suppose instead that $r < k$. Let $\mathcal{A} = a_0, \ldots, a_{n-1}$ be a weakly (s, k, r)-clash-free permutation, for some integer $s \leq n$. For x and y in $\mathbb{Z}_n$ let

$$I_y = \{y, y + 1, \ldots, y + k - 1 \bmod n\}$$

and

$$S_x = \{a_{x'} \ : \ x \le x' \le \min\{x+s-1, n\}\}\,.$$

Note that $|I_y| = k$ for all y and $|S_x| = s$ if $x \le n-s+1$ and $|S_x| = n-x+1$ when $x \ge n-s+2$. If there exists a set $A = \{a_{i_0}, \ldots, a_{i_r}\}$ of $r+1$ elements of $\mathcal{A}$ such that $A \subseteq S_x \cap I_y$ for some $x, y \in \mathbb{Z}_n$, then

$$|i_0, \ldots, i_r| < s \quad \text{and} \quad \|a_{i_0}, \ldots, a_{i_r}\| < k, \tag{5.2}$$

by the definitions of S_x and I_y, respectively. Since $\mathcal{A}$ is weakly (s,k,r)-clash-free, (5.2) cannot hold, so we conclude that $|S_x \cap I_y| \le r$ for all x and y in $\mathbb{Z}_n$.

For each $x \in \mathbb{Z}_n$, consider the sum $\sum_{y\in\mathbb{Z}_n} |I_y \cap S_x|$. As each element of S_x appears in exactly k sets I_y, and $|S_x \cap I_y| \le r$ for all $y \in \mathbb{Z}_n$, it follows that

$$|S_x|\, k = \sum_{y\in\mathbb{Z}_n} |I_y \cap S_x| \le rn. \tag{5.3}$$

In particular, as $|S_0| = s$, we have that $sk \le rn$. Therefore, $\hat{\sigma}(n,k,r) \le \lfloor rn/k \rfloor$, which proves the lemma when rn is not divisible by k.

It only remains to show that if k divides rn then s, and therefore, $\hat{\sigma}(n,k,r)$, cannot be rn/k. Suppose for the sake of contradiction that $s = rn/k$. Let $x \in \mathbb{Z}_n$ be an element such that $|S_x| = s$. Then equality holds in (5.3) and so $|I_y \cap S_x| = r$ for all $y \in \mathbb{Z}_n$. So for any $z \in S_x$, as $z \in I_z \cap S_x$, $z \notin I_{z+1} \cap S_x$ and $|I_{z+1} \cap S_x| = r$, it follows that $(z + k \bmod n) \in S_x$. Therefore, S_x is the union of residue classes of $\mathbb{Z}_n$ modulo k. In particular, as $|S_0| = s = |S_1|$, S_0 and S_1 are each union of residue classes of $\mathbb{Z}_n$ modulo k. However, $S_0 \setminus S_1$ and $S_1 \setminus S_0$ each contain one element while any residue class of $\mathbb{Z}_n$ modulo k contains at least two elements, which is a contradiction. Thus, $s < rn/k$ and the result follows. □

Now we can determine $\hat{\sigma}(n,k,r)$.

Theorem 5.2 *Let n, k and r be integers with $r < k < n$. Then*

$$\hat{\sigma}(n,k,r) = \left\lfloor \frac{rn-1}{k} \right\rfloor.$$

Proof: By Lemma 5.1, we only need to find a weakly (s,k,r)-clash-free permutation with $s = \lfloor \frac{rn-1}{k} \rfloor$. Let $d = \gcd(k,n)$. For $i = 0, \ldots, d-1$ and $j = 0, \ldots, n/d-1$ let $T[i,j] = i + kj \bmod n$. Then we construct the permutation

$$\begin{aligned}\mathcal{A} &= T[0,0], T[0,1], \ldots, T[0, n/d-1], T[1,0], \ldots, T[d-1, n/d-1] \\ &:= a_0, \ldots, a_{n-1}.\end{aligned}$$

It is easy to check that $a_{in/d+j} = T[i,j]$ for all $i = 0, \ldots, d-1$ and $j = 0, \ldots, n/d-1$. We prove that $\mathcal{A}$ is clash-free. For the sake of contradiction, suppose the elements $T[i_0, j_0], \ldots, T[i_r, j_r]$ clash. That is

$$\left| i_0\frac{n}{d} + j_0, \ldots, i_r\frac{n}{d} + j_r \right| < s \quad \text{and} \quad \|i_0 + kj_0, \ldots, i_r + kj_r\| < k. \tag{5.4}$$

Without loss of generality we assume that $0 \le i_0\frac{n}{d} + j_0 < \cdots < i_r\frac{n}{d} + j_r \le n-1$. As $0 \le j_a \le n/d - 1$, it follows that $i_0 \le \cdots \le i_r$. Therefore $|i_0\frac{n}{d} + j_0, \ldots, i_r\frac{n}{d} + j_r| = |i_0\frac{n}{d} + j_0, i_r\frac{n}{d} + j_r| = (i_r - i_0)n/d + (j_r - j_0)$. For $0 \le a \le r$ let $i'_a = i_a$ and $j'_a = (i_a - i_0)\frac{n}{d} + j_a$. Clearly, $\|i_0 + kj_0, \ldots, i_r + kj_r\| = \|i'_0 + kj'_0, \ldots, i'_r + kj'_r\|$. As $0 \le j_a \le n/d - 1$, no two j'_a and j'_b can be the same. So we have that $i'_0 + kj'_0 < i'_1 + kj'_1 < \cdots < i'_r + kj'_r$ where $i'_{a+1} + kj'_{a+1} - (i'_a + kj'_a) \ge k$ for all $0 \le a \le r-1$.

Let I be the smallest interval that contains all of $i_0 + kj_0, \ldots, i_r + kj_r$ modulo n, i.e., I is the smallest set of the form $I = \{a + b \bmod n \,:\, b = 0, \ldots, c\}$ that contains $i_l + kj_l \bmod n$ for all $0 \le l \le r$. By (5.4), $|I| < k$. If $n-1$ and 0 are both in I, we replace each j_a with $j_a - 1$ (and each j'_a with $j'_a - 1$), so that, without loss of generality, I is a set of consecutive integers. For $0 \le a \le r$ let $i'_a + kj'_a = l_a + m_a n$ for integer l_a and m_a such that $0 \le l_a \le n-1$. Clearly $i'_a + kj'_a \bmod n = l_a$ and $l_a \in I$ for all a. Also, as $l_{a+1} + m_{a+1}n - (l_a + m_a n) \ge k$ for any $0 \le a \le r-1$ and I is a set of less than k consecutive integers, $m_a < m_{a+1}$, i.e, each value of m_a is different. Therefore, $l_r + m_r n - (l_0 + m_0 n) = (m_r - m_0)n + (l_r - l_0) > rn - k$. On the other hand by (5.4), $(i_r - i_0)\frac{n}{d} + (j_r - j_0) = |i_0\frac{n}{d} + j_0, \ldots, i_r\frac{n}{d} + j_r| \le s - 1$ and so

$$i'_r + kj'_r - i'_0 - kj'_0 = i_r + (i_r - i_0)\frac{nk}{d} + j_r k - i_0 - kj_0 \le (s-1)k + i_r - i_0\,.$$

Since k and n are multiples of d and $s < \frac{rn}{k}$, $sk \le rn - d$. Thus as $i_r - i_0 < d$, $i'_r + kj'_r - i'_0 - kj'_0 < rn - d - k + d = rn - k$. Hence we have the contradiction

$$rn - k < l_r + m_r n - (l_0 + m_0 n) = i'_r + kj'_r - (i'_0 + kj'_0) < rn - k\,.$$

$\square$

Let $\hat{\nu}(s,k,r)$ be the minimum integer n such that a weakly (s,k,r)-clash-free permutation of length n exists.

Corollary 5.3 *For positive integers s, k and r, with $r < k$ and $r < s$,*

$$\hat{\nu}(s,k,r) = \left\lceil \frac{sk+1}{r} \right\rceil.$$

Proof: By Lemma 5.1, $\hat{\nu}(s,k,r) \ge \lceil \frac{sk+1}{r} \rceil$. By Theorem 5.2 a weakly (s,k,r)-clash-free permutation of $\mathbb{Z}_n$ with $n = \lceil \frac{sk+1}{r} \rceil$ exists, since $n \ge k$ and $n \ge s$. Therefore, $\hat{\nu}(s,k,r) \le \lceil \frac{sk+1}{r} \rceil$ and the result follows. $\square$

One can also consider permutations $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$, that are free of elements $a_{i_0}, \ldots, a_{i_r}$ such that

$$\|i_0, \ldots, i_r\|_n < s \text{ and } |a_{i_0}, \ldots, a_{i_r}| < k. \tag{5.5}$$

Call such a permutation $\mathcal{A}$ *weak dual* (s,k,r)*-clash-free* and let $\tilde{\sigma}(n,k,r)$ be the maximum s for which a weak dual (s,k,r)-clash-free permutation $\mathcal{A}$ of $\mathbb{Z}_n$ exists. We show that the value $\tilde{\sigma}(n,k,r)$ is determined by $\hat{\sigma}(n,k,r)$.

Analogous to Theorem 2.1 , it is easy to check that $\mathcal{A}$ is weak (s,k,r)-clash-free if and only if $\mathcal{A}^{-1}$ is weak dual (k,s,r)-clash-free. Thus, $\tilde{\sigma}(n,k,r) \geq s$ if and only if $\hat{\sigma}(n,s,r) \geq k$. So, as Lemma 5.1 implies a weak (s,k,r)-clash-free permutation of $\mathbb{Z}_n$ exist only if $sk \leq (rn-1)$, we have that $\tilde{\sigma}(n,k,r) \leq \lfloor (rn-1)/k \rfloor$. For $r < k < n$, let s be the largest integer such that $sk \leq rn-1$, i.e., $s = \lfloor (rn-1)/k \rfloor$. By Theorem 5.2, $\hat{\sigma}(n,s,r) = \lfloor (rn-1)/s \rfloor \geq k$. So $\tilde{\sigma}(n,k,r) \geq s$. It follows that $\tilde{\sigma}(n,k,r) = \lfloor (rn-1)/k \rfloor$.

5.3 Variant on the plane

We now briefly consider a variant of clash-free permutations in which we change the topology of the set $[0,n] \times [0,n]$ from a torus to a plane. For a permutation $\mathcal{A} = a_0, a_1, \ldots, a_{n-1}$, a *very weak* (s,k,r)*-clash* occurs between the (distinct) elements $a_{i_0}, \ldots, a_{i_r}$ if

$$|i_0, \ldots, i_r| < s \text{ and } |a_{i_0}, \ldots, a_{i_r}| < k. \tag{5.6}$$

Let $\sigma_{\mathrm{vw}}(n,k,r)$ denote the maximum s for which a very weak (s,k,r)-clash-free permutation of $\mathbb{Z}_n$ exists. As $|a_{i_0}, \ldots, a_{i_r}| < k$ implies $\|a_{i_0}, \ldots, a_{i_r}\| < k$, it follows that $\sigma_{\mathrm{vw}}(n,k,r) \geq \hat{\sigma}(n,k,r)$. In particular, $\sigma_{\mathrm{vw}}(n,k,r) \geq \lfloor (rn-1)/k \rfloor$ for all $r < k < n$, by Theorem 5.2. However, $\sigma_{\mathrm{vw}}(n,k,r)$ can exceed $\hat{\sigma}(n,k,r)$, as the following demonstrates.

Proposition 5.4 *For integers* $1 < k < n$*, such that* $k \mid n$*,*

$$\sigma_{\mathrm{vw}}(n,k) = \frac{n}{k}.$$

Proof: Let $a_{in/k+j} = jk + (k-1-i)$ for $0 \leq i \leq k-1$ and $0 \leq j \leq \frac{n}{k} - 1$ and $\mathcal{A} = a_0, \ldots, a_{n-1}$. We show that $\mathcal{A}$ is very weak $(n/k,k)$-clash-free. If distinct $a_{in/k+j}$ and $a_{i'n/k+j'}$ elements very weakly clash, then

$$\left| i\frac{n}{k} + j - i'\frac{n}{k} - j' \right| < \frac{n}{k} \quad \text{and} \quad |jk + (k-1-i) - j'k - (k-1-i')| < k$$

and with simplification

$$\left| (i-i')\frac{n}{k} + (j-j') \right| < \frac{n}{k} \quad \text{and} \quad |(j-j')k + (i'-i)| < k \tag{5.7}$$

Without loss of generality $j \geq j'$. The latter condition of (5.7), occurs only if $j = j'$ or $j = j'+1$ and $i' < i$. Since $j \geq j'$, the first condition of (5.7), occurs only if $i = i'$ or $i' = i+1$. As $a_{in/k+j}$ and $a_{i'n/k+j'}$ are distinct, both condition of (5.7) cannot occur simultaneously. Thus we have shown that $\sigma_{\mathrm{vw}}(n,k) \geq \lfloor n/k \rfloor$.

Now we show that $\sigma_{\mathrm{vw}}(n,k) \leq \lfloor n/k \rfloor$. Suppose an arbitrary permutation $\mathcal{A} = a_0, \ldots, a_{n-1}$ is very weak (s,k)-clash-free. Let $a_i, a_{i+1}, \ldots, a_{i+s-1}$ be a sequence of elements of $\mathcal{A}$ that contains $n-k$. Relabel the set $\{a_i, a_{i+1}, \ldots, a_{i+s-1}\}$ as $\{b_0, \ldots, b_{s-1}\}$ where $0 \leq b_0 < b_1 < \cdots < b_{s-1} \leq n-1$. As $\mathcal{A}$ is clash-free, $b_{s-1} \leq n-k$, otherwise

$b_j = n - k$ for $j < s - 1$ and so $n - k < b_{s-1} \leq n - 1$ which would imply that $|b_j - b_{s-1}| < k$. So

$$n - k \geq (b_{s-1} - b_0) = \sum_{i=0}^{s-2} (b_{i+1} - b_i) \geq k(s-1).$$

It follows that $\sigma_{\mathrm{vw}}(n, k) \leq \lfloor n/k \rfloor$. Hence, $\sigma_{\mathrm{vw}}(n, k) = n/k$. □

In general we expect the following.

Conjecture 5.5 *Let n, k and r be integers. Then*

$$\sigma_{vw}(n, k, r) = \left\lfloor \frac{rn}{k} \right\rfloor .$$

6 Concluding remarks

We began by considering the function $\sigma(n, k)$, and its sister function $\kappa(n, s)$, and showed that if $\gcd(k, n) = 1$ then $\sigma(n, k) = \lfloor n/k \rfloor$ (Theorem 3.2) and that if k divides n then $\sigma(n, k) = n/k - 1$ (Theorem 3.6). Theorem 3.7 and Corollary 3.8 allowed us to extend these results to some other cases but the values of $\sigma(n, k)$ for general n and k remains open. However, we conjecture (Conjecture 3.9) that

$$\lfloor (n-1)/k \rfloor - 1 \leq \sigma(n, k) \leq \lfloor (n-1)/k \rfloor$$

for all n and k.

We also considered the functions $\hat{\sigma}(n, k, r)$ and $\sigma(n, k, r)$ and their sister functions. In Lemma 4.1 and Theorem 5.2 we showed that if $r < k < n$ then

$$\hat{\sigma}(n, k, r) = \lfloor (rn - 1)/k \rfloor$$

and that

$$\hat{\sigma}(n, k, r) = n$$

if $k \leq r < n$.

For the function $\sigma(n, k, r)$ we showed that if $\gcd(k, n) = 1$ then $\sigma(n, k, r) = \lfloor rn/k \rfloor$ and if k divides n and $\gcd(r, k) = 1$ then $\sigma(n, k, r) = rn/k - 1$ (Theorem 4.2 and Corollary 4.5). Lemma 19 allowed us to extend these results to some other cases but we are unable to obtain values of $\sigma(n, k, r)$ for all n, k, and r. Instead, analogously to Conjecture 3.9 we expect the following.

Conjecture 6.1 *Let n, k and r be integers and $s = \lfloor (rn - 1)/k \rfloor$. Then*

$$s - 1 \leq \sigma(n, k, r) \leq s\,.$$

We also considered the function $\nu(s,k)$ and proved $\nu(s,k) = sk+1$ in Corollary 3.3. As noted at the end Subsection 3.4, it is possible that $n > \nu(s,k)$, yet no (s,k)-clash-free of $\mathbb{Z}_n$ exists. So it is natural to ask, how many such values of n can exists for a given s and k? And for how many pairs (s,k) does there exist an $n > \nu(s,k)$ such that no (s,k)-clash-free of $\mathbb{Z}_n$ exists?

We also consider the function $\nu(s,k,r)$ and proved $\nu(s,k,r) = \lceil (sk+1)/r \rceil$ in certain cases in Corollary 4.3. Can this be extended to all appropriate values of r, s and k?

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References

[1] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Combin. Appl.* **52** (2008), 7–20.

[2] D. Bevan, C. Homberger and B. E. Tenner, Prolific permutations and permuted packings: downsets containing many large patterns, *J. Combin. Theory Ser. A* **153** (2018), 98–121.

[3] S. R. Blackburn, C. Homberger and P. Winkler, The minimum Manhattan distance and minimum jump of permutations, *J. Combin. Theory Ser. A* **161** (2019), 364–386.

[4] R. A. Brualdi, K. P. Kiernan, S. A. Meyer and M. W. Schroeder, Cyclic matching sequencibility of graphs, *Australas. J. Combin.* **53** (2012), 245–256.

[5] D. L. Kreher, A. Pastine, and L. Tollefson, A note on the cyclic matching sequencibility of graphs, *Australas. J. Combin.* **61** (2015), 142–146.

[6] A. Mammoliti, The r-matching sequencibility of complete graphs, *Electron. J. Combin.* **25(1)** (2018), Paper 1.6.

[7] A. Mammoliti, The r-matching sequencibility of complete k-partite k-graphs, *Australas. J. Combin.* **74** (2) (2019), 344–363.

[8] J. Simpson, Disjoint Beatty sequences, *Integers* **4** (2004): A12.

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