A survey of graphs with known or bounded crossing numbers

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Abstract

We present, to the best of the authors' knowledge, all known results for the (planar) crossing numbers of specific graphs and graph families. The results are separated into various categories: specifically, results for general graph families, results for graphs arising from various graph products, and results for recursive graph constructions.

Contents

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1   Introduction

The *crossing number* of a graph is the minimum number of crossings over all possible *drawings* of G. There are many subtleties to consider with this statement, such as, what defines a crossing? What defines a drawing? In a broad survey on the variants of crossing numbers, Marcus Schaefer discusses these subtleties in excellent detail [171]. Here, we are focused only on the standard crossing number in the plane, and as such, simplified definitions suffice. A graph G has the vertex set $V(G)$ and edge set $E(G)$ and a *drawing* is a representation of G in the plane. Vertices are represented as distinct points and each edge $e = \{u, v\}$ is represented as a continuous arc connecting the points associated with u and v in such a way that the interior of the arc does not contain any points associated with vertices. In addition, the interiors of the arcs are only allowed to intersect at a finite number of points and such that each intersection is strictly a crossing between the edges, as opposed to the edges touching and then not crossing. The intersections between the arcs are the *crossings* of the drawing and the *crossing number* of a graph is denoted by $cr(G)$ and is the minimum number of crossings over all possible drawings of G. In what follows, when no confusion is possible, we shall refer to the arcs and points given by a drawing as the 'edges' and 'vertices' of the drawing.

We do not attempt to survey the vast history of crossing numbers, which has been recounted brilliantly in several places, including [171, 172, 14], but we briefly mention three of the most influential lines of research which continue to inspire researchers today. The first line of research is the initial work into the crossing number of complete and complete bipartite graphs, which can be read about in [14]. The subsequent development of Zarankiewicz' Conjecture and the Harary-Hill Conjecture, which both remain largely unsolved, continue to propel crossing numbers to the forefront of research in topological graph theory. The second line of research is Frank Leighton's development of new techniques for bounding the crossing number, including the discovery of the famous Crossing Lemma [115]. Additionally, Leighton's work made an important connection between crossing numbers and VLSI design, and this continues to attract valuable contributions from the computer science community. The third line of research is Garey and Johnson's proof that the *crossing number problem* is NP-complete [48]. Specifically, the version of the problem described in [48] asks whether a given graph has crossing number less than or equal to a given integer k. As has been common with many NP-complete problems, this continues to inspire research into the practical hardness of the problem as well as algorithmic

approaches for computing crossing numbers.

To date, crossing numbers have only been determined for a small number of graph families. For some other graph families, bounds have been established for which the upper bounds are usually conjectured (but not proved) to be exact. In this survey, we summarise all such published results for the standard crossing number in the plane, along with references.

Our motivation for producing and maintaining this survey is threefold. The first motivation is that there appears to be many pockets of active research occurring in this area, but in many cases it seems that researchers in one pocket are unaware of similar research being conducted in another. As a consequence, a number of results have been proved multiple times by a number of different authors. Here, for the first time, we gather all of these results, from the early research into this topic by Richard Guy, to the extensive work on crossing numbers of graph products by Marián Klešč, and the vast field of results published in the Chinese mathematical literature, together in one document. Whenever possible, we have attempted to give credit to the author who published the result first, as well as a summary of the partial results that led to larger results. In cases where two sets of authors independently published a result in the same year, we have credited both sets of authors.

The second motivation for this survey is to highlight the remaining gaps in the literature, where results remain to be determined. These are perhaps best illustrated by the various tables of results for Cartesian and join products of small graphs with paths, cycles, stars and discrete graphs, which have been compiled from dozens of individual publications, and for which some scant holes still remain. Our hope is that this will help researchers to focus on the remaining unsolved problems in this field.

The third motivation for this survey is to provide a comprehensive set of useful instances for benchmarking purposes. One of the few benchmark sets which has been repeatedly used in the crossing minimisation literature is known as the KnownCR instances, which were originally collected by Gutwenger [52] in Section 4.3.2 of his thesis. The KnownCR set of instances can now be expanded upon considerably for use in future research.

Although, to the best of our knowledge, no other similarly extensive survey of this kind exists, there are a number of other valuable resources regarding crossing numbers. In particular, we make note of a few here: Since 2011, Marcus Schaefer [171] has maintained a dynamic survey of different variants of the crossing number. In 2018, Schaefer [172] also released an excellent book about various aspects of the crossing number problem. For over a decade, Imrich Vrt'o [198] has maintained a very extensive, but unannotated, bibliography of papers relating to crossing numbers; the latest update came in 2014. A fascinating discussion of the history of the problem was presented by Lowell Beineke and Robin Wilson in 2010 [14]. In 2005, Wynand Winterbach [211] gave an excellent summary of the state of research into crossing numbers at the time in his master's thesis. In 2010, Yuanqiu Huang and Jing Wang [78] published a survey paper in Chinese, with particular focus on results obtained by Chinese authors.

If one wishes to actually solve instances, Markus Chimani and Tilo Wiedera [22] produced a mixed-integer linear program in 2016 that is able to compute crossing numbers for small graphs, complete with a proof file, and produced an online interface, Crossing Number Web Compute [23], where researchers can submit their own instances. Alternatively, if a fast crossing minimisation heuristic is desired, we refer readers to the crossing minimisation heuristics built into the Open Graph Drawing Framework (OGDF) package [21], or alternatively to our algorithm, QuickCross [29], which is available for download at http://fhcp.edu.au/quickcross.

Care has been taken to ensure no symbol is used to represent two different concepts throughout this survey, even though in rare cases this means using non-standard notation. These are pointed out as they occur. A full glossary of symbols used in this survey is given in Appendix A.3.

Wherever practical, we have attempted to verify every result in this survey in the following way: We have generated some moderate-sized instances from each family, and used QuickCross to attempt to find a drawing with the proposed number of crossings. In the vast majority of cases, the best result from QuickCross matched the proposed number of crossings. In cases where QuickCross was able to find a drawing with fewer crossings than proposed by a paper, that drawing itself is proof that the result is incorrect. In cases where QuickCross was only able to find drawings with more crossings than a result suggested, we searched for minimal counterexamples using Crossing Number Web Compute. We have collated all of these incorrect results in Appendix A.4, not as a means of disparaging the authors, but rather to ensure that other researchers don't use these results as basis for subsequent proofs.

As new results are being published frequently, our intention is to keep this survey regularly updated, with the latest version always available at arXiv:1901.05155, and periodic updates published in Australasian Journal of Combinatorics. We welcome any correspondence alerting us to results which we have either neglected to include, or which have been discovered since the most recent update of this survey. As a general rule, we only include results which have appeared in a peer-reviewed journal other than in exceptional circumstances. Please send any such correspondence to the corresponding author.

1.1 Asterisked results

In an attempt to include as many known results as possible, we have considered results from any recognised journals or University periodicals. However, some of these journals impose no peer review, or that which does occur is inadequate. As such, the results contained within cannot be relied upon, either in their own right, or as a basis for subsequent proofs. Indeed, we have encountered many such cases where the proofs are either incorrect, or incomplete.

To address this, we have marked all results appearing within such journals with an asterisk. We emphasise that this determination is solely based on the journal in which the publication appears, rather than the quality of the publication itself. In the case that the result has been subsequently proved in a fully refereed journal, we

have removed the asterisk and cited only the latter source. Again, marking results in this way is not intended to disparage the authors, but rather to highlight results which should be revisited and submitted to thorough peer review. Our hope is that, in time, we will be able to replace all of the asterisked results in this way.

1.2   General results

Throughout this survey, we will only include results for specific graphs or graph families. There are bounds for the crossing number of general graphs, which we discuss briefly here.

For graphs with sufficiently many edges, the *Crossing Lemma* provides a lower bound on the crossing number which depends upon a constant c. It was independently discovered by Leighton [115] and Ajtai et al. (1982) [5]:

Theorem 1.1 (Leighton, 1983 [115], Ajtai et al., 1982 [5]) *There is an absolute constant $c > 0$ such that for every graph G with n vertices and $m \geq 4n$ edges,*

$$cr(G) \geq \frac{cm^3}{n^2}.$$

The Crossing Lemma is tight, other than for the choice of c, and it was originally shown that it holds for $c = \frac{1}{100}$. This was improved to $\frac{1}{64}$ by Chazelle, Sharir and Welzl in an email conversation summarised in [4]. Further improvements can be found if the number of edges in the graph is increased. Pach and Tóth (1997) [153] showed that c can be increased to $1/33.75$ if $m \geq 7.5n$. Later, Pach et al. (2006) [152] improved this by showing that c can be increased to $1024/31827$ (roughly $1/31.1$) if $m \geq 6.4375n$. Montaron (2005) [142] determined various values of c depending on the ratio of m and n. Finally, using a different approach, Ackerman (2019) [2] showed that if $m \geq 6.95n$, c can be increased to $\frac{1}{29}$.

In 2007, at an AMS special session in Chicago, Albertson stated a conjecture about graphs with chromatic number n:

Conjecture 1.2 (Albertson, 2007) *Consider any graph G with chromatic number n. Then,*

$$cr(G) \geq cr(K_n).$$

The bound is obviously tight since K_n has chromatic number n. The result is trivially true for $n \leq 4$, since $cr(K_n) = 0$ for these cases. For $n = 5$, it is equivalent to the four colour theorem. Oporowski and Zhao (2009) [147] verified the case for $n = 6$. Albertson et al. (2010) [7] then further verified the conjecture for $n = 7, 8, 9, 10, 11, 12$. Barát and Tóth (2010) [12] verified the cases $n = 13, 14, 15, 16$, and Ackerman (2019) [2] verified the cases $n = 17, 18$. For $n \geq 19$ the conjecture is still open, although Ackerman [2] did show that for $n = 19$, counterexamples could only exist for G containing either 37 or 38 vertices.

1.3   Crossing critical graphs

A graph G is *c-crossing-critical* if $cr(G) \geq c$, but every proper subgraph H of G has $cr(H) < c$. Crossing critical graphs are an important family of graphs for which the crossing number is known (or bounded) and they form a large topic on their own with independent theoretical motivations. We direct the interested reader to [19] for an overview of the results from the most influential line of research in this area. We intend to include a more thorough discussion of crossing-critical graphs in a future iteration of this survey.

2   Specific graphs and graph families

2.1   Complete multipartite graphs

The complete n-partite graph $K_{m_1,m_2,...,m_n}$ is the graph on $\sum_{i=1}^{n} m_i$ vertices defined as follows. The vertices are partitioned into disjoint sets $V_1, V_2, \ldots, V_n$, such that $|V_i| = m_i$ for $i = 1, 2, \ldots n$. An edge exists between two vertices precisely when one vertex is in V_i and the other is in V_j where $i \neq j$. An example of the complete bipartite graph $K_{4,5}$ is displayed in Figure 1 in two drawings; the latter drawing is optimal with respect to the number of crossings.

The crossing numbers for complete multi-partite graphs are only known for a small number of cases, however, a general upper bound is known and is due to Harborth [63]. Harborth's upper bound has matched all of the known exact crossing numbers for these graphs thus far.

Theorem 2.1 (Harborth, 1970 [63]) *For the complete n-partite graph $K_{x_1,x_2,...,x_n}$ define $s = \sum_{i=1}^{n} x_i$ and $c = \sum_{i=1}^{n} (x_i \mod 2)$. Then the following holds:*

$$cr(K_{x_1,x_2,...,x_n}) \leq \frac{1}{8} \left(\sum_{1 \leq i < j < k < \ell \leq n} 3 x_i x_j x_k x_\ell + 3 \binom{\lfloor \frac{c}{2} \rfloor}{2} - \lfloor \tfrac{c}{2} \rfloor \sum_{\substack{1 \leq i < j \leq n \\ i=j=0 \pmod 2}} x_i x_j \right.$$

$$\left. - \lfloor \tfrac{c-1}{2} \rfloor \sum_{\substack{1 \leq i < j \leq n \\ i \neq j \pmod 2}} x_i x_j - \lfloor \tfrac{c-2}{2} \rfloor \sum_{\substack{1 \leq i < j \leq n \\ i=j=1 \pmod 2}} x_i x_j \right)$$

$$+ \sum_{i=1}^{n} \lfloor \tfrac{x_i}{2} \rfloor \lfloor \tfrac{x_i-1}{2} \rfloor \lfloor \tfrac{s-x_i}{2} \rfloor \lfloor \tfrac{s-x_i-1}{2} \rfloor - \sum_{1 \leq i < j \leq n} \lfloor \tfrac{x_i}{2} \rfloor \lfloor \tfrac{x_i-1}{2} \rfloor \lfloor \tfrac{x_j}{2} \rfloor \lfloor \tfrac{x_j-1}{2} \rfloor.$$

2.1.1   Complete bipartite graphs

One of the seminal results for crossing numbers was by Zarankiewicz, who attempted to solve the crossing number problem specifically for complete bipartite graphs; this
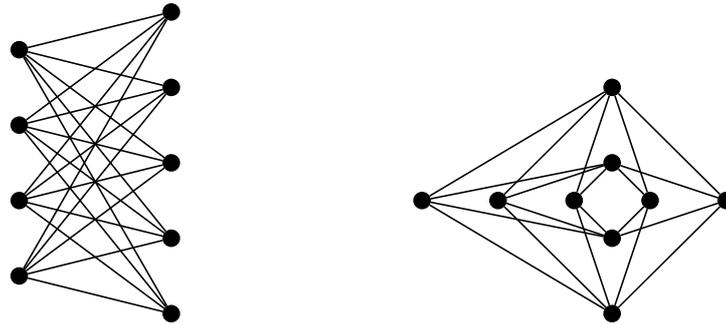
Figure 1: The complete bipartite graph $K_{4,5}$, with the second drawing in the style first described by Zarankiewicz.

special case is known as Turán's brick factory problem. Zarankiewicz claimed to have proved an exact bound [233], but his proof was subsequently found to contain an error [54]. Nonetheless, the result of Zarankiewicz still provides an upper bound:

Theorem 2.2 (Zarankiewicz, 1955 [233]) *The crossing number of the complete bipartite graph $K_{m,n}$ is bounded as follows:*

$$cr(K_{m,n}) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Due to isomorphism, it is clear that $cr(K_{m,n}) = cr(K_{n,m})$ and so in what follows, we will assume that $m \leq n$.

The upper bound has been shown to coincide with the true crossing number for some small cases. In Guy's [54] paper refuting Zarankiewicz's proof, it was shown that the result holds for $m \leq 4$. In 1971, Kleitman [85] verified it for $K_{5,n}$. In 1993, the cases of $K_{7,7}$ and $K_{7,9}$ were verified by Woodall [212] to coincide with Zarankiewicz's formula. In [54], it was shown that if the result holds for $K_{m,n}$, such that m is odd, then it holds for $K_{m+1,n}$. Hence, Kleitman's result also verifies $K_{6,n}$, and Woodall's results also settle the cases $K_{7,8}$, $K_{7,10}$, $K_{8,8}$, $K_{8,9}$ and $K_{8,10}$. Despite detailing a flaw in Zarankiewicz's proof, Guy conjectured that the result would still hold in all cases and this has come to be known as Zarankiewicz' Conjecture:

Conjecture 2.3 (Zarankiewicz, [233], Guy, 1969 [54]) *Theorem 2.2 holds with equality.*

Several asymptotic lower bounds have been proved for complete bipartite graphs. The earliest results by Kleitman [85] imply asymptotic lower bounds, and these were improved upon in 2003 by Nahas [143]. In 2006, de Klerk et al. [32] used semi-definite programming methods to make significant progress, which were refined in 2007 by de Klerk et al. [33], who proved the following:

Theorem 2.4 (de Klerk et al., 2007 [33]) *For the complete bipartite graph $K_{m,n}$ with $m \geq 9$,*

$$\lim_{n \to \infty} \frac{cr(K_{m,n})}{\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor} \geq \frac{0.8594m}{m-1}.$$

In 2013, Norin and Zwols [145] announced that the 0.8594 in Theorem 2.4 could be replaced by 0.905, but this was never published.

In 2013, Christian et al. [25] gave a function $N_0(m)$ and showed that for each m, if Conjecture 2.3 can be confirmed for all $n \leq N_0(m)$, then it is true for all n. Hence for each m, in order to confirm Conjecture 2.3, only finitely many cases need to be considered. Unfortunately, the function $N_0(m)$ is not practical even for small m.

Theorem 2.5 (Christian et al., 2013 [25]) *Let m be a positive integer. Define $N_0(m)$ as follows:*

$$N_0(m) := \left((m!)! \left(2 \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \right)^{m!} \right)^4.$$

If Conjecture 2.3 is true for all $n \leq N_0(m)$, it is true for all n.

2.1.2   Complete tripartite graphs

Complete tripartite graphs have their vertices partitioned into three sets. An example of $K_{2,2,4}$ is displayed in Figure 2.
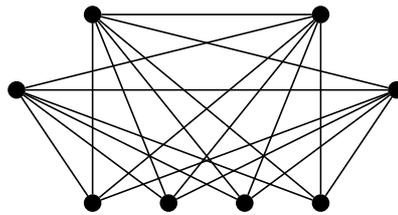
Figure 2: The complete tripartite graph $K_{2,2,4}$.

The crossing number of $K_{a,b,n}$ has been determined for all cases where $a + b \leq 6$, except for $K_{3,3,n}$. The known cases are listed in Table 1, along with the publications where they were first discovered. Note that $K_{1,1,n}$ is planar. In a few cases, the verification came from the perspective of join products, since $K_{a,b} + D_n = K_{a,b,n}$, where D_n is the discrete graph on n vertices. See Section 4 for more information and results on join products. Ho was the first author to explicitly consider $cr(K_{1,2,n})$, although it could be seen as a direct consequence of results from some other authors. First, $P_2 \square S_n$ is a subdivision of $K_{1,2,n}$, and the crossing number of $P_2 \square S_n$ was determined by Bokal (2007) [17]. Second, $K_{1,2,n}$ is a subgraph of $P_2 + P_n$, and a supergraph of

Table 1: Crossing numbers of complete tripartite graphs.

Graph family	Crossing number	Publication
$K_{1,2,n}$	$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$	Ho (2008) [73]
$K_{1,3,n}$	$2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$	Asano (1986) [10]
$K_{1,4,n}$	$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor$	Huang and Zhao (2008) [81]
$K_{1,5,n}$	$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4 \left\lfloor \frac{n}{2} \right\rfloor$ *	Mei and Huang (2007) [141] *
$K_{2,2,n}$	$2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$	Klešč and Schrötter (2011) [109]
$K_{2,3,n}$	$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n$	Asano (1986) [10]
$K_{2,4,n}$	$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$	Ho (2013) [76]

$K_{3,n}$, both of which were shown to have crossing number $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ by Klešč (2007) [96] and Guy (1969) [54] respectively. The result for $K_{1,4,n}$ was also independently proved by Ho (2008) [73].

In 2004, Ho [71] showed that $cr(K_{3,3,n})$ can be determined if Conjecture 2.3 holds for $m = 7$ and $n \leq 20$. To date this is not known to be true, and so the crossing number of $K_{3,3,n}$ can only be given as a conjecture.

Conjecture 2.6 (Ho, 2004 [71]) *For $n \geq 1$,*

$$cr(K_{3,3,n}) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{3n}{2} \right\rfloor + 1.$$

There are some other results depending on the truth of Conjecture 2.3. In particular, Huang and Zhao (2006) [79, 80] proved that $cr(K_{1,6,n})$ can be determined if Conjecture 2.3 holds for $K_{7,k}$ for all $k \geq 1$, and likewise that $cr(K_{1,8,n})$ can be determined if Conjecture 2.3 holds for $K_{9,k}$ for all $k \geq 1$. In 2008, Wang and Huang [203] proved that $cr(K_{1,10,n})$ can be determined if Conjecture 2.3 holds for $K_{11,k}$ for all $k \geq 1$. Finally, Ho (2008) [73] generalised these results by showing the following.

Theorem 2.7 (Ho, 2008 [73]) *If Conjecture 2.3 is true for $K_{2M+1,n}$ then,*

$$cr(K_{1,2M,n}) = M^2 \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - M \left\lfloor \frac{n}{2} \right\rfloor .$$

Currently, Conjecture 2.3 is only known to hold for $M \leq 2$.

In 2017, Gethner et al. [50] gave asymptotic lower bounds on the crossing number of the balanced complete tripartite graphs $K_{n,n,n}$. Let $A(n, n, n)$ denote the right hand side of the inequality in Theorem 2.1, evaluated for $K_{n,n,n}$. Gethner et al. gave asymptotic lower bounds involving $A(n, n, n)$.

Theorem 2.8 (Gethner et al. 2017 [50]) *For the balanced complete tripartite graphs* $K_{n,n,n}$,

$$\lim_{n\to\infty} \frac{cr(K_{n,n,n})}{A(n,n,n)} \geq 0.666.$$

2.1.3   Complete 4-partite graphs

In 2016, Shanthini and Babujee [176] $*$ showed that the crossing number of $K_{1,1,m,n}$ can be expressed in terms of the crossing number of $K_{m+2,n+2}$:

Theorem 2.9 (Shanthini and Babujee, 2016 [176]$*$) *For* $m, n \geq 1$,

$$cr(K_{1,1,m,n}) = cr(K_{m+2,n+2}) + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - mn. \quad *$$

Since Zarankiewicz's conjecture is known to hold for $\min\{m,n\} \leq 6$, Theorem 2.9 settles $cr(K_{1,1,m,n})$ for $m \leq 4$. Each of those four cases had been previously settled. Specifically, the cases $m = 1, 2, 3$ were settled in 2007 by Qian and Huang [158] $*$, and the case $m = 4$ was settled in a separate 2016 paper by Shanthini and Babujee [175] $*$.

There are also some results known for other 4-partite graphs. In particular, He and Huang [67] $*$ determined the crossing number of $K_{1,2,2,n}$ and Ho [74] determined the crossing number of $K_{2,2,2,n}$.

Theorem 2.10 (He and Huang, 2007 [67] $*$) *For* $n \geq 1$,

$$cr(K_{1,2,2,n}) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor. \quad *$$

Theorem 2.11 (Ho, 2008 [74]) *For* $n \geq 1$,

$$cr(K_{2,2,2,n}) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3n.$$

2.1.4   Complete 5-partite graphs

In 2009, Ho [75] determined the crossing number for two 5-partite graphs:

Theorem 2.12 (Ho, 2009 [75]) *For* $n \geq 1$, *the following hold:*

$$cr(K_{1,1,1,1,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n,$$

$$cr(K_{1,1,1,2,n}) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n.$$

2.1.5   Complete 6-partite graphs

In 2008, Lü and Huang [131] determined the crossing number of $K_{1,1,1,1,1,n}$:

Theorem 2.13 (Lü and Huang, 2008 [131]) *For $n \geq 1$,*

$$cr(K_{1,1,1,1,1,n}) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

2.1.6   Complete bipartite graphs minus an edge

In 2011, He et al. [68] $*$ considered the complete bipartite graph $K_{m,n}$ minus one edge, denoted as $K_{m,n} \backslash e$. Due to symmetry, it doesn't matter which edge is removed. They were able to settle the case when $m = 3$ or $m = 4$:

Theorem 2.14 (He et al., 2011 [68] $*$) *For $n \geq 1$, the following hold:*

$$cr(K_{3,n} \setminus e) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor, *$$

$$cr(K_{4,n} \setminus e) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor. *$$

In 2015, Chia and Lee [20] gave a conjecture for larger m, and proved that in addition to $m = 3, 4$, the conjecture is also true for the special case of $K_{5,5} \setminus e$:

Conjecture 2.15 (Chia and Lee, 2015 [20]) *For $m, n \geq 1$,*

$$cr(K_{m,n} \setminus e) = cr(K_{m,n}) - \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

2.2   Complete graphs

The complete graph K_n is the graph on n vertices containing an edge between every pair of vertices. Two examples of complete graphs, K_6 and K_8, are displayed in Figure 3.

Guy, in a 1960 paper [53], describes how the problem of determining the crossing number for complete graphs was brought to his attention by Anthony Hill and C.A. Rogers, but indicates that Paul Erdős had been looking at the problem for over twenty years at that point. Guy gave an upper bound which is conjectured to be equal to the crossing number, and showed that if the conjecture holds for odd n, then it also holds for $n + 1$.

Theorem 2.16 (Guy, 1960 [53]) *The crossing number of the complete graph K_n is bounded above as follows:*

$$cr(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$
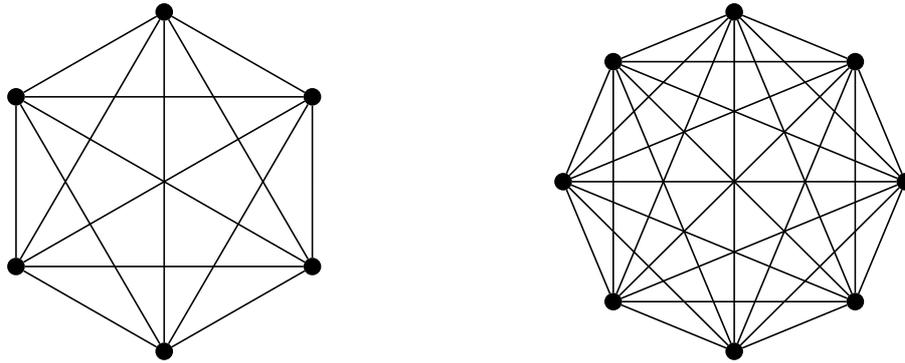
Figure 3: The complete graphs K_6 and K_8.

Conjecture 2.17 (Guy, 1960 [53], Harary and Hill, 1963 [59])
Theorem 2.16 holds with equality.

Guy showed that Conjecture 2.17 holds for $n \leq 6$. The same conjecture was also proposed by Harary and Hill [59] at around the same time and then again four years later by Saaty [165]. In 1972, Guy [55] showed that Conjecture 2.17 holds for $n \leq 10$. This was the best known result for more than three decades, until Pan and Richter [154] proved in 2007 that the conjecture holds for $n = 11$, and hence for $n = 12$ as well. The remaining cases are still open.

The case of $n = 13$ has been considered closely. McQuillan et al. (2015) [139] noted that a simple application of Kleitman's parity theorem [85] implies that $cr(K_{13})$ must be equal to one of the numbers $\{217, 219, 221, 223, 225\}$; McQuillan et al. then proved that $cr(K_{13}) \neq 217$. This result was further improved by Ábrego et al. (2015) [1] who showed that $cr(K_{13}) \neq 219$ and $cr(K_{13}) \neq 221$. Hence, there are only two possibilities remaining for the crossing number of K_{13}; either 223 or 225.

A similar approach as the asymptotic lower bounds for complete bipartite graphs, provides asymptotic lower bounds for complete graphs. In 2019, Balogh et al. [11] gave the current best version of this lower bound.

Theorem 2.18 (Balogh et al., 2019 [11]) *For the complete graph $K(n)$,*

$$\lim_{n \to \infty} \frac{cr(K_n)}{\frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor} > 0.985.$$

The constant in Theorem 2.18 was an improvement on the previous work of Norin and Zwols [145] and de Klerk et al. [33] who gave constants of 0.905 and 0.83 repecitvely.

2.2.1 Complete graphs minus an edge

In 2007, Zheng et al. [239], considered the complete graph with a single edge removed, $K_n \setminus e$. They determined an upper bound for its crossing number, and conjectured

that it would coincide with the crossing number. They also proved that the conjecture holds for $n \leq 8$.

Theorem 2.19 (Zheng et al., 2007 [239]) *For $n \geq 1$,*

$$cr(K_n \setminus e) \leq \frac{1}{4} \left\lfloor \frac{n+2}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor,$$

with equality known to hold for $n \leq 8$.

Conjecture 2.20 (Zheng et al., 2007 [239]) *Theorem 2.19 holds with equality.*

In 2014, Ouyang et al. (2014) [149] proved that Conjecture 2.20 holds for even n whenever Conjecture 2.3 holds for $n - 1$. Hence, Conjecture 2.20 is currently known to hold for $n \leq 12$.

Chia and Lee [20] independently discovered results equivalent to the results in [149] around the same time.

2.2.2 Complete graphs minus a cycle

In 1973, Guy and Hill [57] considered the crossing number of the complement of a cycle $\overline{C}_n$, that is, the complete graph K_n with a simple cycle of length n removed, which is defined for all $n \geq 3$. It is easy to check that $\overline{C}_n$ is planar for $n \leq 6$. They gave an upper bound for the crossing number of $\overline{C}_n$ in general, and conjectured that equality would hold. Guy and Hill also proved that equality does indeed hold for $n \leq 10$.

Theorem 2.21 (Guy and Hill, 1973 [57]) *For $n \geq 3$, the following holds:*

$$cr(\overline{C}_n) \leq \begin{cases} \frac{1}{64}(n-3)^2(n-5)^2, & \text{for odd } n, \\ \frac{1}{64}n(n-4)(n-6)^2, & \text{for even } n, \end{cases}$$

with equality known to hold for $n \leq 10$.

Conjecture 2.22 (Guy and Hill, 1973 [57]) *Theorem 2.21 holds with equality for all $n \geq 3$.*

From the proof of Theorem 2.21, it is known that if the theorem holds for a particular odd n, then it holds for the next even n as well.

Guy and Hill also determined a lower bound for the crossing number of $\overline{C}_n$.

Theorem 2.23 (Guy and Hill, (1973) [57]) *For $n \geq 9$, the following bounds hold:*

$$cr(\overline{C}_n) \geq \frac{\binom{n}{5}(n-15)(n-17)}{4\binom{n-4}{3}} > \frac{1}{80}n(n+2)(n-9)(n-20).$$

Note that Theorem 2.23 is trivial unless $n > 20$.

2.3   Circulant graphs

The circulant graph $Ci_n(L)$ is the graph on n vertices such that, for $i = 1, \ldots, n$ and each j appearing in the list L, vertex i is adjacent to the $(i + j)$-th vertex and the $(i - j)$-th vertex (mod n). Two examples of circulant graphs, $Ci_{10}(\{1, 3\})$ and $Ci_{11}(\{1, 4, 5\})$, are displayed in Figure 4. It is well known that $Ci_n(\{1, 2\})$ is planar for all even n and has crossing number 1 for all odd n. In what follows, we consider circulant graphs for various choices of n and L.
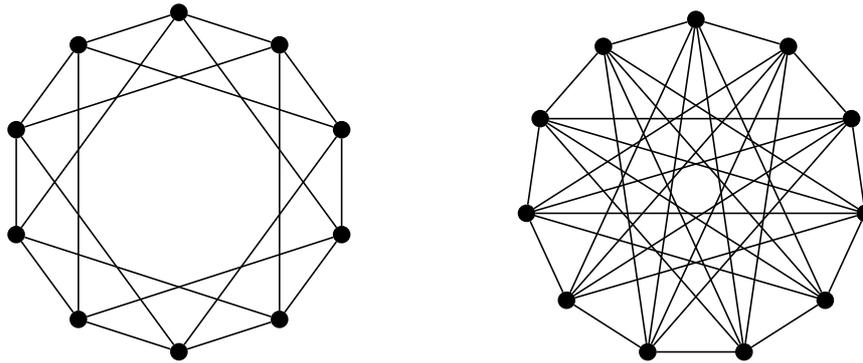
Figure 4: The circulant graphs $Ci_{10}(\{1, 3\})$ and $Ci_{11}(\{1, 4, 5\})$.

In 1986, Fiorini [45] investigated the crossing number of $Ci_n(\{1, 3\})$, primarily as a vessel to establish the crossing number of certain generalized Petersen graphs. While he claimed to have determined the crossing number of $Ci_n(\{1, 3\})$ for $n \geq 8$, his proof was later shown to contain an error. Nonetheless, his proof established a valid upper bound, and also established equality for $n = 8, 10, 12$:

Theorem 2.24 (Fiorini, 1986 [45]) *For $n \geq 8$, the following holds:*

$$cr(Ci_n(\{1, 3\})) \leq \left\lfloor \frac{n}{3} \right\rfloor + (n \mod 3),$$

holding with equality for $n = 8, 10, 12$.

In 2004, Yang et al. [217] proved that equality does indeed hold for all $n \geq 8$:

Theorem 2.25 (Yang et al., 2004 [217]) *For $n \geq 8$, the following holds:*

$$cr(Ci_n(\{1, 3\})) = \left\lfloor \frac{n}{3} \right\rfloor + (n \mod 3).$$

In 2001, Yang and Zhao [220] $*$ determined the crossing number of $Ci_n(\{1, \lfloor n/2 \rfloor\})$ as follows:

Theorem 2.26 (Yang and Zhao, 2001 [220] $*$) *For $n \geq 6$, the following holds:*

$$cr(Ci_n(\{1, \lfloor n/2 \rfloor\})) = 1. *$$

For even n, $Ci_n(\{1, \lfloor n/2 \rfloor\})$ is isomorphic to the Möbius ladder graph on n vertices, for which the crossing number was determined by Guy and Harary [56] in 1967.

In 2006, Lin et al. [127] considered the circulant graph $Ci_n(\{1, \lfloor n/2 \rfloor - 1\})$. They were able to determine the crossing number for even $n \geq 8$, and discover upper bounds for odd $n \geq 13$:

Theorem 2.27 (Lin et al., 2006 [127]) *For even $n \geq 8$, the following holds:*

$$cr(Ci_n(\{1, n/2 - 1\})) = n/2.$$

Theorem 2.28 (Lin et al., 2006 [127]) *For odd $n \geq 13$, the following holds:*

$$cr(Ci_n(\{1, (n-1)/2 - 1\})) \leq \begin{cases} (n+1)/2, & \text{for } n = 1 \pmod 8, \\ (n+3)/2, & \text{for } n = 3 \pmod 8, \\ (n+3)/2, & \text{for } n = 5 \pmod 8, \\ (n+1)/2, & \text{for } n = 7 \pmod 8. \end{cases}$$

In 2005, Salazar [168] proved that the crossing number of $Ci_n(\{1, k\})$ for $k \geq 5$ and $n \geq k^4$ can be bounded above and below and by functions of n and k:

Theorem 2.29 (Salazar, 2005 [168]) *For $k \geq 5$ and $n \geq k^4$, the following bounds hold:*

$$\left(1 - \frac{4}{k}\right) n + (4k^2 + 1 - k^3) \leq cr(Ci_n(\{1, k\})) \leq \left(1 - \frac{2}{k}\right) n + \frac{k^2 + k + 2}{2}.$$

In 2005, Lin et al. [126] considered the circulant graph $Ci_{mk}(\{1, k\})$, for integers $m \geq 3$, $k \geq 3$. They were able to determine the crossing number for $m = 3$, and established upper bounds for larger m:

Theorem 2.30 (Lin et al., 2005 [126]) *For $m, k \geq 3$, the following hold:*

$$cr(Ci_{3k}(\{1, k\})) = k,$$
$$cr(Ci_{4k}(\{1, k\})) \leq 2k + 1, \text{ and for } m \geq 5,$$
$$cr(Ci_{mk}(\{1, k\})) \leq \min\{(m-2)(k+1) - 1, m(k-2)\}.$$

In 2005, Ma et al. [136] considered the circulant graph $Ci_{2m+2}(\{1, m\})$ for $m \geq 2$:

Theorem 2.31 (Ma et al., 2005 [136]) *The following holds:*

$$cr(Ci_{2m+2}(\{1, m\})) = \begin{cases} 0, & \text{for } m = 2, \\ m + 1, & \text{for } m \geq 3. \end{cases}$$

In 2007, Ho [72] considered the circulant graph $Ci_{3m+1}(\{1, m\})$ for $m \geq 3$:

Theorem 2.32 (Ho, 2007 [72]) *For $m \geq 3$, the following holds:*

$$cr(Ci_{3m+1}(\{1, m\})) = m + 1.$$

In 2008, Wang and Huang [205] ∗ considered the circulant graph $Ci_{3m-1}(\{1, m\})$ and derived the following upper and lower bounds, and conjectured that the upper bound would provide the correct crossing number.

Theorem 2.33 (Wang and Huang, 2008 [205] ∗) *For $m \geq 3$, the following holds:*

$$m \leq cr(Ci_{3m-1}(\{1, m\})) \leq m + 1. \ast$$

Conjecture 2.34 (Wang and Huang, 2008 [205]) *For $m \geq 3$, the following holds:*

$$cr(Ci_{3m-1}(\{1, m\})) = m + 1.$$

2.4 Generalized Petersen graphs

The generalized Petersen graphs were first studied by Coxeter [30] and later named by Watkins [210]. The generalized Petersen graph $GP(n, k)$ is constructed by taking the union of the cycle graph C_n and the circulant graph $Ci_n(k)$ (for which the definition is given in Section 2.3) where $n \geq 3$, and then connecting the corresponding vertices in each by an edge. The result is a 3-regular graph containing $2n$ vertices, except in the special case when $n = 2k$ which corresponds to the Möbius ladder graph of size n with the $n/2$ internal edges subdivided twice. Two examples of generalized Petersen graphs, $GP(9, 2)$ and $GP(11, 3)$, are displayed in Figure 5. It is common in the literature to use $P(n, k)$, rather than $GP(n, k)$, to denote generalized Petersen graphs. To avoid confusion with path graphs, we use the latter here.
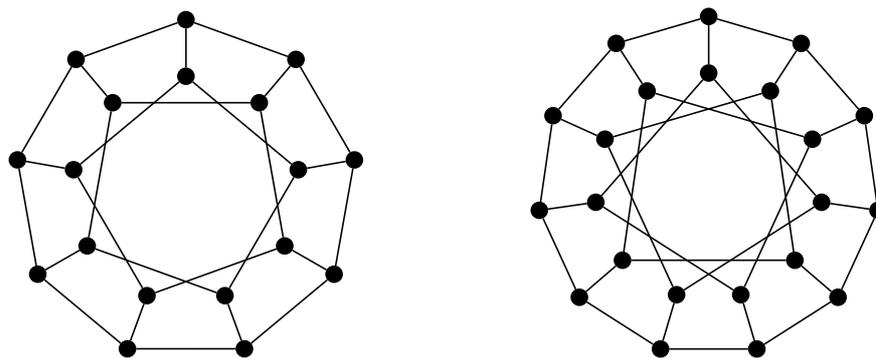
Figure 5: The generalized Petersen graphs $GP(9, 2)$ and $GP(11, 3)$.

A number of results are known about the crossing numbers of certain subfamilies of generalized Petersen graphs. Due to symmetry, $GP(n, k)$ is isomorphic to $GP(n, n-k)$, and so we only need to consider $k \leq \lfloor \frac{n}{2} \rfloor$. Some definitions of $GP(n, k)$

also require that $n \geq 3$. If we allow $n = 1, 2$, then $GP(1, k)$ corresponds to the path graph P_1, and $GP(2, k)$ to the cycle graph C_4, irrespective of the choice of k.

The earliest result in this subsection is due to Guy and Harary (1967) [56], who showed that the crossing number of any Möbius ladder graph is 1. Since $GP(2k, k)$ is a subdivision of a Möbius ladder, we have the following result.

Theorem 2.35 (Guy and Harary, 1967 [56]) *For $k \geq 3$, the following holds:*

$$cr(GP(2k, k)) = 1.$$

$GP(2k, k)$ is also known as the generalised Wagner graph, V_{2k}.

In 1981, Exoo et al. [42] considered $GP(n, k)$ for the special cases where $k = 1, 2$. They showed the former were always planar, and for $k = 2$ they gave the following theorem:

Theorem 2.36 (Exoo et al., 1981 [42]) *The following holds:*

$$cr(GP(n, 2)) = \begin{cases} 0, & \text{for even } n, \\ 0, & \text{for } n = 3, \\ 2, & \text{for } n = 5, \\ 3, & \text{for odd } n \geq 7. \end{cases}$$

In 1986, Fiorini [45] considered the crossing number of $GP(n, k)$ for the special case where $k = 3$, and settled the cases when $n = 0 \bmod 3$ and $n = 2 \bmod 3$. Fiorini also claimed to have solved the special case of $GP(10, 3)$, but the proof was later refuted by McQuillan and Richter [140] in 1992, who also provided a nicer proof of the special case of $GP(8, 3)$. They conjectured the crossing number for the case when $n = 1 \bmod 3$, which was later proved in 2002 by Richter and Salazar [162], who also corrected some errors in Fiorini's proofs, to settle the case where $k = 3$. The results are summarised in the following theorem:

Theorem 2.37 (Fiorini, (1986) [45], Richter and Salazar, (2002) [162])
For $h \in \{0, 1, 2\}$, the following holds:

$$cr(GP(3m + h, 3)) = \begin{cases} 2, & \text{for } h = 0, \, m = 3, \\ m, & \text{for } h = 0, \, m \geq 4, \\ 3, & \text{for } h = 1, \, m = 2, \\ m + 3, & \text{for } h = 1, \, m \geq 3, \\ m + 2, & \text{for } h = 2, \, m \geq 2. \end{cases}$$

For $k = 4$, Fiorini [45] also considered the crossing number of the special case of $GP(4n, 4)$, while in 1997 Saražin [170] settled the special case of $GP(10, 4)$.

Theorem 2.38 (Fiorini, 1986 [45]) *For $n \geq 4$, the following holds:*

$$cr(GP(4n, 4)) = 2n.$$

Theorem 2.39 (Saražin, 1997 [170]) $cr(GP(10, 4)) = 4.$

In 2009, Lin et al. [128] then settled every remaining graph for $n \leq 16$ (see the upcoming Table 2), and gave some conjectures for two remaining cases:

Conjecture 2.40 (Lin et al., 2009 [128]) *For $k \geq 3$, the following hold:*

$$cr(GP(4k + 2, 2k)) = 2k + 1,$$

$$cr(GP(4k + 2, 4)) = 2k + 2.$$

The known crossing numbers of $GP(n, k)$ for $n \leq 17$ are summarised in Table 2 and Appendix A.2.1 lists the publications where these results were first proved.

Table 2: Crossing numbers for $GP(n, k)$, $5 \leq n \leq 17$. For $n \leq 4$, $GP(n, k)$ is planar. Appendix A.2.1 lists the publications where these results were first proved. The numbers in this table (other than for $n = 17$) were first collated by Lin et al. [128].

n / k	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	0	3	0	3	0	3	0	3	0	3	0	3
3		1	3	4	2	6	5	4	7	6	5	8	7
4				1	3	4	5	4	7	8	10	8	
5						1	3	8	9	6	5	8	
6								1	3	7	10	12	7
7										1	3	9	
8												1	3

There are also some results for arbitrarily large k. In 2005, Ma et al. [135] settled the case for $GP(2k + 1, k)$ for $k \geq 3$:

Theorem 2.41 (Ma et al., 2005 [135]) *For $k \geq 3$, the following holds:*

$$cr(GP(2k + 1, k)) = 3.$$

In 2003, Fiorini and Gauci [46] settled the case for $GP(3k, k)$ for $k \geq 4$:

Theorem 2.42 (Fiorini and Gauci, 2003 [46]) *For $k \geq 4$, the following holds:*

$$cr(GP(3k, k)) = k.$$

In 2019, Gauci and Xuereb settle the cases for $GP(3k-1, k)$ and $GP(3k+1, k)$ for $k \geq 3$.

Theorem 2.43 (Gauci and Xuereb, 2019 [49]) *For $k \geq 3$, the following hold:*

$$cr(GP(3k-1, k)) = k+1,$$

$$cr(GP(3k+1, k)) = k+3.$$

For general $GP(n, k)$ with $k \geq 5$, there are also lower and upper bounds known. In particular, in 2004, Pinontoan and Richter [156] determined bounds for $k \geq 6$ and $n \geq 2k+1$:

Theorem 2.44 (Pinontoan and Richter, 2004 [156]) *For $k \geq 6$, $n \geq 2k+1$, there exists a nonnegative constant c_k such that:*

$$\frac{nk}{3} - c_k \leq cr(GP(n, k)) \leq 2n - \frac{4n}{k} + c_k.$$

Also, in 2005, Salazar [168] derived bounds via a different approach for $n \geq k$ and $k \geq 5$:

Theorem 2.45 (Salazar, 2005 [168]) *For $n \geq k \geq 5$, the following bounds hold:*

$$\frac{2}{5}\left[\left(1 - \frac{4}{k}\right)(n - k^4)\right] + (4k^2 + 1 - k^3) \leq cr(GP(n, k)) \leq \left(2 - \frac{2}{k}\right)n + \frac{k^2 + k + 2}{2}.$$

2.5   Path powers

Consider the path graph P_n on $n+1$ vertices. The graph P_n^k, called the k-power of the graph P_n, is a graph on the same vertex set as P_n. An edge $\{a, b\}$ exists in P_n^k if and only if the distance between a and b on P_n is at most k. Two examples of path powers, P_6^2 and P_6^3, are displayed in Figure 6.

Figure 6: The path powers P_6^2 and P_6^3.

The crossing number of P_n^k is known for some values of k. First, if $k \leq 3$, then P_n^k is planar. If $k = n$, then P_n^k is isomorphic to K_n; see Section 2.2. If $k = n-1$, then P_n^k is isomorphic to K_n minus an edge; see Section 2.2.1.

At this point the only other case which has been fully settled is $k = 4$, which was considered in 1993 by Harary and Kainen [60]:

Theorem 2.46 (Harary and Kainen, 1993 [60]) *For $n \geq 5$, the following holds:*

$$cr(P_n^4) = n - 4.$$

In 1999, Harary et al. [61] extended this result by providing lower and upper bounds for P_n^5.

Theorem 2.47 (Harary et al., 1999 [61]) *For $n \geq 6$, the following bounds hold:*

$$2n - 9 \leq cr(P_n^5) \leq 4n - 21.$$

In 2009, Zheng et al. [242] gave some upper bounds for higher powers, and conjectured that they would coincide with the exact crossing number. These results are listed in Table 3.

Table 3: Results for paths of higher powers, from [242].

$cr(P_6^5) = 3$	$cr(P_7^5) = 6$	$cr(P_8^5) = 9$	$cr(P_n^5) \leq 4n - 23,$ for $n \geq 9$
$cr(P_7^6) = 9$	$cr(P_8^6) = 15$	$cr(P_9^6) \leq 22$	$cr(P_n^6) \leq 8n - 51,$ for $n \geq 10$
$cr(P_8^7) = 18$	$cr(P_9^7) \leq 30$	$cr(P_{10}^7) \leq 42$	$cr(P_{11}^7) \leq 57$
			$cr(P_n^7) \leq 15n - 109,$ for $n \geq 12$

Conjecture 2.48 (Zheng et al., 2009 [242]) *All upper bounds in Table 3 hold with equality.*

2.6   Knödel graphs

The Knödel graph $\mathcal{W}_{\Delta,n}$, for even $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$ is the graph on n vertices which is defined as follows. The vertices are labelled (i, j) where $i = 1, 2$ and $0 \leq j \leq \frac{n}{2} - 1$. For every j, there is an edge between $(1, j)$ and every vertex $(2, (j + 2^p - 1) \mod (n/2))$ for $p = 0, \ldots, \Delta - 1$. An example of the Knödel graph $\mathcal{W}_{3,10}$ is displayed in Figure 7.

Zheng et al. [241] considered the special case of $\mathcal{W}_{3,n}$, for even $n \geq 8$, and determined the crossing number in all cases:

Theorem 2.49 (Zheng et al., 2008 [241]) *The following holds:*

$$cr(\mathcal{W}_{3,n}) = \begin{cases} 0, & \text{for } n = 8, \\ 1, & \text{for } n = 10, \\ \lfloor \frac{n}{6} \rfloor + \frac{(n \bmod 6)}{2}, & \text{for even } n \geq 12. \end{cases}$$
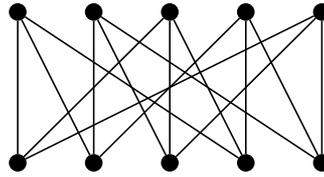
Figure 7: The Knödel graph $\mathcal{W}_{3,10}$.

2.7   Flower Snarks

Snarks are simple, connected, bridgeless cubic graphs with edge chromatic number equal to 4. Most definitions also demand that Snarks have girth at least five. The Flower Snarks are an infinite family of Snarks discovered by Isaacs [82]. The following construction is valid for any $n \geq 3$, but only odd $n \geq 5$ yield Flower Snarks. The results encompass both constructions from odd and even n and collectively we shall denote the graphs as I_n. In particular, the graph I_n on $4n$ vertices is produced by first taking n copies of $K_{1,3}$. Suppose that in copy i of $K_{1,3}$ the three vertices with degree 1 are labelled a_i, b_i and c_i. The copies are joined together via a cycle $a_1, a_2, \ldots, a_n, a_1$, and a cycle $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n, b_1$. Two examples of Flower Snarks, I_5 and I_7, are displayed in Figure 8.
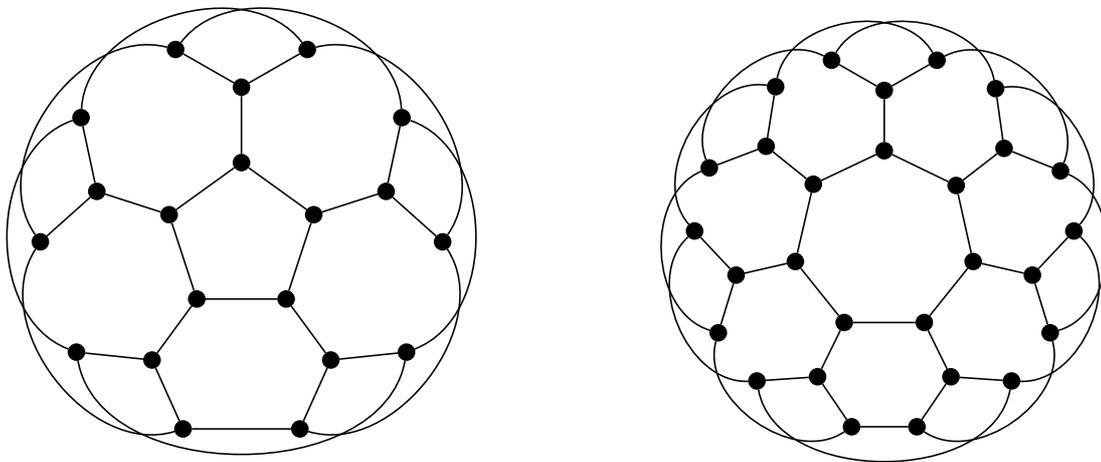
Figure 8: The Flower Snarks I_5 and I_7.

Zheng et al. [243] determined the crossing numbers of the graphs I_n (for both odd and even n):

Theorem 2.50 (Zheng et al., 2008 [243]) *For $n \geq 3$, the following holds:*

$$cr(I_n) = \begin{cases} n - 1, & \text{for } 3 \leq n \leq 5, \\ n, & \text{for } n \geq 6. \end{cases}$$

2.8 Hexagonal graph $H_{3,n}$

The Hexagonal graph $H_{3,n}$ is defined as follows for $n \geq 2$. Take the union of three cycles of length $2n$, and label their vertices a_i, b_i and c_i respectively, for $i = 1, \ldots, 2n$. Then add the edges $\{a_{2i-1}, b_{2i-1}\}$, $\{b_{2i}, c_{2i}\}$, $\{a_{2i}, c_{2i-1}\}$, for $i = 1, \ldots, n$. An example of the Hexagonal graph $H_{3,3}$ is displayed in Figure 9.
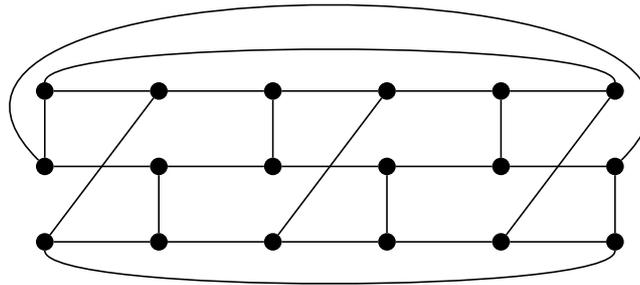
Figure 9: The Hexagonal graph $H_{3,3}$, in an optimal drawing.

In 2019, Wang et al. [206] considered $H_{3,n}$ and determined its crossing number for all $n \geq 2$:

Theorem 2.51 (Wang et al. (2019) [206]) *For $n \geq 2$, the following holds:*

$$cr(H_{3,n}) = n.$$

3 Cartesian products of graphs

The Cartesian product of two graphs G and H is written as $G \square H$. The result is a graph with vertex set $V(G) \times V(H)$, and edges between vertices (u, u') and (v, v') if and only if either $u = v$ and $(u', v') \in E(H)$, or $u' = v'$ and $(u, v) \in E(G)$. An example of the Cartesian product of two paths, $P_3 \square P_4$, is displayed in Figure 10.
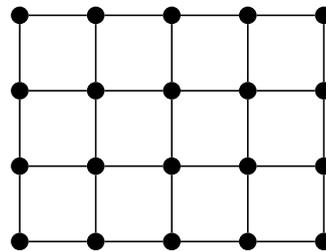
Figure 10: The Cartesian product $P_3 \square P_4$.

The Cartesian product was the first graph product for which the crossing number was investigated in detail, beginning in the 1970s and continuing to this day. Investigations typically fall into two categories; either the Cartesian product of two families

of graphs of unbounded size is considered, or the Cartesian product of specific graphs with a family of graphs of unbounded size. In the latter case, researchers all over the world have dedicated themselves to finding the crossing numbers of Cartesian products involving the various connected graphs of small orders, each often involving an ad hoc proof or technique unique to that graph.

If a graph G is disconnected, then $G \square H$ is equivalent to taking the union of the Cartesian products of each component of G with H. Hence, the crossing number of $G \square H$ is simply equal to the sum of the Cartesian products of H with each of its connected components. To that end, in what follows, all results are for connected graphs.

3.1 Cartesian products of cycles, paths and stars

A number of results have been determined for the Cartesian product of families of graphs. In particular, much attention has been paid to Cartesian products involving cycles, paths, and stars. The star graph S_n is simply the complete bipartite graph $K_{1,n}$. Since $K_{1,1} = P_1$ and $K_{1,2} = P_2$, we only consider S_n for $n \geq 3$. It is easy to check that the Cartesian product of a cycle with a path, or a path with a path, will result in a planar graph. It can also easily be checked that the Cartesian product of two stars $S_n \square S_m$ is isomorphic to a subdivision of the complete tripartite graph $K_{1,m,n}$, which is discussed in Section 2.1.2. We now consider the three remaining cases.

3.1.1 Two cycles

The first publication to consider the Cartesian product of two cycles was by Harary et al. [62] in 1973:

Conjecture 3.1 (Harary et al., 1973 [62]) *For $n \geq m \geq 3$, the following holds:*

$$cr(C_m \square C_n) = (m-2)n.$$

Conjecture 3.1 has been verified for $m \leq 7$. Most of the proofs first rely on finding the crossing number for the special case when $n = m$ and then extending it for $n > m$. The first such result was by Ringeisen and Beineke [164] who verified Conjecture 3.1 for $m = 3$ in 1978.

In 1970, Eggleton and Guy [40] proved that $cr(C_4 \square C_4) = 8$ but the paper contained a (separate) error and was never published. In 1980, Beineke and Ringeisen [13] verified Conjecture 3.1 for $m = 4$ but relied on Eggleton and Guy's result which had still not appeared in the literature. A proof for $cr(C_4 \square C_4) = 8$ was finally published in 1995 by Dean and Richter [34].

In 1995, Richter and Thomassen [163] verified Conjecture 3.1 for $C_5 \square C_5$, and the general case was subsequently verified for $m = 5$ by Kleŝĉ et al. [108] in 1996.

In 1996, Anderson et al. [8] verified Conjecture 3.1 for $C_6 \square C_6$, and the general case was subsequently verified for $m = 6$ by Richter and Salazar [161] in 2001, although the bulk of the work for doing so was first detailed in 1997 in Salazar's PhD thesis [166].

In 1996, Anderson et al. [9] verified Conjecture 3.1 for $C_7 \square C_7$, and the general case was subsequently verified for $m = 7$ by Adamsson and Richter [3] in 2004. Adamsson and Richter also claimed (providing only a sketch of the proof) that the conjecture can be shown to hold for $m = 8$ if it can first be shown that $cr(C_8 \square C_8) = 48$.

In 2004, Glebsky and Salazar [51] provided a breakthrough by showing that for each m, the conjecture must be true for all but a finite number of cases.

Theorem 3.2 (Glebsky and Salazar, 2004 [51]) *For $n \geq m(m+1)$ and $m \geq 3$, the following holds:*

$$cr(C_m \square C_n) = (m - 2)n.$$

Some lower bounds have also been developed for the Cartesian product of two cycles. In 1973, Harary et al. [62] showed that $cr(C_m \square C_n) \geq m$ and asked if this could be improved. In 1995, Shahrokhi et al. [173] showed that $cr(C_m \square C_n) \geq \frac{mn}{90}$, and that $cr(C_m \square C_n) \geq \frac{mn}{6}$ if $n = m$ or $n = m + 1$. In 1998, Shahrokhi et al. [174] improved their result further to show that $cr(C_m \square C_n) \geq \frac{mn}{9}$, and that $cr(C_m \square C_n) \geq \frac{3mn}{5}$ if $n \leq 5(m - 1)/4$. In 2000, the lower bound was improved again to $cr(C_m \square C_n) \geq \frac{(m-2)n}{3}$ by Salazar [167]. Finally, this was further considered in 2004 by Salazar and Ugalde [169], who gave what is currently the best asymptotic lower bound:

Theorem 3.3 (Salazar and Ugalde, 2004 [169]) *For every $\epsilon > 0$, there is an N_ϵ with the following property. For all $n \geq m \geq N_\epsilon$, the following holds:*

$$cr(C_m \square C_n) \geq (0.8 - \epsilon)mn.$$

3.1.2   Paths and stars

This case has now been completely settled. The first results were due to Jendrol' and Ščerbová [83] in 1982, where they determined an upper bound for the crossing number of $S_m \square P_n$, and conjectured that it held with equality. Their conjecture would ultimately prove to be correct. They also verified their conjecture for $m = 3$, as well as the special case of $m = 4, n = 2$. In 1991, Klešč [86] verified the conjecture for $m = 4$. Finally, in 2007, Bokal verified the conjecture for all cases:

Theorem 3.4 (Bokal, 2007 [17]) *For $m \geq 3$ and $n \geq 1$, the following holds:*

$$cr(S_m \square P_n) = (n - 1) \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor.$$

3.1.3 Cycles and stars

Results for $S_m \square C_n$ are only known for $\min(m, n) \leq 4$. For arbitrarily large cycles, the crossing numbers of $S_3 \square C_n$ and $S_4 \square C_n$ are due to Jendrol' and Ščerbová (1982) [83] and Klešč (1991) [86] respectively.

Theorem 3.5 (Jendrol' and Ščerbová, 1982 [83]) *The following holds:*

$$cr(S_3 \square C_n) = \begin{cases} 1, & \text{for } n = 3, \\ 2, & \text{for } n = 4, \\ 4, & \text{for } n = 5, \\ n, & \text{for } n \geq 6. \end{cases}$$

Theorem 3.6 (Klešč, 1991 [86]) *The following holds:*

$$cr(S_4 \square C_n) = \begin{cases} 2, & \text{for } n = 3, \\ 4, & \text{for } n = 4, \\ 8, & \text{for } n = 5, \\ 2n, & \text{for } n \geq 6. \end{cases}$$

For arbitrarily large stars, the crossing numbers of $S_m \square C_3$ and $S_m \square C_4$ are known. The former does not appear to have been explicitly stated anywhere, but it is trivial to exhibit drawings of $S_m \square C_3$ with $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ crossings, which is also a lower bound because it contains $S_m \square P_2$ as a subgraph. The latter case was settled by Klešč (1994) [87].

Theorem 3.7 (Klešč, 1994 [87]) *For $m \geq 1$, the following holds:*

$$cr(S_m \square C_4) = 2 \left\lfloor \frac{(m-1)^2}{2} \right\rfloor.$$

3.2 Cartesian products of paths with other graphs

3.2.1 Paths and cycles with one or two extra edges

In 2007, Yuan and Huang [222] $*$ considered the Cartesian product of a path with a graph consisting of a cycle with one or two extra edges added. Consider the cycle graph C_m on $m \geq 5$ vertices. From this, construct a new graph H, by adding a chord between two vertices of distance two. Similarly, consider a new graph B, constructed by taking C_m and adding in two chords, whose endpoints have distance at least two.

Theorem 3.8 (Yuan and Huang, 2007 [222] $*$) *For $n \geq 1$, the following hold:*

$$cr(P_n \square H) = n - 1, \ *$$

$$cr(P_n \square B) = 2n - 2. \ *$$

3.2.2   Paths and double cones

The join product $G + H$ (considered further in Section 4) is equal to the union of G and H, plus edges linking every vertex of G to every vertex of H. A generalisation of the wheel graph is the cone graph $W_{l,n}$, which is equal to the join product $C_m + D_l$, where D_l is the discrete graph on l isolated vertices with zero edges. Then, $W_{1,n}$ is simply the wheel graph. In this context, $W_{2,n}$ is referred to as the double cone. An example of the double cone $W_{2,6}$ is displayed in Figure 11. In 2011, Zheng et al. [244] considered the Cartesian product of double cones with paths:

Theorem 3.9 (Zheng et al., 2011 [244]) *For $m \geq 3$ and $n \geq 1$, the following holds:*

$$cr(W_{2,m} \square P_n) = 2n \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2n.$$
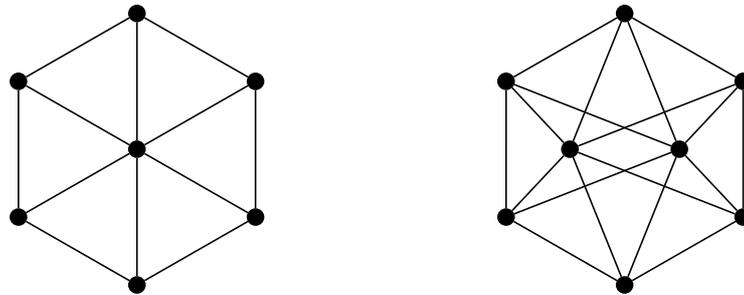
Figure 11: The wheel graph W_6 and the double cone $W_{2,6}$.

3.2.3   Paths and complete graphs

Consider the Cartesian product of the complete graph and the path graph, $K_n \square P_m$. It is clear that the graph is planar for $n = 3$, so we now consider $n > 3$.

The cases for $n = 4$ and $n = 5$ were settled by Klešč [87, 90] in 1994 and 1999 respectively:

Theorem 3.10 (Klešč, 1994 [87]) *For $m \geq 1$, $cr(K_4 \square P_m) = 2m$.*

Theorem 3.11 (Klešč, 1999 [90]) *For $m \geq 1$, $cr(K_5 \square P_m) = 6m$.*

In 2007, Zheng et al. [239] settled the case for $n = 6$, and gave an upper bound the general case which they conjectured would hold with equality:

Theorem 3.12 (Zheng et al., 2007 [239]) *For $m \geq 1$, $cr(K_6 \square P_m) = 15m + 3$.*

Theorem 3.13 (Zheng et al., 2007 [239]) *For $n \geq 4$ and $m \geq 1$,*

$$cr(K_n \square P_m) \leq \frac{1}{4} \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left(m \left\lfloor \frac{n+4}{2} \right\rfloor + \left\lfloor \frac{n-4}{2} \right\rfloor \right).$$

Conjecture 3.14 (Zheng et al., 2007 [239]) *Theorem 3.13 holds with equality.*

In 2014, Ouyang et al. [149] confirmed that Conjecture 3.14 holds for $n \leq 10$.

3.2.4   Paths and circulant graphs

Consider the path graph on $n+1$ vertices P_n, and the circulant graph $Ci_m(\{1,k\})$. The latter is sometimes referred to as $C(m,k)$. There has been some effort to find the crossing number of $P_n \square Ci_m(1,k)$, with the cases for $m \leq 6$ being covered in Sections 3.6–3.9. Here, we include the known results for larger m.

Theorem 3.15 (Yuan et al., 2008 [227] ∗) *For $n \geq 1$, $cr(P_n \square Ci_7(1,2)) = 8n$. ∗*

Theorem 3.16 (Yuan et al., 2008 [229]) *For $n \geq 1$, $cr(P_n \square Ci_8(1,2)) = 8n$.*

Theorem 3.17 (Wang and Ma, 2017 [208] ∗) *For $n \geq 1$, $cr(P_n \square Ci_8(1,4)) = 9n - 1$. ∗*

Theorem 3.18 (Yuan et al., 2013 [228]) *For $n \geq 1$, $cr(P_n \square Ci_9(1,2)) = 10n$.*

Theorem 3.19 (Yuan et al., 2009 [230]) *For $n \geq 1$, $cr(P_n \square Ci_{10}(1,2)) = 10n$.*

Yuan et al. [230] also claim, without providing a proof, that $cr(P_n \square Ci_{11}(1,2)) = cr(P_n \square Ci_{12}(1,2)) = 12n$. Later, a separate paper was published proving the latter result:

Theorem 3.20 (Yuan and Huang, 2011 [225]) *For $n \geq 1$, $cr(P_n \square Ci_{12}(1,2)) = 12n$.*

3.2.5   Paths and complete multipartite graphs

There are a few results relating to the Cartesian products of paths and complete multipartite graphs. Theorem 3.21 was also rediscovered a year later by Zhang et al. [240].

Theorem 3.21 (Tang et al., 2007 [196]) *For $m \geq 2$ and $n \geq 1$,*

$$cr(P_n \square K_{2,m}) = 2n \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Theorem 3.22 (Ouyang et al., 2014 [148]) *For $m \geq 2$ and $n \geq 1$,*

$$cr(P_n \square K_{1,1,m}) = 2n \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + (n-1) \left\lfloor \frac{m}{2} \right\rfloor.$$

3.3   Cartesian products of cycles with other graphs

3.3.1   Cycles and 2-powers of paths

For a description of 2-powers of paths, see Section 2.5. In 2012, Klešč and Kravecová [101] considered the Cartesian product of P_n^2 and cycles. They conjectured the crossing number, and proved it for the special case where the cycle has three vertices:

Conjecture 3.23 (Klešč and Kravecová, 2012 [101]) *For $n \geq 2$ and $m \geq 3$,*

$$cr(P_n^2 \square C_m) = m(n-1).$$

Theorem 3.24 (Klešč and Kravecová, 2012 [101]) *For $n \geq 2$,*

$$cr(P_n^2 \square C_3) = 3n - 3.$$

Later in 2012, Kravecová and Petrillová [113] also proved the result for C_4:

Theorem 3.25 (Kravecová and Petrillová, 2012 [113]) *For $n \geq 2$,*

$$cr(P_n^2 \square C_4) = 4n - 4.$$

3.4   Cartesian products of stars with other graphs

3.4.1   Stars and trees

In 2007, Bokal [18] showed that the crossing number of $S_n \square T$, where T is a tree, can be written in terms of the crossing numbers of complete tripartite graphs, as follows:

Theorem 3.26 (Bokal, 2007 [18]) *Consider a tree T, and define $d(v)$ to be the degree of vertex v in T. Then,*

$$cr(S_n \square T) = \sum_{v \in V(T)} cr(K_{1,d(v),n}).$$

At the time Bokal published his result, $cr(K_{1,m,n})$ was only known for $m \leq 3$. In the ensuing years, it was determined for $m = 4, 5$ (see Section 2.1.2). Hence, the following corollary can be stated.

Corollary 3.27 *Consider a tree T with maximum degree 5. Let n_i be the number of vertices of degree i contained in T, and set $a = n_2 + 2n_3 + 4n_4 + 6n_5$ and $b = n_3 + 2n_4 + 4n_5$. Then,*

$$cr(S_n \square T) = a \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + b \left\lfloor \frac{n}{2} \right\rfloor.$$
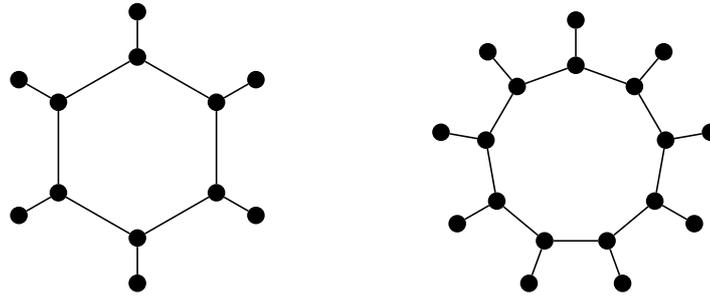
Figure 12: The sunlet graphs $\mathcal{S}_6$ and $\mathcal{S}_9$.

3.4.2   Stars and sunlet graphs

The sunlet graph $\mathcal{S}_n$ is the graph on $2n$ vertices constructed by taking the cycle C_n and adding a pendant edge to each vertex. Two examples of sunlet graphs, $\mathcal{S}_6$ and $\mathcal{S}_9$ are displayed in Figure 12.

In 2019, Haythorpe and Newcombe [64] considered the Cartesian product of a sunlet graph $\mathcal{S}_n$ and a star graph S_m. They were able to determine the crossing number for $m \leq 3$ and gave an upper bound for larger m which they conjecture coincides with the crossing number.

Theorem 3.28 (Haythorpe and Newcombe, 2019 [64]) *For $n \geq 3$ and $m \geq 1$, the following holds:*

$$cr(\mathcal{S}_n \square S_m) = \begin{cases} 0, & \text{for } m = 1, \\ n, & \text{for } m = 2, \\ 3n, & \text{for } m = 3. \end{cases}$$

Additionally, for $m \geq 4$, $cr(\mathcal{S}_n \square S_m) \leq \frac{nm(m-1)}{2}$.

Conjecture 3.29 (Haythorpe and Newcombe, 2019 [64]) *The upper bound in Theorem 3.28 holds with equality.*

3.5   Cartesian products of other graph families

3.5.1   Complete graphs and complete bipartite graphs

Zheng et al. [240] proved the following bounds related to Cartesian products involving complete graphs with cycles, and complete bipartite graphs with paths:

Theorem 3.30 (Zheng et al., 2008 [240]) *The following hold:*

1. *$cr(K_m \square C_n) \geq n \cdot cr(K_{m+2})$, for $n \geq 3$, $m \geq 5$.*

2. *$cr(K_m \square C_n) \leq \frac{n}{4} \left\lfloor \frac{m+2}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$, for $m = 5, 6, 7$, $n \geq 3$, and for $m \geq 8$ with even $n \geq 4$.*

3. $cr(K_{m,l}\square P_n) \leq (n-1)(\lfloor\frac{m+2}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{l+2}{2}\rfloor\lfloor\frac{l+1}{2}\rfloor - ml)$
$$+2(\lfloor\tfrac{m+1}{2}\rfloor\lfloor\tfrac{m}{2}\rfloor\lfloor\tfrac{l+1}{2}\rfloor\lfloor\tfrac{l}{2}\rfloor - \lfloor\tfrac{m}{2}\rfloor\lfloor\tfrac{l}{2}\rfloor), \text{ for } m,l \geq 2 \text{ and } n \geq 1.$$

They also showed that equality holds for item 2 for some small values of m:

Theorem 3.31 (Zheng et al., 2008 [240]) *For $n \geq 1, m \geq 2$,*

$$cr(K_m\square C_n) = \begin{cases} 9n & m = 5, n \geq 3, \\ 18n & m = 6, n \geq 3, \\ 36n & m = 7, n \geq 3, \\ 60n & m = 8, n \geq 4 \text{ and even}, \\ 100n & m = 9, n \geq 4 \text{ and even}, \\ 150n & m = 10, n \geq 4 \text{ and even}. \end{cases}$$

3.5.2 Wheels and trees

Consider the wheel graph W_n on $n+1$ vertices. In 2017, Klešč et al. [107] considered the Cartesian product of wheel graphs with trees of maximum degree no larger than 5:

Theorem 3.32 (Klešč et al., 2017 [107]) *Let W_n be the wheel graph on $n + 1$ vertices, and T be a tree with maximum degree $\triangle(T) \leq 5$. Let n_i be the number of vertices of degree i in T, and set $a = n_1 + n_2 + 2n_3 + 2n_4 + 3n_5$ and $b = n_2 + 2n_3 + 4n_4 + 6n_5$ and $c = n_3 + 2n_4 + 4n_5$. Then, for $n \geq 3$,*

$$cr(W_n\square T) = a + b\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + c\left\lfloor\frac{n}{2}\right\rfloor.$$

This result extended an earlier result of Cartesian products of wheels with sub-cubic trees by Bokal [18] in 2007.

Wang and Huang [209] gave a similar result, except for the situation when the tree can have any degree, but the wheel is restricted to maximum degree 5.

Theorem 3.33 (Wang and Huang, (to appear) [209]) *Let W_n be the wheel graph on $n + 1$ vertices, for $n = 3, 4, 5$, and T be a tree with maximum degree $\triangle(T)$. Let n_i be the number of vertices of degree i in T. Then,*

$$cr(W_n\square T) = \begin{cases} \sum_{i=1}^{\triangle(T)} n_i(2\lfloor\frac{i}{2}\rfloor\lfloor\frac{i-1}{2}\rfloor + i) & \text{for } n = 3, \\ \sum_{i=1}^{\triangle(T)} n_i(4\lfloor\frac{i}{2}\rfloor\lfloor\frac{i-1}{2}\rfloor + i + \lfloor\frac{i}{2}\rfloor) & \text{for } n = 4, \\ \sum_{i=1}^{\triangle(T)} n_i(6\lfloor\frac{i}{2}\rfloor\lfloor\frac{i-1}{2}\rfloor + i + 3\lfloor\frac{i}{2}\rfloor) & \text{for } n = 5. \end{cases}$$

3.6 Cartesian products with 3-vertex graphs

There are only two connected non-isomorphic graphs on 3 vertices; the path P_2 and the cycle C_3. Although no author made a point of considering Cartesian products

with 3-vertex graphs specifically, the results presented in Table 4 are all known from other works, as follows.

The Cartesian products of paths and paths, or paths and cycles, always produce planar graphs. The result for $P_2 \square S_n$ is due to Bokal (2007) [17]. The result for $C_3 \square C_n$ is due to Ringeisen and Beineke (1978) [164]. The result for $C_3 \square S_n$ was discussed in Section 3.1.3.

Table 4: Crossing numbers of Cartesian products of 3-vertex graphs with paths, cycles and stars. The results for P_n are for $n \geq 1$, and the results for C_n and S_n are for $n \geq 3$.

	G	$cr(G \square P_n)$	$cr(G \square C_n)$	$cr(G \square S_n)$
	P_2	0	0	$\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$
	C_3	0	n	$\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$

3.7   Cartesian products with 4-vertex graphs

There are six connected non-isomorphic graphs on 4 vertices. We denote the graphs on 4 vertices as G_i^4, and note that there are eleven such graphs. However, only six of these are connected, which correspond to $i = 5, 7, 8, 9, 10, 11$ as displayed in Table 5. In 1980, Beineke and Ringeisen [13] considered the crossing numbers of the Cartesian product of each of them with the cycle graph on n vertices. They were able to settle all cases except for $G_7^4 \square C_n$, which was subsequently settled in 1982 by Jendrol' and Šcerbová [83]. The latter authors also settled $G_7^4 \square P_n$. The result for $G_7^4 \square C_n$ is only valid when $n \geq 6$. For smaller values of n, we have $cr(G_7^4 \square C_3) = 1$, $cr(G_7^4 \square C_4) = 2$ and $cr(G_7^4 \square C_5) = 4$.

In 1994, Klešč [87] then derived crossing numbers of the remaining Cartesian products of each of the six connected four-vertex graphs with path graphs and star graphs. The results are summarised in Table 5.

3.8   Cartesian products with 5-vertex graphs

There are 21 connected non-isomorphic graphs on 5 vertices. We denote the connected graphs on 5 vertices as G_i^5, as defined in Table 6. For most of them, the crossing numbers of their Cartesian products with cycles, stars and paths are known. The results are summarised in Table 6, with the graphs indexed in the order originally designated by Klešč in [93]. The main contributions are discussed below and a list of publications where each result was first proved is displayed in Appendix A.2.2.

For paths, the crossing number has been determined for all 21 graphs. The majority of the results were first determined in Klešč (2001) [93]. In particular, $cr(G_i^5 \square P_n)$ was first determined in [93] for $i = 3, 4, 5, 6, 7, 9, 11, 13, 14, 17, 19, 20$. In addition, it is easy to check that $G_1^5 \square P_n$ and $G_8^5 \square P_n$ are planar.

Table 5: Crossing numbers of Cartesian products of 4-vertex graphs with paths, cycles and stars. The results for P_n are for $n \geq 1$, and the results for C_n and S_n are for $n \geq 3$.

i	G_i^4	$cr(G_i^4 \square P_n)$	$cr(G_i^4 \square C_n)$		$cr(G_i^4 \square S_n)$
5		0	0		$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$
7		$n-1$	$n \; (n \geq 6)$,	$\begin{matrix}1 & (n=3)\\ 2 & (n=4)\\ 4 & (n=5)\end{matrix}$	$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$
8		0	$2n$		$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$
9		$n-1$	n		$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$
10		$n-1$	$2n$		$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$
11		$2n$	$3n$		$2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n$

For cycles, the crossing number has been determined for 15 graphs, with the cases $i = 10, 15, 17, 18, 19, 20$ still unknown. The majority of cases were settled in Klešč (2001) [93] ($i = 4, 5, 9, 12$) and Klešč (2005) [95] ($i = 3, 6, 7, 13, 14$), although each of the latter were first stated (without proof) in [93]. For $G_9^5 \square C_n$, the result was claimed in [93] to hold only for $n \geq 6$, but was then later shown to hold for all $n \geq 3$ in Klešč (2005) [95]. In addition, Klešč (2002) [94] also determined an upper bound for $G_{10}^5 \square C_n$.

Theorem 3.34 (Klešč, 2002 [94]) $cr(G_{10}^5 \square C_3) = 9$, and for $n \geq 4$,

$$cr(G_{10}^5 \square C_n) \leq 4n.$$

For stars, the crossing number has been determined for 16 graphs, with the cases $i = 5, 8, 12, 13, 16$ still unknown. The result for $G_2^5 \square S_n$ was first determined by Huang and Zhao (2008) [81] and independently by Ho (2008) [73] by recognizing it as a subdivision of $K_{1,4,n}$.

Table 6: Crossing numbers of Cartesian products of 5-vertex graphs with paths, cycles and stars. Unless otherwise stated, the results for P_n are for $n \geq 1$, and the results for C_n and S_n are for $n \geq 3$. Empty cells imply the crossing number is not known.

i	G_i^5	$cr(G_i^5 \square P_n)$	$cr(G_i^5 \square C_n)$		$cr(G_i^5 \square S_n)$	
1		0	0		$3 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$	
2		$2n-2$	$2n \ (n \geq 6),$	$\begin{array}{ll} 2 & (n=3) \\ 4 & (n=4) \\ 8 & (n=5) \end{array}$	$n(n-1)$	
3		$n-1$	$n \ (n \geq 6),$	$\begin{array}{ll} 1 & (n=3) \\ 2 & (n=4) \\ 4 & (n=5) \end{array}$	$3 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$	
4		$n-1$	n		$3 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$	
5		$n-1$	n			
6		$2n-2$	$2n \ (n \geq 6),$	$\begin{array}{ll} 4 & (n=3) \\ 6 & (n=4) \\ 9 & (n=5) \end{array}$	$n(n-1)$	
7		$n-1$	$2n \ (n \geq 4),$	$4 \quad (n=3)$	$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$	$*$
8		0	$3n \ (n \geq 5),$	$\begin{array}{ll} 5 & (n=3) \\ 10 & (n=4) \end{array}$		
9		$2n-2$	$2n$		$n(n-1)$	
10		$2n$		$9 \quad (n=3)$	$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$	
11		$2n-2$	$3n \ (n \geq 4),$	$7 \quad (n=3)$	$n(n-1)$	
12		$2n-2$	$2n$			
13		$n-1$	$3n \ (n \geq 4),$	$7 \quad (n=3)$		
14		$2n-2$	$3n$		$n(n-1)$	
15		$3n-1$			$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor$	
16		$3n-1$	$2 \left(n + \left\lfloor \frac{n+1}{2} \right\rfloor \right)$			
17		$2n$			$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n$	
18		$3n-1$			$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor$	
19		$3n-1$			$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n + \left\lfloor \frac{n}{2} \right\rfloor$	
20		$4n$			$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4n$	$*$
21		$6n$	$9n$		$4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 5n + 1$	

3.9 Cartesian products with 6-vertex graphs

In this section, we gather the known crossing numbers for Cartesian products of various 6-vertex graphs with paths, cycles and stars. There are, up to isomorphism, 112 connected graphs on 6 vertices. In order to refer to these graphs in a consistent manner, we give each 6-vertex graph a label, in the order originally designated by Harary [58]. We denote the graphs on 6 vertices as G_i^6, as defined for all indices, in Appendix A.1 and also repeated for the graphs of interest in the Tables of this section. The list in Appendix A.1 includes disconnected graphs, which brings the total to 156 graphs, but the disconnected graphs are only used in Section 4.

3.9.1 Paths

In 2013, Klešč and Petrillová [105] gave a summary of known results, including the crossing numbers of Cartesian products of path graphs with forty different graphs on 6 vertices. The majority of those results were first determined in [105], including $cr(G_i^6 \square P_n)$ for $i = 26, 27, 28, 29, 41, 43, 44, 45, 46, 47, 48, 53, 54, 59, 60, 61, 64,$ 66, 72, 73, 74, 77, 79, 80, 83, 85, 86, 94, 104, 111, 121. For G_{121}^6, the result relied on a previous paper which has not undergone peer review, and so we mark that result with an asterisk. These, along with the remaining settled cases are displayed in Table 7. A list of publications where each result was first proved is displayed in Appendix A.2.3. In total, the crossing number of $G_i^6 \square P_n$ has been settled for 48 of the 6-vertex graphs to date, with an additional 8 results claimed in papers which have not undergone adequate peer review. The latter are marked with asterisks.

3.9.2 Cycles

To date, results are only known for graphs with few edges, with the only exception being the result with K_6. To date, the crossing number has been determined for 15 cases and 10 of these are due to Draženská and Klešč (2011) [39]. These results are listed in Table 8 and a list of publications where each result was first proved is displayed in Appendix A.2.4.

3.9.3 Stars

The crossing numbers for the Cartesian product of many 6-vertex graphs and stars were provided in 2013 by Klešč and Schrötter [111] who made an attempt to gather the results known to them at the time. In that paper, they determined the crossing number of $G_i^6 \square S_n$, for seventeen graphs, however some of them had been previously determined. Only the cases $i = 27, 31, 43, 47, 48, 53, 59, 72, 73, 77, 79, 80, 104$ were newly settled in [111], although the case of $i = 27$ could also be seen as a corollary of Bokal (2007) [18] and Huang and Zhao (2008) [81]. In total, the crossing number of $G_i^6 \square S_n$ has been settled for 20 graphs to date and these are displayed in Table 9. A further 8 cases have had results claimed in papers which have not undergone adequate peer review; these are marked by asterisks. A list of publications where each result was first proved is displayed in Appendix A.2.5.

Table 7: Known crossing numbers of Cartesian products of 6-vertex graphs with paths. All results are for $n \geq 1$.

i	G_i^6	$cr(G_i^6 \square P_n)$	i	G_i^6	$cr(G_i^6 \square P_n)$	i	G_i^6	$cr(G_i^6 \square P_n)$
25		0	60		$n-1$	89		$3n-3$
26		$n-1$	61		$2n$	91		$3n-1$
27		$2n-2$	64		$2n-2$	93		$4n$ $\quad *$
28		$n-1$	65		$3n-3$	94		$2n-2$
29		$2n-2$	66		$2n-2$	103		$6n-2$
31		$4n-4$	68		$3n-1$	104		$4n-4$
40		0	70		$3n-3$	109		$4n$ $\quad *$
41		$n-1$	71		$3n-1$	111		$3n-1$
42		$2n-4$	72		$4n-4$	113		$4n-4$
43		$n-1$	73		$4n-4$	119		$7n-1$ $*$
44		$2n-2$	74		$2n-2$	120		$3n-3$ $*$
45		$2n-2$	75		$2n$	121		$4n$ $\quad *$
46		$n-1$	77		$2n-2$	125		$5n-3$
47		$2n-2$	79		$4n-4$	146		$5n-1$ $*$
48		$4n-4$	80		$4n-4$	152		$6n$ $\quad *$
51		$3n-3$	83		$2n-2$	154		$9n-1$
53		$2n-2$	85		$2n$	155		$12n$ $\quad *$
54		$2n-2$	86		$3n-1$	156		$15n+3$
59		$2n-2$	87		$3n-1$			

Table 8: Crossing numbers of Cartesian products of 6-vertex graphs with cycles.

i	G_i^6	$cr(G_i^6 \square C_n)$		i	G_i^6	$cr(G_i^6 \square C_n)$	
25		0		49		$2n$ $(n \geq 4)$,	4 $(n=3)$
40		$4n$ $(n \geq 6)$,	$\begin{array}{ll} 6 & (n=3) \\ 12 & (n=4) \\ 18 & (n=5) \end{array}$	53		$2n$ $(n \geq 6)$,	$\begin{array}{ll} 4 & (n=3) \\ 6 & (n=4) \\ 9 & (n=5) \end{array}$
41		$3n$ $(n \geq 5)$,	$\begin{array}{ll} 5 & (n=3) \\ 10 & (n=4) \end{array}$	54		$2n$ $(n \geq 6)$,	$\begin{array}{ll} 4 & (n=3) \\ 6 & (n=4) \\ 9 & (n=5) \end{array}$
42		$2n$ $(n \geq 4)$,	4 $(n=3)$	67		$3n$ $(n \geq 4)$,	7 $(n=3)$
43		n $(n \geq 3)$		78		$3n$ $(n \geq 6)$,	$\begin{array}{ll} 7 & (n=3) \\ 10 & (n=4) \\ 14 & (n=5) \end{array}$
44		$2n$ $(n \geq 4)$,	4 $(n=3)$	113		$4n$ $(n \geq 3)$	
46		n $(n \geq 3)$		156		$18n$ $(n \geq 3)$	
47		$2n$ $(n \geq 6)$,	$\begin{array}{ll} 4 & (n=3) \\ 6 & (n=4) \\ 9 & (n=5) \end{array}$				

3.10 Cartesian products with graphs on 7 or more vertices

3.10.1 7-vertex graphs with paths, cycles and stars

In 2005, He and Huang [70] * considered six graphs on seven vertices and determined the crossing number of their Cartesian products with paths. The results are summarised in Table 10, with those graphs labelled G_1^7 to G_6^7. The Cartesian product of two additional graphs with paths were considered by Liu et al. (2012) [129] * and Ding et al. (2018) [36]; those graphs are labelled G_7^7 and G_8^7 respectively. There are also a number of known results for Cartesian products of named 7-vertex graphs with paths, cycles and stars. These are provided in Table 11 along with a list of the publications where the result was proved.

Additionally, the crossing number of the Cartesian product of a star with any 7-vertex tree except S_6 is known due to Bokal (2007) [18] (see Section 3.4).

3.10.2 8-vertex graphs with paths, cycles and stars

The results in Table 12 are due to Yuan and Huang (2007) [222] *, with the exception of G_5^8 which is due to Ding et al. (2018) [36]. There are also a number of known results for Cartesian products of named 8-vertex graphs with paths, cycles and stars. These are provided in Table 13 along with a list of the publications where the result was proved.

Table 9: Crossing numbers of Cartesian products of 6-vertex graphs with stars.

i	G_i^6	$cr(G_i^6 \square S_n)$	i	G_i^6	$cr(G_i^6 \square S_n)$
25		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$	77		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$
26		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor$	79		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$
27		$5\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$	80		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$
28		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor$	85		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n$ $*$
29		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$	93		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4n$ $*$
31		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$	94		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$
43		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor$	104		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$
47		$5\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$	111		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor+2n\,*$
48		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$	120		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+3\left\lfloor\frac{n}{2}\right\rfloor$ $*$
53		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$	124		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n+3\left\lfloor\frac{n}{2}\right\rfloor*$
59		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor$	125		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+3\left\lfloor\frac{n}{2}\right\rfloor+2n$
61		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2n\,*$	130		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4n$ $*$
72		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$	137		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4n$ $*$
73		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+4\left\lfloor\frac{n}{2}\right\rfloor$	152		$6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+6n$

Table 10: Crossing numbers of Cartesian products of 7-vertex graphs with paths. All results are for $n \geq 1$.

i	G_i^7	$cr(G_i^7 \square P_n)$	i	G_i^7	$cr(G_i^7 \square P_n)$	i	G_i^7	$cr(G_i^7 \square P_n)$
1		$3n-3$ $*$	4		$n-1$ $*$	7		$5n-1$ $*$
2		$4n-4$ $*$	5		$2n$ $*$	8		$4n-4$
3		$2n-2$ $*$	6		$2n-2$ $*$			

Table 11: Crossing numbers of Cartesian products of named 7-vertex graphs with paths, cycles and stars.

Graph family	Crossing number	Publication
$S_6 \Box P_n$	$6n - 6$	Bokal (2007) [17]
$Ci_7(\{1,2\}) \Box P_n$	$8n$ *	Yuan et al. (2008) [227] *
$W_6 \Box P_n$	$7n - 5$	Bokal (2007) [18]
$K_{2,5} \Box P_n$	$8n$	Tang et al. (2007) [196]
$K_{1,1,5} \Box P_n$	$10n - 2$	Ouyang et al. (2014) [148]
$K_7 \Box P_n$	$30n + 6$	Ouyang et al. (2014) [149]
$P_6 \Box S_n$	$5 \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$	Bokal (2007) [17]
$C_7 \Box C_n$	$5n$	Adamsson and Richter (2004) [3]
$K_7 \Box C_n$	$36n$	Zheng et al. (2008) [240]

Table 12: Crossing numbers of Cartesian products of 8-vertex graphs with paths. All results are for $n \geq 1$.

i	G_i^8	$cr(G_i^8 \Box P_n)$	i	G_i^8	$cr(G_i^8 \Box P_n)$	i	G_i^8	$cr(G_i^8 \Box P_n)$
1		$n - 1$ *	3		$2n - 2$ *	5		$4n - 4$
2		$2n - 2$ *	4		$2n - 2$ *			

Table 13: Crossing numbers of Cartesian products of named 8-vertex graphs with paths, cycles and stars.

Graph family	Crossing number	Publication
$S_7 \square P_n$	$9n - 9$	Bokal (2007) [17]
$W_7 \square P_n$	$10n - 8$	Bokal (2007) [18]
$Ci_8(\{1, 2\}) \square P_n$	$8n$	Yuan et al. (2008) [229]
$Ci_8(\{1, 4\}) \square P_n$	$9n - 1$ *	Wang and Ma (2017) [208] *
$GP(4, 1) \square P_n$	$8n$ *	Yuan and Huang (2011) [226] *
$K_{2,6} \square P_n$	$12n$	Tang et al. (2007) [196]
$K_{1,1,6} \square P_n$	$15n - 3$	Ouyang et al. (2014) [148]
$K_8 \square P_n$	$54n + 18$	Ouyang et al. (2014) [149]
$P_7 \square S_n$	$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$	Bokal (2007) [17]

Additionally, the crossing number of the Cartesian product of a star with any 8-vertex tree with maximum degree 5 is known due to Bokal (2007) [18] (see Section 3.4).

3.11   Variants of toroidal grid graphs

Toroidal grid graphs are another name for the Cartesian product of two cycles. In 2002, Foley et al. [47] considered the crossing numbers of two variants of toroidal grid graphs; specifically, the twisted toroidal grid graph $\mathcal{T}(m, n)$ and the crossed toroidal grid graphs $\mathcal{X}(m, n)$. The twisted toroidal graph $\mathcal{T}(3, n)$ is equivalent to $C_3 \square C_n$, except vertices 1, 2, 3 in the final cycle link to vertices 3, 1, 2 in the first cycle, respectively. The crossed toroidal graph $\mathcal{X}(3, n)$ is equivalent to $C_3 \square C_n$, except vertices 1, 2, 3 in the final cycle link to vertices 1, 3, 2 in the first cycle, respectively. Examples of $\mathcal{T}(3, 5)$ and $\mathcal{X}(3, 5)$ are displayed in Figure 13.

Theorem 3.35 (Foley et al., 2002 [47]) *For $n \geq 3$, the following holds:*

$$cr(\mathcal{T}(3, n)) = cr(\mathcal{X}(3, n)) = n.$$

4   Join products of graphs

The join product of two graphs G and H, denoted $G + H$ and sometimes simply called the join of G and H, is equal to the union of G and H, plus edges linking every vertex of G to every vertex of H. An example of the join product of two paths,
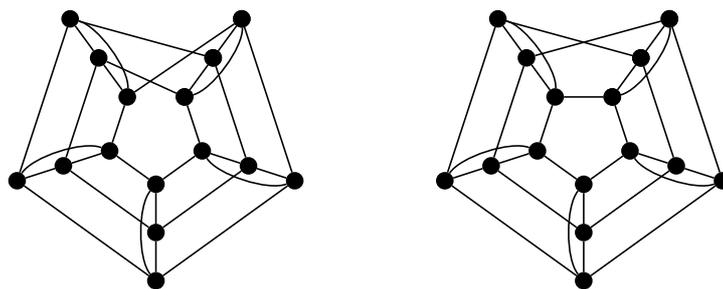
Figure 13: The twisted toroidal grid graph $\mathcal{T}(3,5)$ and the crossed toroidal grid graph $\mathcal{X}(3,5)$, respectively.

$P_3 + P_4$, is displayed in Figure 14 in two drawings, with the latter drawing being crossing-optimal.
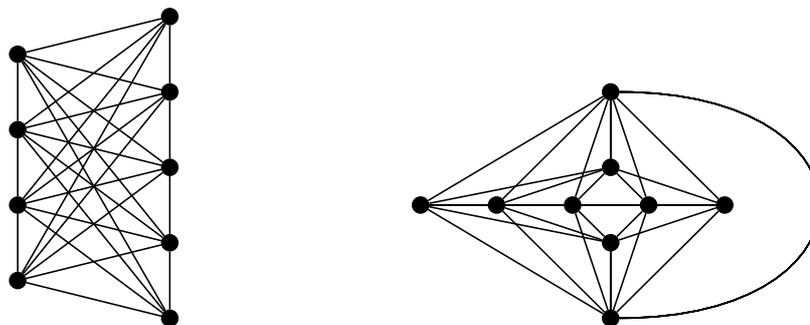

Figure 14: Two drawings of the join product $P_3 + P_4$, with the second being optimal.

In effect, the join product connects G and H by the edges of the complete bipartite graph $K_{|G|,|H|}$. Hence, the crossing number of a join product is bounded below by the crossing number of the corresponding complete bipartite graph. Accordingly, the crossing number of the complete bipartite graph has appeared in all known results for join products to date. Within this section we will adopt the common practice of defining $Z(m,n)$ to be the conjectured crossing number of $K_{m,n}$ (see Section 2.1.1).

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Recall from Section 2.1.1 that $Z(m,n)$ is only known to coincide with $cr(K_{m,n})$ if $\min\{m,n\} \leq 6$, or for the special cases when $7 \leq \min\{m,n\} \leq 8$ and $7 \leq \max\{m,n\} \leq 10$. The crossing numbers of families of graphs resulting from join products have, to date, only been computed exactly when at least one graph involved has at most six vertices. Hence, this section will be divided into subsections corresponding to the size of the fixed graph in the join product. We highlight a few important points regarding this section before continuing.

Unlike for Cartesian products, the join product always results in a connected graph, even if one or both of the graphs involved is disconnected. Hence, we will also consider disconnected graphs in this section. In addition to cycles and paths which were considered extensively in Section 3, it is also common to consider the crossing number of join products of graphs with the discrete graph. The discrete graph D_n is the graph with n isolated vertices and no edges. It is also often denoted nK_1. To date it has been rare to consider the crossing number of join products involving arbitrarily large stars.

A number of interesting graphs can be viewed as being the result of join products. Most notably, complete multipartite graphs can be viewed as resulting from join products, in the following way. Consider the complete k-partite graph $K_{a_1,a_2,...,a_k}$. Then $K_{a_1,a_2,...,a_k} + D_n$ is isomorphic to the complete $(k+1)$-partite graph $K_{a_1,a_2,...,a_k,n}$. Hence, a number of results for join products can be taken from the various publications on crossing numbers of complete multipartite graphs, and vice versa. We include such results in what follows, even if they were not originally presented as join product results.

Finally, when referring to join products, it is common in the literature to use the notation P_n to refer to the path graph on n vertices; this is contrary to the more standard usage of P_n to refer to the path graph on $n+1$ vertices. The reason for this is that the order of each input graph is an important variable in join products, as the crossing number will inevitably contain $Z(m,n)$ for input graphs of size m and n. Despite this common practice in the literature, for the sake of consistency we will maintain our notation from the rest of this survey, and use P_{n-1} to refer to the path graph on n vertices.

4.1   Join products with 3-vertex graphs

There are four graphs, up to isomorphism, on three vertices; see Table 14. The crossing numbers of the join products with discrete graphs, paths and cycles has been found for each of them. Each of the results for join products with P_{n-1} and C_n was found by Klešč (2007) [96]. Since every join product with a three vertex graph contains $K_{3,n}$ and hence a lower bound is $Z(3,n)$, and D_n is a subgraph of P_{n-1}, the results for join products with D_n can be seen as immediate corollaries.

4.2   Join products with 4-vertex graphs

There are eleven graphs, up to isomorphism, on four vertices; see Table 15. The crossing numbers of the join products with discrete graphs, paths and cycles has been found for each of them.

Each of the results for join products with P_{n-1} and C_n were first shown by Klešč (2007) [96]. The results for join products with D_n were first shown by Klešč and Schrötter (2011) [109], with three exceptions. First, $G_1^4 + D_n = K_{4,n}$, for which the crossing number was first determined by Guy (1969) [54]. Second, $G_7^4 + D_n = K_{1,3,n}$, for which the crossing number was first determined by Asano (1986) [10]. Finally,

Table 14: Crossing numbers of joins of 3-vertex graphs with discrete graphs, paths and cycles. The results for P_{n-1} are for $n \geq 2$ and the results for C_n are for $n \geq 3$.

i	G_i^3	$cr(G_i^3 + D_n)$	$cr(G_i^3 + P_{n-1})$	$cr(G_i^3 + C_n)$
1		$Z(3,n)$	$Z(3,n)$	$Z(3,n)$
2		$Z(3,n)$	$Z(3,n)$	$Z(3,n)$
3		$Z(3,n)$	$Z(3,n)$	$Z(3,n)+1$
4		$Z(3,n)$	$Z(3,n)+1$	$Z(3,n)+2$

Table 15: Crossing numbers of joins of 4-vertex graphs with discrete graphs, paths and cycles. The results for D_n are for $n \geq 1$, the results for P_{n-1} are for $n \geq 2$, and the results for C_n are for $n \geq 3$.

i	G_i^4	$cr(G_i^4 + D_n)$	$cr(G_i^4 + P_{n-1})$	$cr(G_i^4 + C_n)$
1		$Z(4,n)$	$Z(4,n)$	$Z(4,n)$
2		$Z(4,n)$	$Z(4,n)$	$Z(4,n)$
3		$Z(4,n)$	$Z(4,n)$	$Z(4,n)$
4		$Z(4,n)$	$Z(4,n)$	$Z(4,n)+1$
5		$Z(4,n)$	$Z(4,n)$	$Z(4,n)+1$
6		$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor+2$
7		$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor+2$
8		$Z(4,n)$	$Z(4,n)+1$	$Z(4,n)+2$
9		$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor+2$
10		$Z(4,n)+\lfloor\frac{n}{2}\rfloor$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor+1$	$Z(4,n)+\lfloor\frac{n}{2}\rfloor+3$
11		$Z(4,n)+n$	$Z(4,n)+n+1$	$Z(4,n)+n+4$

$G_{11}^4 + D_n = K_{1,1,1,1,n}$, for which the crossing number was first determined by Ho (2009) [75].

4.3 Join products with 5-vertex graphs

We first consider the 21 connected graphs on five vertices; see Table 16. We use the same graph identifiers as first given by Klešč [93] to describe the 5-vertex graphs. For many of the graphs, the crossing numbers of their join products with discrete graphs, paths and cycles have been determined. A list of publications where each result was first proved is displayed in Appendix A.2.6, except for the cases $G_1^5 + D_n$ and $G_8^5 + D_n$ which are immediately corollaries of Kleitman [85]. It is worth noting that $n(n-1)$, a common expression in Table 16, is equal to $Z(5,n) + 2\left\lfloor \frac{n}{2} \right\rfloor$.

In addition, in 2018, Su [188] gave a conjecture about $cr(G_{21}^5 + C_n)$:

Conjecture 4.1 (Su, 2018 [188]) *For $n \geq 3$,*

$$cr(G_{21}^5 + C_n) = Z(5,n) + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 7.$$

4.3.1 Disconnected 5-vertex graphs

In 2014, Li [121] ∗ considered the disconnected graph constructed by taking the union of C_4 and one isolated vertex:

Theorem 4.2 (Li, 2014 [121] ∗) *Let G be $C_4 \cup K_1$, then the following hold:*

$$cr(G + D_n) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor, \ for \ n \geq 1, \ast$$
$$cr(G + P_{n-1}) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor + 1, \ for \ n \geq 2, \ast$$
$$cr(G + C_n) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor + 2, \ for \ n \geq 3. \ast$$

The result for $cr(G + D_n)$ was independently confirmed in 2018 by Ding and Huang [35].

In 2019, Staš [179] considered the disconnected graph constructed by taking the union of G_9^4 with an isolated vertex.

Theorem 4.3 (Staš, 2019 [179]) *Let G be $G_9^4 \cup K_1$, then for $n \geq 1$, the following holds:*

$$cr(G + D_n) = Z(5,n) + \left\lfloor \frac{n}{2} \right\rfloor.$$

4.4 Join products with 6-vertex graphs

So far, join products involving 6-vertex graphs have only been considered in-depth for connected 6-vertex graphs, and are only known for some cases. Specifically, for the graphs displayed in Table 17. A list of publications where each result was first proved is displayed in Appendix A.2.7, except for the cases $G_{25}^6 + D_n$ and $G_{40}^6 + D_n$ which are immediately corollaries of Kleitman [85]. As in Section 3.9, we use the graph indices from Appendix A.1 to denote each graph for which a result has been determined.

Table 16: Crossing numbers of joins of connected 5-vertex graphs with discrete graphs, paths and cycles. The results for D_n are for $n \geq 1$, the results for P_{n-1} are for $n \geq 2$, and the results for C_n are for $n \geq 3$. Empty cells imply that the crossing number has not yet been determined.

i	G_i^5	$cr(G_i^5 + D_n)$	$cr(G_i^5 + P_{n-1})$	$cr(G_i^5 + C_n)$
1		$Z(5,n)$	$Z(5,n)$	$Z(5,n)+1$
2		$n(n-1)$	$n(n-1)$ *	$n(n-1)+2$ *
3		$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+1$
4		$Z(5,n)+\lfloor\frac{n}{2}\rfloor$ *	$Z(5,n)+\lfloor\frac{n}{2}\rfloor$ *	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+1$ *
5		$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+2$
6			$n(n-1)$ *	$n(n-1)+2$ *
7		$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+1$ *	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+2$
8		$Z(5,n)$	$Z(5,n)+1$	$Z(5,n)+2$
9			$n(n-1)$ *	$n(n-1)+2$ *
10		$Z(5,n)+n$	$Z(5,n)+n+1$	$Z(5,n)+n+3$
11		$n(n-1)$	$n(n-1)+1$ *	$n(n-1)+3$ *
12		$n(n-1)$ *	$n(n-1)$	
13		$Z(5,n)+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+1$	$Z(5,n)+\lfloor\frac{n}{2}\rfloor+2$
14		$n(n-1)$	$n(n-1)+1$ *	$n(n-1)+3$ *
15		$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor$ *	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+2$	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+4$
16		$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor$ *	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+1$*	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+3$*
17			$Z(5,n)+n+1$	
18		$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor$ *	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+2$*	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+4$
19		$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor$	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+1$*	$Z(5,n)+n+\lfloor\frac{n}{2}\rfloor+4$
20		$Z(5,n)+2n$	$Z(5,n)+2n+2$	
21		$Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+1$	$Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+4$	

Table 17: Crossing numbers of joins of particular 6-vertex graphs with discrete graphs, paths and cycles. The results for D_n are for $n \geq 1$, the results for P_{n-1} are for $n \geq 2$, and the results for C_n are for $n \geq 3$. Empty cells imply that the crossing number has not yet been determined.

i	G_i^6	$cr(G_i^6 + D_n)$	$cr(G_i^6 + P_{n-1})$	$cr(G_i^6 + C_n)$
25		$Z(6,n)$	$Z(6,n)$	$Z(6,n)+1$
31		$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor+3$　$*$
40		$Z(6,n)$	$Z(6,n)+1$	$Z(6,n)+2$
44		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$　$*$	
45		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$		
48			$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	
49		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+2$
59		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$　$*$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+1$	
60		$Z(6,n)+\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+\left\lfloor\frac{n}{2}\right\rfloor+1$	$Z(6,n)+\left\lfloor\frac{n}{2}\right\rfloor+2$
61		$Z(6,n)+n$	$Z(6,n)+n+1$	$Z(6,n)+n+3$
66		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$		
72			$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	
73			$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	
74		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$		
79			$Z(6,n)+4\left\lfloor\frac{n}{2}\right\rfloor$　$*$	
83		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+1$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+2$
85		$Z(6,n)+n$　$*$		
93		$Z(6,n)+2n$		
94		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+1$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+3$
103		$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+2n$　$*$	$Z(6,n)+2\left\lfloor\frac{n}{2}\right\rfloor+2n+2$　$*$	
109		$Z(6,n)+2n$	$Z(6,n)+2n+1$	$Z(6,n)+2n+3$
111		$Z(6,n)+n+\left\lfloor\frac{n}{2}\right\rfloor$　$*$	$Z(6,n)+n+\left\lfloor\frac{n}{2}\right\rfloor+1$　$*$	$Z(6,n)+n+\left\lfloor\frac{n}{2}\right\rfloor+3$　$*$
120		$Z(6,n)+3\left\lfloor\frac{n}{2}\right\rfloor$　$*$	$Z(6,n)+3\left\lfloor\frac{n}{2}\right\rfloor+2$	$Z(6,n)+3\left\lfloor\frac{n}{2}\right\rfloor+4$
124		$Z(6,n)+n+3\left\lfloor\frac{n}{2}\right\rfloor$　$*$		
125		$Z(6,n)+n+3\left\lfloor\frac{n}{2}\right\rfloor$	$Z(6,n)+n+3\left\lfloor\frac{n}{2}\right\rfloor+1$　$*$	

Table 17 (continued):

130		$Z(6,n) + 2n *$		
137		$Z(6,n) + 2n *$		
152		$Z(6,n) + 3n$		

4.5 Other join products

4.5.1 Triangular snakes

The triangular snake graph TS_n is the graph with n vertices, for odd n, defined as follows. Start by taking the path graph P_{n-1} and add the edges $\{2i-1, 2i+1\}$ for $i = 1, \ldots, \frac{n-1}{2}$. An example of TS_{11} is displayed in Figure 15.
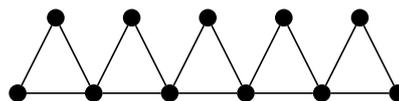

Figure 15: The triangular snake TS_{11}.

Rajan et al. [160] considered the join product of TS_n with discrete graphs, paths, and cycles with at most six vertices:

Theorem 4.4 (Rajan et al., 2012 [160]) *For odd $n \geq 5$ and $m \leq 6$, the following hold:*

$$cr(TS_n + D_m) = Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor,$$
$$cr(TS_n + P_{m-1}) = Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor,$$
$$cr(TS_n + C_m) = Z(n,m) + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + 2.$$

4.5.2 Cycles and wheels

In 2014, Yue et al. [231] considered the join product of the wheel graph W_m with the cycle graph C_n, and gave a conjecture as to its crossing number. They proved the conjecture was correct for $m \leq 4$, although $m = 3$ was already known from Klešč (2007) [96].

Conjecture 4.5 (Yue et al., 2014 [231]) *For $m, n \geq 3$, the following holds:*

$$cr(W_m + C_n) = Z(m+1, n) + \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 2,$$

with the conjecture known to hold for $m = 3$ and $m = 4$.

5   Other kinds of graph products

5.1   Strong products

The strong product of two graphs G and H, denoted $G \boxtimes H$, is the graph with the vertex set $V(G) \times V(H)$ and edge set $\{\big((u,v),(x,y)\big) : u = x$ and $(v,y) \in E(H)$, or $v = y$ and $(u,x) \in E(G)$, or $(u,x) \in E(G)$ and $(v,y) \in E(H)\}$. An example of $P_3 \boxtimes P_4$ is displayed in Figure 16.

5.1.1   Two paths

In 2013, Klešč et al. [106] considered the strong product of two path graphs, $P_n \boxtimes P_m$ for $n, m \geq 3$. They first determined the crossing number of $P_n \boxtimes P_3$, and then proposed a conjecture about the crossing number of $P_n \boxtimes P_m$, which was subsequently proved by Ma [134] to be correct in all cases except $P_4 \boxtimes P_4$.
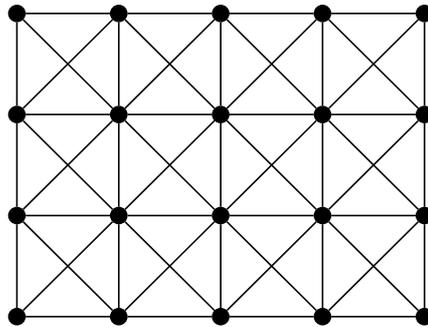

Figure 16: The strong product $P_3 \boxtimes P_4$.

Theorem 5.1 (Klešč et al., 2013 [106]) *For $n \geq 3$, the following holds:*
$$cr(P_n \boxtimes P_3) = n - 3.$$

Theorem 5.2 (Ma, 2017 [134]) *The following holds:*
$$cr(P_n \boxtimes P_m) = \begin{cases} 4, & \text{for } n = m = 4, \\ (m-1)(n-1) - 4, & \text{for } n > m \geq 4. \end{cases}$$

5.1.2   Paths and cycles

In 2018, Ouyang et al. [151] considered the strong product of a path and a cycle:

Theorem 5.3 (Ouyang et al., 2018 [151]) *For $m \geq 1$ and $n \geq 3$, the following holds:*
$$cr(P_m \boxtimes C_n) \leq (m-1)n,$$
with equality for $m = 1$ and $m = 2$.

Conjecture 5.4 (Ouyang et al., 2018 [151]) *Theorem 5.3 holds with equality.*

6 Hypercubes, meshes, and other recursive constructions

The graph families in this section have the common property that their number of vertices grow exponentially in terms of their parameters. Many, but not all, of these graph families arose due to their relation to network topologies in VLSI design. Leighton's pioneering work in this area continues to motivate the study of these graphs today [115, 116].

6.1 Hypercubes and related constructions

The n-dimensional hypercube, Q_n, contains 2^n vertices and $n \cdot 2^{n-1}$ edges. Each vertex is labelled by a different n-digit binary number and an edge exists between two vertices when their labels differ in exactly one digit. An example of the 4-cube, Q_4, is displayed in Figure 17 in two different drawings.
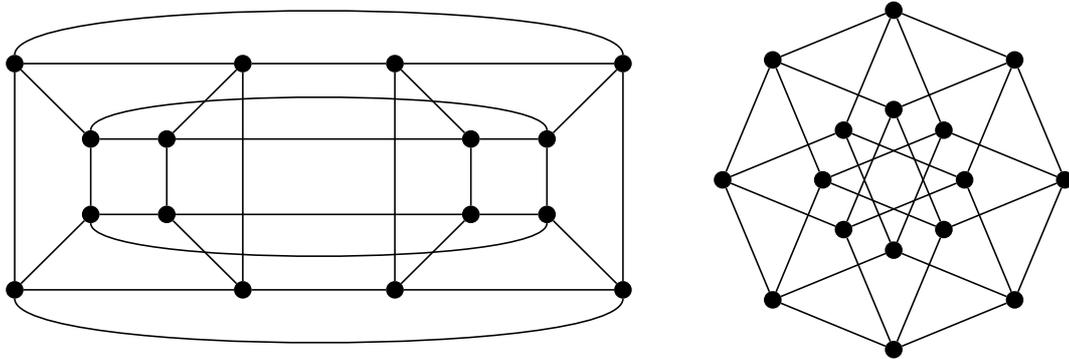

Figure 17: The hypercube Q_4 in two different drawings.

6.1.1 n-cube

The crossing number of the n-dimensional hypercube, often called the n-cube, was first considered by Eggleton and Guy [40] in 1970, who claimed to have discovered an upper bound. By 1973, it was known that their proof contained an error, as was discussed by Erdős and Guy [41]. Nonetheless, Erdős and Guy conjectured that not only would the upper bound discovered in [40] be proved to be correct, but that the crossing number would meet this bound exactly. In the following years, some weaker bounds were proved.

Theorem 6.1 (Madej, 1991 [137]) *For $n \geq 1$, the following holds:*

$$cr(Q_n) \leq \frac{1}{6}4^n - n^2 2^{n-3} - 3 \cdot 2^{n-4} + \frac{1}{48}(-2)^n.$$

Theorem 6.2 (Faria and De Figueiredo, 2000 [43]) *For $n \geq 1$, the following holds:*

$$cr(Q_n) \leq \frac{165}{1024}4^n - \frac{2n^2 - 11n + 34}{2}2^{n-2}.$$

Finally, in 2008 the original upper bound from Eggleton and Guy was proved to be correct by Faria et al. [44] who established that:

Theorem 6.3 (Faria et al., 2008 [44]) *For $n \geq 1$, the following holds:*

$$cr(Q_n) \leq \frac{5}{32} 4^n - \left\lfloor \frac{n^2 + 1}{2} \right\rfloor 2^{n-2}.$$

It has since been claimed by Yang et al [219] that Faria et al. had an error in their proof as well. Indeed, Yang et al. [219] give the following updated upper bounds for the n-cube, which implies that the previous upper bound is not tight. At the current time, their paper has not undergone peer review and exists only on ArXiv. We are not in a position to check the following result, however we have verified that their drawing of the 7-cube, provided in their paper, is valid and does improve on the upper bound given in Theorem 6.3 for $n = 7$. This refutes the conjecture by Erdős and Guy that equality would hold. In particular, their drawing contains 1744 crossings, compared to the 1760 crossings suggested by Theorem 6.3. We have also independently discovered our own drawings of the 7-cube with fewer than 1760 crossings.

Theorem 6.4 (Yang et al., 2017 [219]) *The following holds:*

$$cr(Q_n) \leq \begin{cases} \frac{139}{896} 4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2} + \left(\frac{4}{7}\right) 2^{3\left\lfloor \frac{n}{2} \right\rfloor - n}, & \text{for } 5 \leq n \leq 10, \\ \frac{26695}{172032} 4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2} - \left(\frac{n^2+2}{3}\right) 2^{n-2} + \left(\frac{4}{7}\right) 2^{3\left\lfloor \frac{n}{2} \right\rfloor - n}, & \text{for } n \geq 11. \end{cases}$$

There have also been some lower bounds proved for the n-cube. Most notably, in 1993, Sýkora and Vrt'o [195] showed that:

Theorem 6.5 (Sýkora and Vrt'o, 1993 [195]) *For $n \geq 1$, the following holds:*

$$cr(Q_n) > \frac{4^n}{20} - (n+1)2^{n-2}.$$

The lower bound is trivial for $n \leq 4$. Finally, Q_1, Q_2 and Q_3 are planar, and Q_4 is isomorphic to $C_4 \square C_4$, so it is known from Dean and Richter [34] that $cr(Q_4) = 8$. The crossing number of Q_n for $n \geq 5$ has still not been determined.

6.1.2   Locally twisted cubes

A variation of the hypercube is the n-dimensional locally twisted cube LTQ_n, proposed by Yang et al. [215]. It is defined as follows: Let LTQ_2 be identical to Q_2. For $n \geq 3$, LTQ_n is built from two disjoint copies of LTQ_{n-1}. In the first copy, augment the labeling of each node by adding 0 to the front, and in the second copy, augment the label of each node by adding 1 to the front. For each node $0x_2x_3 \ldots x_n$ in the
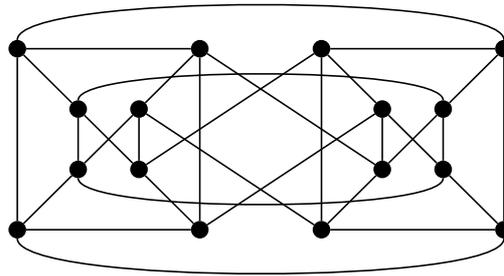
Figure 18: The locally twisted cube LTQ_4.

first copy, add an edge to node $1(x_2 + x_n)x_3 \ldots x_n$ in the second copy, where the addition is modulo 2. The result is an n-regular graph on 2^n vertices. An example of the locally twisted cube LTQ_4 is displayed in Figure 18.

Wang et al. [200] determined the following bounds for $cr(LTQ_n)$:

Theorem 6.6 (Wang et al., 2017 [200]) *For $n \geq 6$, the following bounds hold:*

$$\frac{4^n}{20} - (n^2 + 1)2^{n-1} < cr(LTQ_n) \leq \frac{265}{6}4^{n-4} - \left(n^2 + \frac{15 + (-1)^{n-1}}{6} \right) 2^{n-3}.$$

6.1.3 Folded hypercube

The folded hypercube FQ_n is obtained by taking Q_n and adding in edges between all pairs of vertices with complementary labels (that is, their labels differ in all digits). The resulting graph is $(n + 1)$-regular. An example of the folded hypercube FQ_4 is displayed in Figure 19.
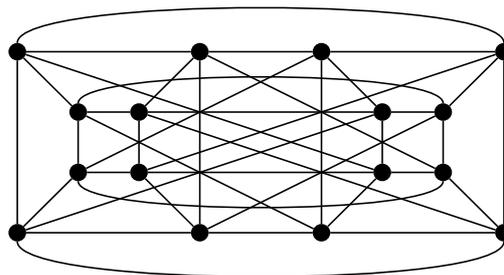

Figure 19: The folded hypercube FQ_4.

Wang et al. [201] provided lower and upper bounds for the crossing numbers of these graphs:

Theorem 6.7 (Wang et al., 2015 [201]) *For $n \geq 1$, the following bounds hold:*

$$\frac{1}{20}4^n(1 - (\pi/2(2\lceil n/2 \rceil + 1))^{-1/2})^{-2} < cr(FQ_n) \leq \frac{11}{32}4^n - (n^2 + 3n)2^{n-3}.$$

6.1.4   Augmented cube

The augmented cube AQ_n is defined as follows. Let AQ_1 be isomorphic to K_2, with vertices labelled 0 and 1. AQ_n is generated by taking two copies of AQ_{n-1}, and prepending 0 to the labels of the first copy and 1 to the labels of the second copy. That is, their new labels in the first copy all begin with 0, and in the second copy all begin with 1. Two vertices are joined by a new edge if and only if their labels differ in only the first position (e.g. identical vertices from each copy), or if the labels differ in all positions. The augmented cube was first proposed by Choudum and Sunitha (2002) [24], who also provided a non-recursive definition, as follows. AQ_n is the graph containing vertices labelled with n-digit binary numbers, and any two vertices are connected by an edge if and only if there exists an l, $1 \le l \le n$, such that either (1) the two labels are different in position l, and identical in all other positions, or (2) the two labels are identical in the first $l - 1$ positions and then different in all subsequent positions. The resulting graph is $(2n - 1)$-regular. An example of the augmented cube AQ_3 is displayed in Figure 20.
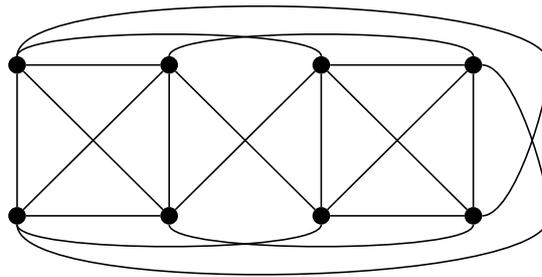
Figure 20: The augmented cube AQ_3.

In 2013, Wang et al. [199] investigated the crossing number of the augmented cube and discovered lower and upper bounds:

Theorem 6.8 (Wang et al., 2013 [199]) *The following hold:* $cr(AQ_3) = 4$, $cr(AQ_4) \le 46$, $cr(AQ_5) \le 328$, $cr(AQ_6) \le 1848$, $cr(AQ_7) \le 9112$, *and for* $n \ge 8$,

$$\frac{4^n}{5(1 + 2^{2-n})^2} - (4n^2 + 4n + \frac{17}{5})2^{n-1} < cr(AQ_n) < \frac{13}{16}4^n - (2n^2 + \frac{7}{2}n - 6)2^{n-2}.$$

The lower bound is valid for all n, but is only meaningful for $n \ge 11$.

6.1.5   Cube connected cycle graphs

The cube connected cycle graph CCC_n is obtained from Q_n by replacing vertices of Q_n with cycles of length n, as described in Sýkora and Vrt'o (1993) [195]. An example of the cube connected cycle graph CCC_4 is displayed in Figure 21.

In [195] Sýkora and Vrt'o establish lower and upper bounds for $cr(CCC_n)$:
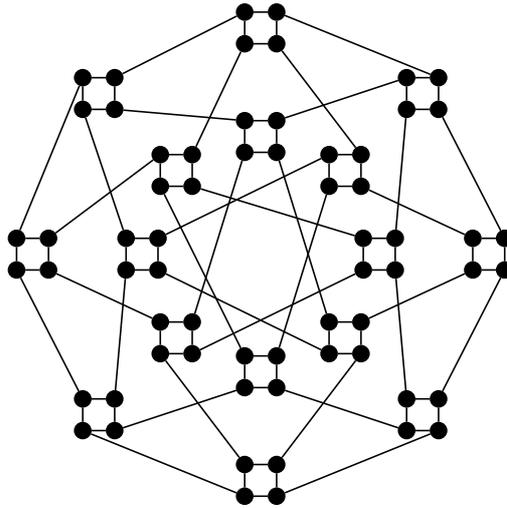
Figure 21: The cube connected cycle CCC_4.

Theorem 6.9 (Sýkora and Vrt'o, 1993 [195]) *For $n \geq 1$, the following bounds hold:*

$$\frac{4^n}{20} - 3(n+1)2^{n-2} < cr(CCC_n) < \frac{4^n}{6} + 3n^2 2^{n-3}.$$

6.2 Complete mesh of trees

The complete (2-dimensional) mesh of trees M_n is defined for any n which is a power of 2, as follows: start with a square $n \times n$ grid. Identify the n vertices in each row and each column with the n leaves of a full, complete, balanced binary tree. What results is a graph on $3n^2 - 2n$ vertices and $4n^2 - 4n$ edges. For a detailed description, see [27]. It is easy to check that M_2 is just the cycle on eight vertices, and is hence planar. An example of M_4 is displayed in Figure 22.

Figure 22: The complete mesh of trees M_4.

The earliest result on the crossing number of M_n is due to Leighton [116], who in

1984 derived a lower bound. The bound is only interesting in an asymptotic sense since it holds trivially for $n \leq 2^{121}$.

Theorem 6.10 (Leighton, 1984 [116]) *For $n \geq 2$ and a power of 2, the following holds:*

$$cr(M_n) \geq \frac{n^2 \log_2(n) - 121n^2 + 121n}{40}.$$

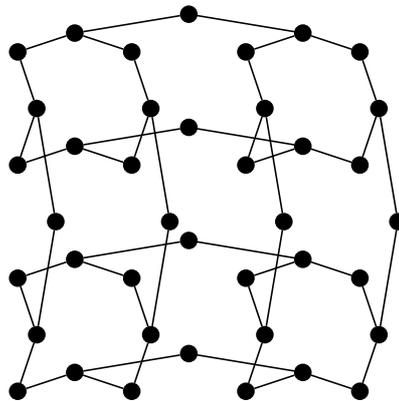In 1996, Cimikowski [26] claimed to have determined an upper bound for the crossing number of M_n. He later pointed out there had been an error in his proof, and in 1998 presented an amended upper bound which he conjectured would hold with equality, as well as a practical lower bound, which coincides with the upper bound for the case $n = 4$:

Theorem 6.11 (Cimikowski, 1998 [27]) *For $n \geq 4$ and a power of 2, the following bounds hold:*

$$\frac{n^2}{4} \leq cr(M_n) \leq n^2 \left(\frac{3 \log_2(n) - 5}{8}\right) + \frac{n}{2}.$$

Conjecture 6.12 (Cimikowski, 1998 [27]) *The upper bound in Theorem 6.11 holds with equality.*

In 2003, Cimikowski and Vrt'o [28] published a paper which included the results from [27], and also gave an alternative lower bound, which is superior to their previous lower bound for $n \geq 512$:

Theorem 6.13 (Cimikowski and Vrt'o, 2003 [28]) *For $n \geq 2$ and a power of 2, the following holds:*

$$cr(M_n) \geq \frac{5n^2 \log_2(n) - 44n^2}{80}.$$

The crossing number for M_n is only known for $n = 1, 2, 4$. Specifically, $cr(M_1) = cr(M_2) = 0$, and $cr(M_4) = 4$.

6.3   Butterfly graphs

The Butterfly graph $BF(r)$ is the graph with $(r + 1)2^r$ vertices and $r \cdot 2^{r+1}$ edges, defined as follows: The vertices are labelled $\langle w, i \rangle$ where $i = 0, \ldots, r$ and w is an r-bit binary number. Two vertices $\langle w, i \rangle$ and $\langle w', i' \rangle$ are adjacent if and only if $i' = i + 1$ and either $w = w'$, or w and w' differ in precisely the i-th bit. An example of $BF(3)$ is displayed in Figure 23.

Cimikowski [26] considered the Butterfly graphs in 1996 and proved the following upper bound for their crossing numbers:
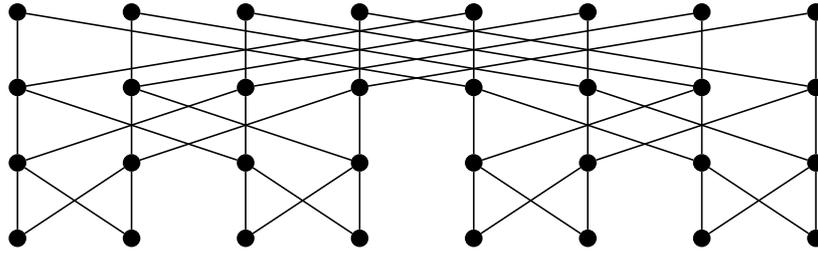
Figure 23: The Butterfly graph $BF(3)$.

Theorem 6.14 (Cimikowski, 1996 [26]) *For $r \geq 1$, the following holds:*

$$cr(BF(r)) \leq \frac{3}{2}4^r - 3 \cdot 2^r - r \cdot 2^r + 1.$$

In 2013, Manuel et al. [138] improved this upper bound, and also provided a lower bound:

Theorem 6.15 (Manuel et al., 2013 [138]) *For $r \geq 3$, the following bounds hold:*

$$\frac{1}{59}4^r - r \cdot 2^r + 2^{r-1} \leq cr(BF(r)) \leq \frac{1}{4}4^r - r \cdot 2^{r-1},$$

and $cr(BF(3)) = 4$.

6.3.1 Wrapped butterfly graphs

The Wrapped Butterfly graph, denoted $WBF(r)$, is derived from the butterfly network $BF(r)$ by merging the first and last rows into a single row; that is, merging vertex $\langle w, 0 \rangle$ with vertex $\langle w, r \rangle$ for all w [26]. Then $WBF(r)$ has $r \cdot 2^r$ vertices and $r \cdot 2^{r+1}$ edges. An example of the Wrapped Butterfly graph $WBF(3)$ is displayed in Figure 24.

Cimikowski [26] gave an upper bound for the crossing number of $WBF(r)$:

Theorem 6.16 (Cimikowski, 1996 [26]) *For $r \geq 1$, the following holds:*

$$cr(WBF(r)) \leq \frac{3}{2}4^r - 3 \cdot 2^r - r \cdot 2^r.$$

6.3.2 Benes networks

The Benes network $B(r)$ is formed by taking two copies of the Butterfly network BF_r. For each w, vertex $\langle w, r \rangle$ from the first copy is merged with $\langle w, r \rangle$ from the second copy [15]. As an example, $B(3)$ is displayed in Figure 25.

In 1996, Cimikowski investigated the Benes network, and determined an upper bound for its crossing number:

Theorem 6.17 (Cimikowski, 1996 [26]) *For $r \geq 1$, the following holds:*

$$cr(B(r)) \leq 3 \cdot 4^r - 5 \cdot 2^r - 2r \cdot 2^r + 2.$$
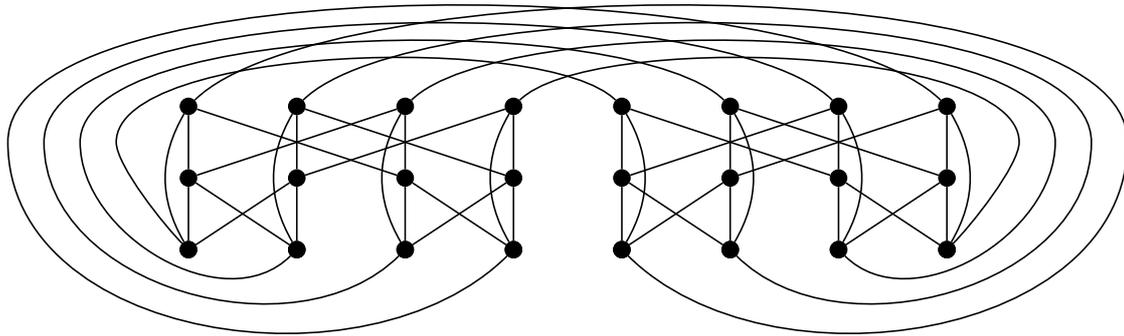
Figure 24: The Wrapped Butterfly graph $WBF(3)$.
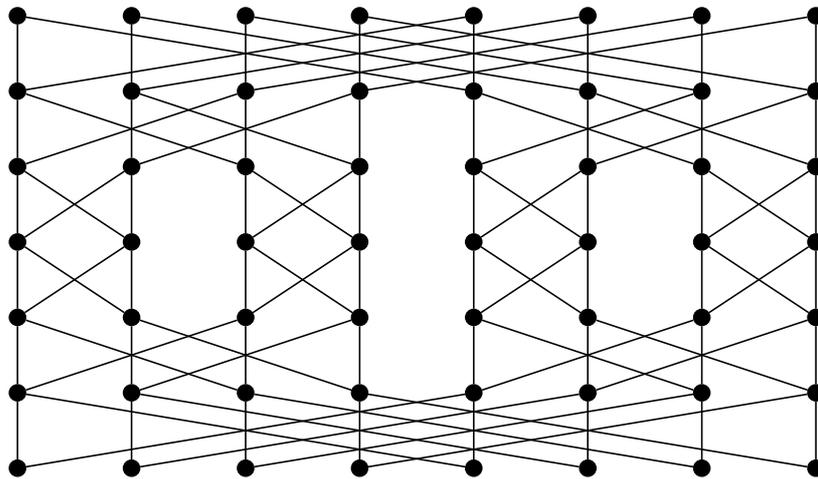
Figure 25: The Benes network $B(3)$.

6.4 Generalized fat trees

Generalized fat trees, $GFT(h, m, w)$ were introduced in 1995 by Ohring et al. [146] as a new network topology. They are a three parameter family of graphs with $\sum_{i=0}^{h} m^i w^{h-i}$ vertices and $\sum_{i=0}^{h-1} m^{h-i} w^{i+1}$ edges. The definition of generalized fat trees is complicated, and so we refer the interested reader to [146] for the full definition.

In 2011, Rajan et al. [159] considered the crossing number of generalized fat trees, and determined an upper bound for the special case $GFT(h, 3, 3)$, as well as a conjecture for an upper bound for the general case:

Theorem 6.18 (Rajan et al., 2011 [159]) *For $h \geq 1$, the following holds:*

$$cr(GFT(h, 3, 3)) \leq 3^h + \frac{5}{4}3^{2h} - \frac{1}{4}3^{h+2} - \frac{h}{2}3^{h+1}.$$
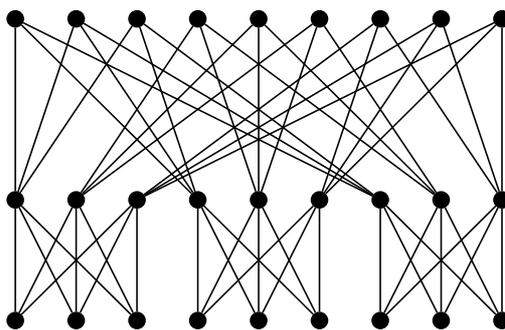
Figure 26: The generalized fat tree $GFT(2, 3, 3)$.

Conjecture 6.19 (Rajan et al., 2011 [159]) *For $h \geq 2$ and $m, w \geq 1$, the following hold:*

$$
cr(GFT(h, m, w)) \leq \begin{cases} \frac{w^h m(m-1)}{4} \left(w^{h+2} \frac{1 - \left(\frac{m}{w^2}\right)^h}{w^2 - m} - h \right), & \text{if } m = w, \\[3mm] \frac{w^h m(m-1)}{4} \left(w^h h - w \frac{1 - \left(\frac{m}{w}\right)^h}{w - m} \right), & \text{if } m = w^2, \\[3mm] \frac{w^{h+1} m(m-1)}{4} \left(w^{h+1} \frac{1 - \left(\frac{m}{w^2}\right)^h}{w^2 - m} - \frac{1 - \left(\frac{m}{w}\right)^h}{w - m} \right), & \text{otherwise.} \end{cases}
$$

6.5 Sierpiński and Sierpiński-like graphs

In 2005, Klavžar and Mohar [84] considered Sierpiński graphs, and two of their regularizations. The Sierpiński graph $S(n, k)$ for $n, k \geq 1$ is defined as follows: The vertex set is $\{1, \ldots, k\}^n$, and two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent if and only if there exists an integer $h \in \{1, \ldots, n\}$, such that

(i) $u_t = v_t$ for $t = 1, \ldots, h - 1$;

(ii) $u_h \neq v_h$; and

(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

In this construction, any vertex of the form $(i, i, \ldots, i)$ is called an extreme vertex. From $S(n, k)$, two regularizations can be defined. First, $S^+(n, k)$ is obtained from $S(n, k)$ by adding a single vertex w which is connected to all extreme vertices in $S(n, k)$. Second, $S^{++}(n, k)$ is obtained by taking the union of $k + 1$ copies of $S(n - 1, k)$, and then connecting the extreme vertices as a K_{k+1}. Both regularizations produce a regular graph of degree k. Examples of $S(2, 4)$, $S^+(2, 4)$ and $S^{++}(2, 4)$ are displayed in Figure 27. A full definition is given in Klavžar and Mohar [84].

For $S(n, k)$, Klavžar and Mohar only determined upper and lower bounds for the case $k = 4$, but for $S^+(n, k)$ and $S^{++}(n, k)$, they determined the crossing numbers in terms of the crossing numbers of complete graphs K_{k+1}. Hence, the latter are known precisely only for $k \leq 11$. For $k \leq 3$, all three constructions are planar.
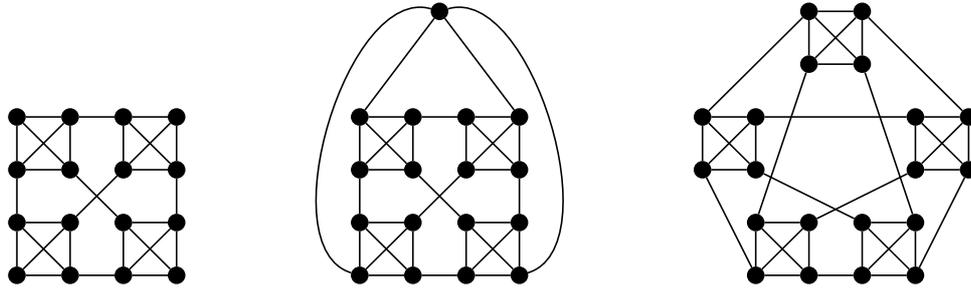
Figure 27: The Sierpiński graph $S(2,4)$, left, along with the extended Sierpiński graphs $S^+(2,4)$, centre, and $S^{++}(2,4)$, right.

Theorem 6.20 (Klavžar and Mohar, 2005 [84]) *For $k \geq 2$, the following hold:*

$$\frac{3}{16}4^n \leq cr(S(n,4)) \leq \frac{1}{3}4^n - \frac{12n-8}{3}, \ for \ n \geq 3,$$

$$cr(S^+(n,k)) = \frac{k^n - 1}{k-1}cr(K_{k+1}), \ for \ n \geq 1,$$

$$cr(S^{++}(n,k)) = \frac{(k+1)k^{n-1} - 2}{k-1}cr(K_{k+1}), \ for \ n \geq 1.$$

6.6   Star maps and pancake graphs

In 1989, Akers and Krishnamurthy [6] proposed two recursive graph families as special cases of Cayley graphs, namely the Star maps (also known as Star graphs) and the pancake graphs. To avoid confusion with stars $S_n = K_{1,n}$, we will use the term Star map and a calligraphy $\mathcal{S}^*$.

The Star map $\mathcal{S}_n^*$ contains $n!$ vertices labelled with the $n!$ permutations on the set of symbols $1, 2, \ldots, n$. An edge $\{i, j\}$ exists if and only if the label for j can be obtained from the label for i by exchanging the first symbol with any other symbol. For example, in $\mathcal{S}_3^*$, the vertex labelled 123 would be adjacent to 213 and 321.

In 2014, Lü et al. [130] considered $\mathcal{S}_4^*$, the smallest non-planar Star map, and determined its crossing number:

Theorem 6.21 (Lü et al., 2014 [130]) *For the Star map $\mathcal{S}_4^*$,*

$$cr(\mathcal{S}_4^*) = 8.$$

The pancake graph $\mathcal{P}_n$ contains $n!$ vertices labelled with the $n!$ permutations on the set of symbols $1, 2, \ldots, n$. An edge $\{i, j\}$ exists if and only if the label for j is a permutation of the label for i such that $i = i_1 i_2 i_3 \cdots i_k i_{k+1} \cdots i_n$ and $j = i_k \cdots i_2 i_1 i_{k+1} \cdots i_n$ for some k where $2 \leq k \leq n$.

In 2017, Yang et al. [218] considered $\mathcal{P}_4$, the smallest non-planar pancake graph, and determined its crossing number:
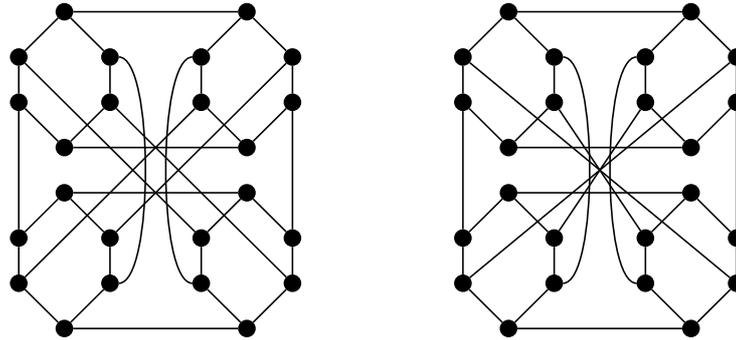
Figure 28: The Star map $\mathcal{S}_4$ and the pancake graph $\mathcal{P}_4$.

Theorem 6.22 (Yang et al., 2017 [218]) *For the pancake graph $\mathcal{P}_4$,*

$$cr(\mathcal{P}_4) = 6.$$

Results for larger Star maps or larger pancake graphs are currently unknown. $\mathcal{S}_4$ and $\mathcal{P}_4$ are displayed in Figure 28.

Acknowledgements

We are greatly indebted to the several anonymous referees whose suggestions significantly improved the flow and quality of this survey.

A   Appendices

A.1   List of graphs on six vertices

There are 156 graphs up to isomorphism on six vertices, including 112 connected graphs and 44 disconnected graphs. We list them here in the order proposed in Frank Harary's classic textbook, Graph Theory [58]. The graphs are ordered by their number of edges.
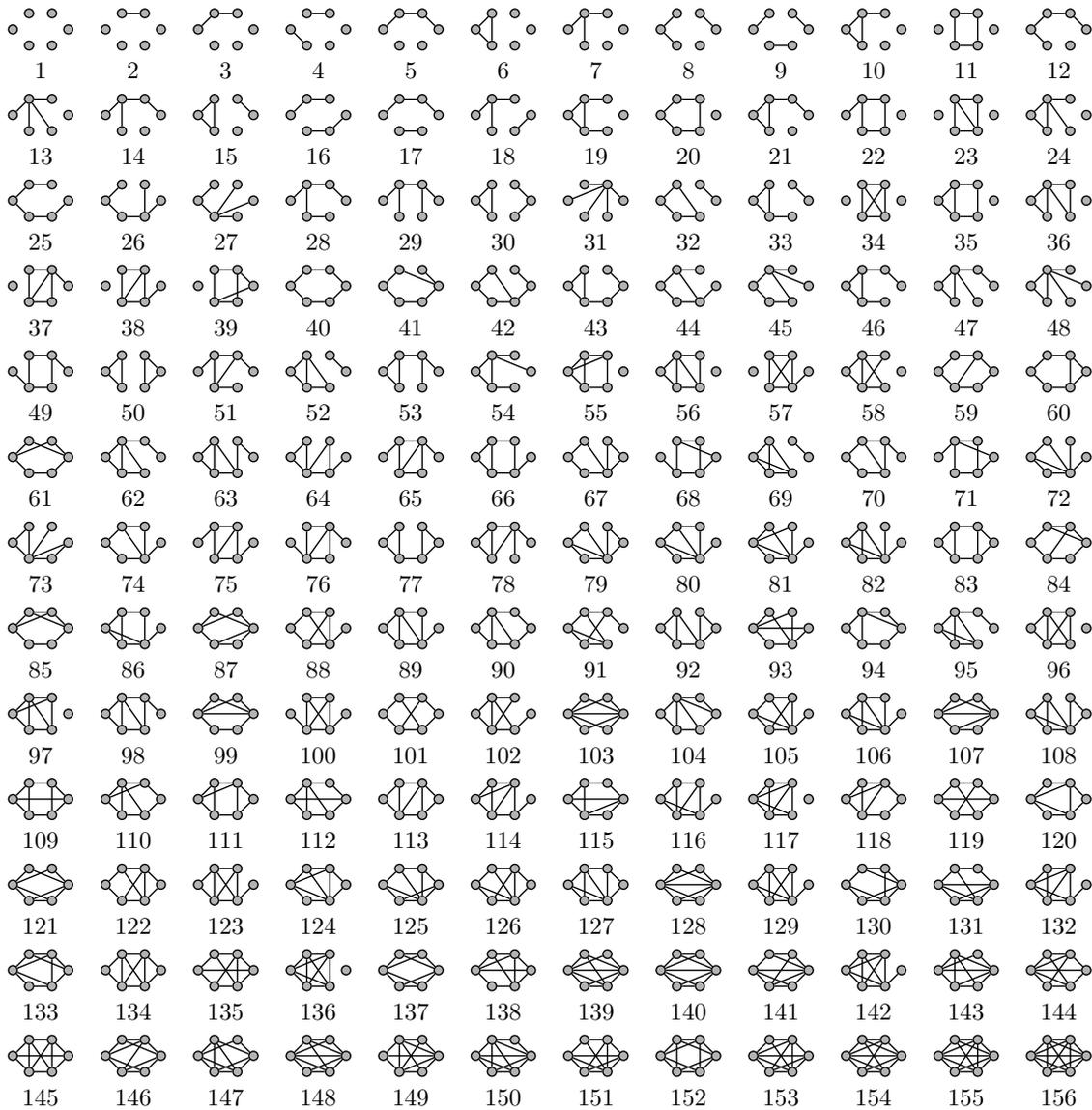
Figure 29: All 156 graphs on six vertices and their indices, ordered by number of edges.

A.2   Tables of citations

A.2.1   Generalized Petersen graphs

Table 18: Publications where the crossing numbers for $GP(n,k)$ were first proved.

n \ k	5	6	7	8	9	10	11	12	13	14	15	16	17
1	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]
2	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]	[42]
3		[56]	[162]	[45]	[45]	[162]	[45]	[45]	[162]	[45]	[45]	[162]	[45]
4				[56]	[135]	[170]	[128]	[45]	[128]	[128]	[128]	[45]	
5						[56]	[135]	[128]	[128]	[128]	[45]	[128]	
6								[56]	[135]	[128]	[128]	[128]	[45]
7										[56]	[135]	[128]	
8												[56]	[42]

A.2.2   Cartesian products with 5-vertex graphs

Table 19: Publications where the crossing numbers of Cartesian products of 5-vertex graphs with paths, cycles and stars were first proved. Empty cells imply the crossing number is not known and a dash indicates that the graphs are planar.

i	G_i^5	$cr(G_i^5 \square P_n)$	$cr(G_i^5 \square C_n)$	$cr(G_i^5 \square S_n)$
1		–	–	[17]
2		[86]	[86]	[81], [73]
3		[93]	[95]	[18]
4		[93]	[93]	[97]
5		[93]	[93]	
6		[93]	[95]	[97]
7		[93]	[95]	[234] ∗
8		–	[108]	
9		[93]	[95]	[97]
10		[89]	[94]	[89]
11		[93]	[99]	[92]
12		[88]	[93]	
13		[93]	[95]	
14		[93]	[95]	[92]
15		[88]		[97]
16		[91]	[91]	
17		[93]		[93]
18		[88]		[97]
19		[93]		[209]
20		[93]		[133] ∗
21		[90]	[240]	[131]

A.2.3 Cartesian products of paths with 6-vertex graphs

Table 20: Publications where the crossing numbers of Cartesian products of 6-vertex graphs with paths were first proved.

i	G_i^6	$cr(G_i^6 \square P_n)$	i	G_i^6	$cr(G_i^6 \square P_n)$	i	G_i^6	$cr(G_i^6 \square P_n)$
25		-	60		[105]	89		[103]
26		[105]	61		[105]	91		[155]
27		[105]	64		[105]	93		[204] ∗
28		[105]	65		[103]	94		[105]
29		[105]	66		[105]	103		[148]
31		[17]	68		[155]	104		[105]
40		-	70		[103]	109		[213] ∗
41		[105]	71		[155]	111		[105]
42		[104]	72		[105]	113		[114]
43		[105]	73		[105]	119		[245] ∗
44		[105]	74		[105]	120		[189] ∗
45		[105]	75		[104]	121		[105] ∗
46		[105]	77		[105]	125		[18]
47		[105]	79		[105]	146		[223] ∗
48		[105]	80		[105]	152		[66] ∗
51		[103]	83		[105]	154		[224]
53		[105]	85		[105]	155		[216] ∗
54		[105]	86		[105]	156		[239]
59		[105]	87		[155]			

A.2.4   Cartesian products of cycles with 6-vertex graphs

Table 21: Publications where the crossing numbers of Cartesian products of 6-vertex graphs with cycles were first proved.

i	G_i^6	$cr(G_i^6 \square C_n)$	i	G_i^6	$cr(G_i^6 \square C_n)$
25		-	49		[39]
40		[161]	53		[39]
41		[39]	54		[39]
42		[39]	67		[39]
43		[39]	78		[37]
44		[39]	113		[100]
46		[39]	156		[240]
47		[39]			

A.2.5   Cartesian products of stars with 6-vertex graphs

Table 22: Publications where the crossing numbers of Crossing numbers of Cartesian products of 6-vertex graphs with stars were first proved.

i	G_i^6	$cr(G_i^6 \square S_n)$	i	G_i^6	$cr(G_i^6 \square S_n)$
25		[17]	77		[111]
26		[18]	79		[111]
27		[111]	80		[111]
28		[18]	85		[236] *
29		[18]	93		[132] *
31		[111]	94		[207]
43		[111]	104		[111]
47		[111]	111		[182] *
48		[111]	120		[119] *
53		[111]	124		[183] *
59		[111]	125		[209]
61		[214] *	130		[191] *
72		[111]	137		[235] *
73		[111]	152		[38]

A.2.6 Join products with 5-vertex graphs

Table 23: Publications where the crossing numbers of joins of connected 5-vertex graphs with discrete graphs, paths and cycles were first proved. Empty cells imply that the crossing number has not yet been determined.

i	G_i^5	$cr(G_i^5 + D_n)$	$cr(G_i^5 + P_{n-1})$	$cr(G_i^5 + C_n)$
1		[85]	[96]	[96]
2		[81], [73]	[117] *	[237] *
3		[180]	[180]	[180]
4		[122] *	[122] *	[122] *
5		[16]	[181]	[181]
6			[117] *	[232] *
7		[35]	[238] *	[110]
8		[85]	[96]	[96]
9			[117] *	[232] *
10		[10]	[190]	[221]
11		[92]	[124] *	[124] *
12		[157] *	[194]	
13		[97]	[110]	[110]
14		[92]	[124] *	[124] *, [232] *
15		[158] *	[194]	[187]
16		[125] *	[125] *	[125] *
17			[190]	
18		[123] *	[123] *	[187]
19		[35]	[192] *	[231]
20		[75]	[185]	
21		[131]	[188]	

A.2.7   Join products with 6-vertex graphs

Table 24: Publications where the crossing numbers of joins of particular 6-vertex graphs with discrete graphs, paths and cycles were first proved.

i	G_i^6	$cr(G_i^6 + D_n)$	$cr(G_i^6 + P_{n-1})$	$cr(G_i^6 + C_n)$
25		[85]	[96]	[96]
31		[141] *	[118] *	[193] *
40		[85]	[96]	[96]
44		[177]	[118] *	
45		[178]		
48			[118] *	
49		[246]	[246]	[246]
59		[120] *	[249]	
60		[102]	[102]	[102]
61		[112]	[112]	[112]
66		[177]		
72			[118] *	
73			[118] *	
74		[177]		
79			[118] *	
83		[98]	[98]	[98]
85		[236] *		
93		[76]		
94		[207]	[248]	[248]
103		[175] *	[176] *	
109		[150]	[150]	[150]
111		[182] *	[186] *	[186] *
120		[119] *	[184]	[184]
124		[183] *		
125		[209]	[192] *	
130		[191] *		
137		[235] *		
152		[74]		

A.3   Glossary of symbols

Table 25 contains an alphabetized list of symbols which are used in this survey. Care has been taken to not use any symbols for multiple purposes.

Table 25: List of symbols used in this survey, with descriptions.

Symbol	Description	Symbol	Description
AQ_n	Augmented cube	$\mathcal{P}_n$	Pancake graph
$B(r)$	Benes network	Q_n	n-dimensional hypercube
$BF(r)$	Butterfly graph	S_n	Star on $n+1$ vertices
C_n	Cycle on n vertices	$\mathcal{S}_n$	Sunlet graph
$\overline{C}_n$	Complement of C_n	$\mathcal{S}_n^*$	Star map
CCC_n	Cube connected cycle graph	$S(n,k)$	Sierpiński graph
$Ci_n(L)$	Circulant graph	$S^+(n,k)$	Extended Sierpiński graph
D_n	Discrete graph on n vertices	$S^{++}(n,k)$	Extended Sierpiński graph
FQ_n	Folded hypercube	T	Tree
G_i^n	Graph i on n vertices	$\mathcal{T}(m,n)$	Twisted toroidal graph
$GFT(h,m,w)$	Generalized fat tree	TS_n	Triangular snake graph
$GP(n,k)$	Generalized Petersen graph	W_n	Wheel on $n+1$ vertices
$H_{3,n}$	Hexagonal graph	$W_{2,n}$	Double cone on $n+2$ vertices
I_n	Flower Snark	$WBF(r)$	Wrapped Butterfly graph
K_n	Complete graph	$\mathcal{W}_{\Delta,n}$	Knödel graph
$K_{m,n}$	Complete bipartite graph	$\mathcal{X}(m,n)$	Crossed toroidal graph
$K_{a_1,a_2,\ldots,a_k}$	Complete k-partite graph	$Z(m,n)$	Conjectured value of $cr(K_{m,n})$
LTQ_n	Locally twisted cube	$G \square H$	Cartesian product of G and H
M_n	Complete mesh of trees	$G + H$	Join product of G and H
P_n	Path on $n+1$ vertices	$G \boxtimes H$	Strong product of G and H
P_n^k	k-power of P_n		

A.4 Incorrect results

Throughout the history of research into crossing numbers, it is unfortunately common for papers to contain errors. Indeed, the inaugural result relating to crossing numbers, Zarankiewicz's proof for the crossing number of complete bipartite graphs, was later found to have an error and the result is still not resolved to this day. In this section we attempt to detail any publications which contain incorrect proofs or results. In some cases, the flaws were found and either simply noted, or in some cases corrected. In other cases, the flaws have not been widely recognised, in which case we provide a counterexample demonstrating the incorrect result. The intention is not to disparage the authors, but rather to ensure that researchers do not use the flawed results as a basis for future proofs, and to provide references containing the corrected results when such results exist.

The incorrect results are listed in order of appearance. When the result is known to be incorrect due to a later publication, references to the publications are given. When the result is known to be incorrect due to the existence of a better drawing, we show such a counterexample as obtained by QuickCross [29]. When the result is known to be incorrect because a drawing does not exist with the proposed number of crossings, we use Crossing Number Web Compute [23] to determine the true crossing number for a minimal counterexample.

- In 1955, Zarankiewicz [233] claimed to have determined that the crossing number of complete bipartite graphs $K_{m,n}$. The flaw in the proof was noted and communicated privately, and subsequently described by Guy [54] in 1969. The result is now known to be correct for $\min\{m,n\} \leq 6$, and the special cases $m \leq 8$, $n \leq 10$; see Section 2.1.1.

- In 1986, Fiorini [45] claimed to have determined that the crossing number of the generalized Petersen graph $GP(10,3)$ was equal to four. This was shown to be false in 1992 by McQuillan and Richter [140] who conjectured that the true value would be six, which was finally proved by Richter and Salazar [162] in 2002. Fiorini (along with co-author Gauci) also corrected his earlier proofs in a paper published in 2003 [46]; see Section 2.4.

- In 1996, Cimikowski [26] claimed to have determined an upper bound for the crossing number of the complete mesh of trees M_n, but in 1998 he reported the error himself and provided a corrected upper bound [27]; see Section 6.2.

- In 2005, Wang and Huang [202] determined the crossing number of the Cartesian product of four six-vertex graphs with paths. For the first of these graphs, G_{131}^6 (which they label G_1), they claimed that $cr(G_{131}^6 \square P_n) = 3n - 1$ for $n \geq 1$. This is not correct. Indeed, we used Crossing Number Web Compute to show that $cr(G_{131}^6 \square P_1) = 4$, rather than 2 as would be suggested by [202].

- In 2008, He [65] considered the crossing number of $K_{4,n}$ with two edges deleted. In particular, if the 4 vertices are denoted y_1, y_2, y_3, y_4 and the n vertices

are denoted x_1, x_2, ..., x_n, then they remove edges $\{x_1, y_1\}$ and $\{x_1, y_2\}$. He claimed to have proved the crossing number is equal to $Z(4, n) - 2 \left\lfloor \frac{n}{2} \right\rfloor + 2$ for $n \geq 4$. The result appears to be incorrect for odd values of n. In particular, in Figure 30 we display a drawing of $K_{4,5}$ with two edges removed as described above, with only four crossings, rather than the six suggested by [65].
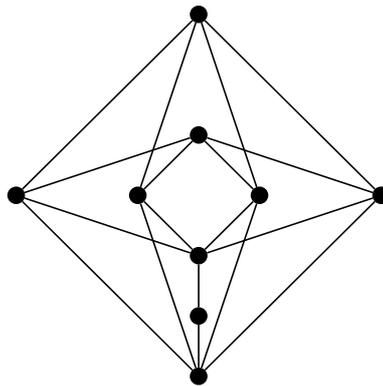

Figure 30: A drawing of $K_{4,5}$ minus two edges, with only four crossings.

- In 2011, He [69] considered the crossing number of $G_3^5 \square S_n$ and $G_4^5 \square S_n$ and claimed that both were equal to $4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 1$. These results are contradicted by Bokal (2007) [18] and Klešč (2009) [97] respectively, who showed that each has crossing number equal to $3 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$. Tests with QuickCross have confirmed that it is possible to find drawings with the latter number of crossings, e.g. see Figure 31 in which drawings of $G_3^5 \square S_3$ and $G_4^5 \square S_3$ are drawn with four crossings, rather than the five suggested by [69].
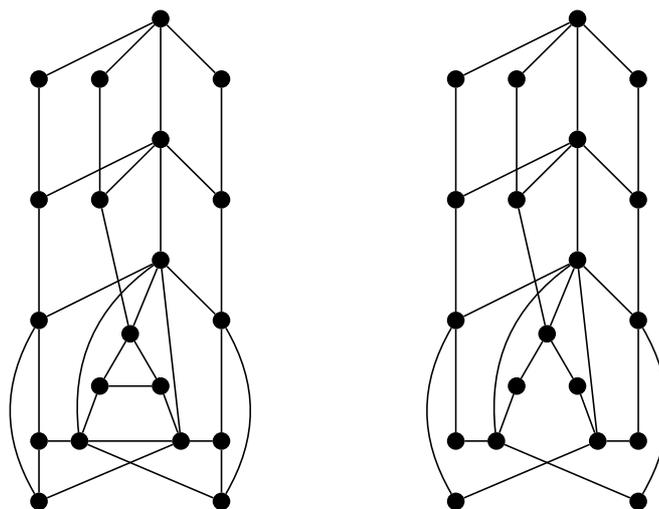

Figure 31: Drawings of $G_3^5 \square S_3$ and $G_4^5 \square S_3$, with only four crossings.

- In 2015, Cruz and Japson [31] considered the crossing number of $GP(17,5)$ and claimed it was equal to 14. This result is contradicted by Newcombe (2019) [144], who presented a drawing of $GP(17,5)$ with only 13 crossings.

- In 2016, Hsieh and Lin [77] claimed to have determined the crossing number of the join product of various path powers with discrete graphs and path graphs. All of the claimed results appear to be incorrect, as they all rely on an intermediate result, Lemma 5 of [77], which claims that $cr(P_m^{m-1}+D_n) = cr(P_m^{m-2}+D_n)$ for $3 \leq m \leq 6$ and $n \geq 1$. This result is incorrect. In [77], P_m^k is defined as the k-th power on the path graph with m vertices, rather than m edges. The minimal counterexample can be seen by considering graphs on four vertices. By Hsieh and Lin's definition, $P_4^3 = K_4$ and $P_4^2 = K_4 \setminus e$. However, we know from Ho [75] that $cr(K_4+D_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n$, and from Klešč and Schrötter [109] that $cr((K_4 \setminus e) + D_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$.

- In 2016, Vijaya et al. [197] considered the join product of G_{133}^6 with discrete graphs, paths and cycles. They provided detailed proofs for the cases of discrete graphs and paths, but omitted the proof for cycles, for which they claimed that $cr(G_{133}^6+C_n) = Z(6,n)+n+2 \left\lfloor \frac{n}{2} \right\rfloor +5$, for $n \geq 3$. This appears to be incorrect. In particular, in Figure 32 we display a drawing of $G_{133}^6 + C_3$ which has fifteen crossings, rather than sixteen as suggested by [197].
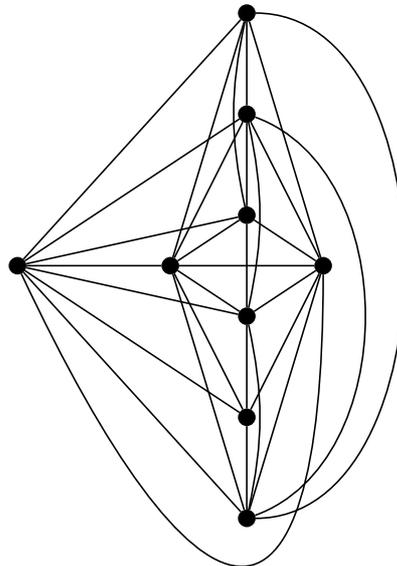

Figure 32: A drawing of $G_{133}^6 + C_3$, with only fifteen crossings.

- In 2016, Zhou and Li [247] considered the join product of G_{111}^6 with discrete graphs, paths and cycles. They claimed that $cr(G_{111}^6 + D_n) = Z(6,n) + 2 \left\lfloor \frac{n}{2} \right\rfloor$,

$cr(G_{111}^6 + P_{n-1}) = Z(6, n) + 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $cr(G_{111}^6 + C_n) = Z(6, n) + 2 \left\lfloor \frac{n}{2} \right\rfloor + 3$. These appear to be incorrect, and Crossing Number Web Compute shows that the minimal counterexamples are $cr(G_{111}^6 + D_1) = 1$, $cr(G_{111}^6 + P_1) = 4$ and $cr(G_{111}^6 + C_3) = 13$, rather than the zero, three and eleven crossings suggested by [247] respectively.

The proof files for the crossing numbers determined by Crossing Number Web Compute are available as follows:

$cr(G_{131}^6 \square P_1) = 4$: http://crossings.uos.de/job/KN4vnYbb797WV0C5TQdTNA

$cr(G_{111}^6 + D_1) = 1$: http://crossings.uos.de/job/MuRSG12mxlzDmJz4cc9eFA

$cr(G_{111}^6 + P_1) = 4$: http://crossings.uos.de/job/dhr5Vl_8pWDL-jAARvBNzw

$cr(G_{111}^6 + C_3) = 13$: http://crossings.uos.de/job/SHHUcOtUyGpQFNY-Nqmwxg

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