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A Turán-type generalization of Tuza's triangle edge cover problem

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Abstract

We investigate the smallest number $\lambda_k(G)$ of edges that can be removed from a non-empty graph G so that the resulting graph contains no k-clique. Turán's theorem tells us that $\lambda_k(K_n)$ is the number of edges missing from the Turán graph $T(n, k-1)$. The investigation of $\lambda_3(G)$ was initiated by Tuza. Let $G(k)$ be the union of k-cliques of G. Let m, t, and κ be the number of edges of $G(k)$, the number of k-cliques of G, and $\binom{k}{2}$, respectively. We prove that $\lambda_k(G) \leq \frac{2m+\kappa t}{3\kappa}$, and that equality holds if and only if the k-cliques of G are pairwise edge-disjoint. We also prove that $\lambda_k(G) \leq m\left(1 - (\frac{\kappa-1}{\kappa})(\frac{m}{\kappa t})^{\frac{1}{\kappa-1}}\right)$, and this bound is also attained by unions of pairwise edge-disjoint k-cliques.

1 Introduction

Unless stated otherwise, we use small letters such as x to denote non-negative integers or elements of a set, and capital letters such as X to denote sets or graphs. The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any $n \in \mathbb{N}$, the set $\{1, \dots, n\}$ is denoted by $[n]$. For a set X, the set of all 2-element subsets of X is denoted by $\binom{X}{2}$. Every arbitrary set is taken to be finite. For standard terminology in graph theory, we refer the reader to [6, 23]. Every graph G is taken to be *simple*, that is, its vertex set $V(G)$ and edge set $E(G)$ satisfy $E(G) \subseteq \binom{V(G)}{2}$. We may represent an edge $\{v, w\}$ by vw. For $X \subseteq V(G)$, $G[X]$ denotes the *subgraph of G induced by X*, that is, $G[X] = (X, E(G) \cap \binom{X}{2})$. For $S \subseteq E(G)$, $G - S$ denotes the subgraph of G obtained by removing the edges in S from G, that is, $G - S = (V(G), E(G) \backslash S)$. We may abbreviate $G - \{e\}$ to $G - e$.

A graph G is *complete* if every two distinct vertices of G are neighbours (that is, $E(G) = \binom{V(G)}{2}$). The complete graph $([n], \binom{[n]}{2})$ is denoted by K_n. If G is a complete

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graph and $|V(G)| = k$, then G is also called a *k-clique*. If G contains a k-clique H, then we call H a *k-clique of G*. We denote the set of k-cliques of G by $\mathcal{C}_k(G)$. The size of a largest clique of G is called the *clique number of G* and is denoted by $\omega(G)$.

The classical Turán problem is to determine, for a given graph H, the maximum number of edges a graph can have if it contains no copy of H. Turán [20] solved this problem for $H = K_k$. The special case $H = K_3$ had been settled by Mantel [18].

For $1 \leq k \leq n$, let $I_1, \dots, I_k$ be sets that partition an n-element set X such that, for each $i \in [k]$, $|I_i|$ is $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$, and let $T(n, k)$ be the graph with vertex set X and edge set $\{vw \colon (v, w) \in I_i \times I_j \text{ for some } i, j \in [k] \text{ with } i \neq j\}$. We call $T(n, k)$ a *Turán graph*. Clearly, $T(n, k-1)$ contains no k-clique. Note that $|E(T(n, k-1))| \leq \left(\frac{k-2}{k-1}\right)\frac{n^2}{2}$. Turán's theorem is the following.

Theorem 1.1 (Turán [20]) *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, $k \geq 2$, and G contains no k-clique, then*

$$m \leq |E(T(n, k-1))|.$$

Moreover, equality holds if and only if G is a copy of $T(n, k-1)$.

Several known proofs are provided and discussed in Aigner's recommended expository article [2]. Turán's theorem is a fundamental result that gave rise to extremal graph theory (see [6]).

We consider a generalization of the Turán problem in the same spirit of our work in [8, 9, 10]. We investigate the smallest number of edges that can be removed from a non-empty graph G so that the resulting graph contains no k-clique. We call a subset L of $E(G)$ a *k-clique edge cover of G* if $\omega(G - L) < k$, that is, if the graph obtained from G by removing the edges in L contains no k-clique. Thus, L is a k-clique edge cover of G if and only if each k-clique of G has an edge in L. The size of a smallest k-clique edge cover of G will be denoted by $\lambda_k(G)$ and called the *k-clique edge cover number of G*.

The study of $\lambda_3(G)$ was initiated by Tuza [21, 22]. In [21], Tuza conjectured that $\lambda_3(G)$ is at most twice the size of a largest set of pairwise edge-disjoint triangles (3-cliques) of G. This popular conjecture has been treated by many authors and verified for certain cases (see, for example, [4, 11, 12, 13, 14, 16, 19, 22, 24]) but is still open.

Turán's theorem is the solution to our problem for the case $G = K_n$. Clearly, $\lambda_k(K_n)$ is the number of edges missing from $T(n, k-1)$, that is,

$$\lambda_k(K_n) = |E(K_n)| - |E(T(n, k-1))| = \binom{n}{2} - |E(T(n, k-1))|. \qquad (1)$$

Turán's theorem actually yields the following general result, giving a lower bound on $\lambda_k(G)$.

Corollary 1.2 *If G is a graph, $n = |V(G)|$, $m = |E(G)|$, and $k \geq 2$, then*

$$\lambda_k(G) \geq m - |E(T(n, k-1))|.$$

Moreover, equality holds if G contains $T(n, k-1)$.

Proof. Let L be a k-clique edge cover of G of size $\lambda_k(G)$. By Theorem 1.1, we have $|E(T(n,k-1))| \geq |E(G-L)| = m - \lambda_k(G)$, and the bound follows.

Suppose that G contains $T(n, k-1)$. Let $L = E(G) \backslash E(T(n, k-1))$. Then, L is a k-clique edge cover of G. We have $m - |E(T(n,k-1))| \leq \lambda_k(G) \leq |L| = m - |E(T(n,k-1))|$, so $\lambda_k(G) = m - |E(T(n,k-1))|$. □

By taking $G = K_n$ in Corollary 1.2, we obtain (1).

For a graph G, let $G(k)$ denote the subgraph of G given by the union of k-cliques of G. Note that the edges of G that are not edges of $G(k)$ are redundant for our problem; their removal does not affect $\lambda_k(G)$ (see Proposition 2.3). Thus, in our main results, we will only consider the edges in $G(k)$.

The following is our first main result, proved in Section 3.

Theorem 1.3 *If G is a graph, $k \geq 2$, $m = |E(G(k))|$, and $t = |\mathcal{C}_k(G)|$, then*

$$\lambda_k(G) \leq \frac{2m + \binom{k}{2}t}{3\binom{k}{2}}.$$

Moreover, equality holds if and only if the k-cliques of G are pairwise edge-disjoint.

Trivially,

$$m \leq \binom{k}{2}t. \tag{2}$$

Proposition 2.5 gives us the straightforward bound $\lambda_k(G) \leq t$. By (2), this follows immediately from the bound in Theorem 1.3, and the smaller m is, the better the latter bound is. If $m = \binom{k}{2}t$, then the k-cliques of G are pairwise edge-disjoint, and the two bounds are attained.

By adapting the probabilistic argument of Alon in [3] to our problem, we obtain the next sharp upper bound, also proved in Section 3.

Theorem 1.4 *If G is a graph, $k \geq 3$, $m = |E(G(k))|$, $t = |\mathcal{C}_k(G)|$, and $\kappa = \binom{k}{2}$, then*

$$\lambda_k(G) \leq m\left(1 - \left(\frac{\kappa - 1}{\kappa}\right)\left(\frac{m}{\kappa t}\right)^{\frac{1}{\kappa-1}}\right).$$

Moreover, equality holds if the k-cliques of G are pairwise edge-disjoint.

In the next section, we investigate $\lambda_k(G)$ from a structural point of view. Some of the structural results obtained are then used in the proof of Theorem 1.3. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4. In Section 4, we compare the bounds in these theorems, using real analysis.

2 Structural results

In this section, we mainly observe how $\lambda_k(G)$ is affected by the removal of edges from G.

Let $\mathcal{C}_k^1(G)$ denote the set $\{C \in \mathcal{C}_k(G)\colon E(C) \cap E(K) \neq \emptyset$ for some $K \in \mathcal{C}_k(G)\backslash\{C\}\}$. Let $\mathcal{C}_k^2(G)$ denote $\mathcal{C}_k(G)\backslash\mathcal{C}_k^1(G)$. Then, a k-clique C of G is a member of $\mathcal{C}_k^2(G)$ if and only if, for each k-clique K of G such that $K \neq C$, C and K have no common edge.

Lemma 2.1 *If G is a graph, H is a subgraph of G, and L is a k-clique edge cover of G, then $L \cap E(H)$ is a k-clique edge cover of H.*

Proof. The result is immediate if $\mathcal{C}_k(H) = \emptyset$. Suppose $\mathcal{C}_k(H) \neq \emptyset$. Let $J = L \cap E(H)$. We need to show that, for each $C \in \mathcal{C}_k(H)$, $e \in E(C)$ for some $e \in J$. Since H is a subgraph of G, $C \in \mathcal{C}_k(G)$. Thus, $e \in E(C)$ for some $e \in L$. Since $C \in \mathcal{C}_k(H)$, $e \in E(H)$. Thus, $e \in J$. □

We point out that $|L| = \lambda_k(G)$ does not guarantee that $|L \cap E(H)| = \lambda_k(H)$. Indeed, if $k \geq 3$, G is a copy of K_k, $e_1 = \{1,2\}$, $e_2 = \{1,k\}$, $L = \{e_1\}$, and $H = G - e_2$, then L is a k-clique edge cover of G of size $\lambda_k(G)$, $L \cap E(H) = L$, and $\emptyset$ is a k-clique edge cover of H.

Corollary 2.2 *If H is a subgraph of G, then $\lambda_k(H) \leq \lambda_k(G)$.*

Proof. Let L be a k-clique edge cover of G of size $\lambda_k(G)$. By Lemma 2.1, $\lambda_k(H) \leq |L \cap E(H)| \leq |L|$. □

Proposition 2.3 *If G is a graph and $e \in E(G)\backslash \bigcup_{C\in\mathcal{C}_k(G)} E(C)$, then $\lambda_k(G - e) = \lambda_k(G)$.*

Proof. Since $e \notin \bigcup_{C\in\mathcal{C}_k(G)} E(C)$, $\mathcal{C}_k(G - e) = \mathcal{C}_k(G)$. Thus, a set of edges is a k-clique edge cover of $G - e$ if and only if it is a k-clique edge cover of G. The result follows. □

Proposition 2.4 *If G is a graph and $e \in E(G)$, then $\lambda_k(G) \leq 1 + \lambda_k(G - e)$.*

Proof. Let L be a k-clique edge cover of $G - e$ of size $\lambda_k(G - e)$. Since $e \in E(C)$ for each $C \in \mathcal{C}_k(G)\backslash\mathcal{C}_k(G - e)$, $\{e\} \cup L$ is a k-clique edge cover of G. Therefore, $\lambda_k(G) \leq |\{e\} \cup L| = 1 + |L|$. □

Proposition 2.5 *For any graph G, $\lambda_k(G) \leq |\mathcal{C}_k(G)|$, and equality holds if and only if $\mathcal{C}_k(G) = \mathcal{C}_k^2(G)$.*

Proof. For each $C \in \mathcal{C}_k(G)$, let e_C be an edge of C. Let $L = \{e_C \colon C \in \mathcal{C}_k(G)\}$. Since L is a k-clique edge cover of G, $\lambda_k(G) \leq |L| \leq |\mathcal{C}_k(G)|$. Suppose $\mathcal{C}_k(G) \neq \mathcal{C}_k^2(G)$. Then, $E(C_1) \cap E(C_2) \neq \emptyset$ for some $C_1, C_2 \in \mathcal{C}_k(G)$. Let $e' \in E(C_1) \cap E(C_2)$. Let $L' = \{e_C \colon C \in \mathcal{C}_k(G) \backslash \{C_1, C_2\}\} \cup \{e'\}$. Since L' is a k-clique edge cover of G, $\lambda_k(G) \leq |L'| \leq |\mathcal{C}_k(G)| - 1$. Now suppose $\mathcal{C}_k(G) = \mathcal{C}_k^2(G)$. Then, the k-cliques of G are pairwise edge-disjoint. Let L^* be a smallest k-clique edge cover of G. Then,

$$\begin{aligned} \lambda_k(G) &= |L^*| = |L^* \cap \bigcup_{C \in \mathcal{C}_k(G)} E(C)| = |\bigcup_{C \in \mathcal{C}_k(G)} (L^* \cap E(C))| = \sum_{C \in \mathcal{C}_k(G)} |L^* \cap E(C)| \\ &\geq \sum_{C \in \mathcal{C}_k(G)} 1 = |\mathcal{C}_k(G)| \geq \lambda_k(G), \end{aligned}$$

and hence $\lambda_k(G) = |\mathcal{C}_k(G)|$. $\square$

3 Proofs of the main results

In this section, we prove Theorems 1.3 and 1.4.

Lemma 3.1 *If A, B, and C are distinct k-cliques with $E(A) \cap E(B) \cap E(C) = \emptyset$, then*

$$|(E(A) \cup E(B)) \backslash E(C)| \geq \binom{k}{2}.$$

For the proof of Lemma 3.1, we first prove the following.

Lemma 3.2 *If $2 \leq t \leq k$, $0 \leq p \leq k - t + 1$, $0 \leq q \leq k - t + 1$, and $p + q \leq k + 1$, then*

$$\binom{k}{2} \geq \binom{t}{2} + \binom{p}{2} + \binom{q}{2}.$$

Proof. We use induction on k. The base case $k = 2$ is trivial. Consider $k > 2$. If $p \leq 1$ or $q \leq 1$, then $\binom{p}{2} + \binom{q}{2} \leq \binom{k-t+1}{2}$, and the result follows since $\binom{k}{2} \geq \binom{t}{2} + \binom{k-t+1}{2}$ (indeed, for a k-clique A, a t-element subset T of $V(A)$, and an element x of T, the number $\binom{k}{2}$ of edges of A is at least the sum of the number $\binom{t}{2}$ of edges of $A[T]$ and the number $\binom{k-t+1}{2}$ of edges of $A[(V(A) \backslash T) \cup \{x\}]$). Suppose $p \geq 2$ and $q \geq 2$. Let $k' = k - 1$, $p' = p - 1$, and $q' = q - 1$. Then, $2 \leq t \leq k'$ (as $2 \leq p \leq k - t + 1$ implies $k \geq t + 1$), $0 < p' \leq k' - t + 1$, $0 < q' \leq k' - t + 1$, and $p' + q' \leq k'$. By the induction hypothesis, $\binom{k'}{2} \geq \binom{t}{2} + \binom{p'}{2} + \binom{q'}{2}$. Now $\binom{k}{2} = \binom{k'}{2} + k' \geq \binom{t}{2} + \binom{p'}{2} + \binom{q'}{2} + (p' + q') = \binom{t}{2} + \binom{p}{2} + \binom{q}{2}$. $\square$

Proof of Lemma 3.1. Let $T = V(A) \cap V(B)$, $P = V(C) \cap V(A)$, and $Q = V(C) \cap V(B)$. Let $t = |T|$, $p = |P|$, and $q = |Q|$. We have $\max\{t, p, q\} \leq k$. Let $r = |(E(A) \cup E(B)) \backslash E(C)|$. If $t \leq 1$, then $E(A) \cap E(B) = \emptyset$, and hence

$$r \geq |E(A)| + |E(B)| - |E(C)| = \binom{k}{2}.$$

Suppose $t \geq 2$. Since

$$\emptyset = (E(A) \cap E(B)) \cap E(C) = \binom{T}{2} \cap E(C),$$

$|V(C) \cap T| \leq 1$. We have $p = |V(C) \cap T| + |V(C) \cap (V(A) \backslash T)| \leq 1 + |V(A) \backslash T| \leq k - t + 1$. Similarly, $q \leq k - t + 1$. Since

$$k = |C| \geq |P \cup Q| = p + q - |P \cap Q| = p + q - |V(C) \cap T| \geq p + q - 1,$$

$p + q \leq k + 1$. Now, $E(C) \cap (E(A) \cup E(B)) = \binom{P}{2} \cup \binom{Q}{2}$. We have

$$\begin{aligned} r &= |(E(A) \cup E(B)) \backslash (E(C) \cap (E(A) \cup E(B)))| \\ &\geq |E(A) \cup E(B)| - |E(C) \cap (E(A) \cup E(B))| \\ &\geq 2\binom{k}{2} - \binom{t}{2} - \binom{p}{2} - \binom{q}{2}. \end{aligned}$$

By Lemma 3.2, $\binom{k}{2} - \binom{t}{2} - \binom{p}{2} - \binom{q}{2} \geq 0$, and hence $r \geq \binom{k}{2}$. □

Proof of Theorem 1.3. We may assume that $G = G(k)$. Let $\kappa = \binom{k}{2}$. If the t k-cliques of G are pairwise edge-disjoint, then $m = \kappa t$ and $\lambda_k(G) = t = \frac{2m+\kappa t}{3\kappa}$. We now prove the bound in the theorem and show that it is attained only if the k-cliques of G are pairwise edge-disjoint.

We use induction on t. If $t = 0$, then $\lambda_k(G) = 0$ and the result is trivial. Suppose $t \geq 1$. Then, $m \geq \kappa$. If $k = 2$, then $t = m$ and $\lambda_k(G) = m = \frac{2m+\kappa t}{3\kappa}$. Suppose $k \geq 3$.

Suppose first that $\mathcal{C}_k^2(G) \neq \emptyset$. Let $K \in \mathcal{C}_k^2(G)$. Let $e^* \in E(K)$. If $\mathcal{C}_k(G - e^*) = \emptyset$, then $\lambda_k(G) = 1 \leq \frac{2m+\kappa t}{3\kappa}$, and equality holds only if $\mathcal{C}_k(G) = \{K\}$. Suppose $\mathcal{C}_k(G - e^*) \neq \emptyset$. Since $K \in \mathcal{C}_k^2(G)$ and $K \notin \mathcal{C}_k(G - e^*)$, $e \notin \bigcup_{C \in \mathcal{C}_k(G-e^*)} E(C)$ for each $e \in E(K) \backslash \{e^*\}$. Taking $H = G - E(K)$, we thus have $\mathcal{C}_k(H) = \mathcal{C}_k(G - e^*) = \mathcal{C}_k(G) \backslash \{K\}$. By repeated application of Proposition 2.3, $\lambda_k(H) = \lambda_k(G - e^*)$. Thus, by Proposition 2.4 and the induction hypothesis,

$$\lambda_k(G) \leq 1 + \lambda_k(H) \leq 1 + \frac{2(m-\kappa) + \kappa(t-1)}{3\kappa} = \frac{2m + \kappa t}{3\kappa},$$

and equality holds throughout only if the k-cliques of H are pairwise edge-disjoint, in which case the k-cliques of G are pairwise edge-disjoint.

Now suppose $\mathcal{C}_k^2(G) = \emptyset$. Then, $\mathcal{C}_k(G) = \mathcal{C}_k^1(G)$.

Suppose that G has three distinct k-cliques C_1, C_2, and C_3 that have a common edge e^*. Then, $\mathcal{C}_k(G - e^*) \subseteq \mathcal{C}_k(G) \backslash \{C_1, C_2, C_3\}$. By Proposition 2.4 and the induction hypothesis,

$$\lambda_k(G) \leq 1 + \lambda_k(G - e^*) \leq 1 + \frac{2(m-1) + \kappa(t-3)}{3\kappa} < \frac{2m + \kappa t}{3\kappa}.$$

Now suppose that

$$\text{no three distinct } k\text{-cliques of } G \text{ have a common edge.} \tag{3}$$

Let $r = \lambda_k(G)$. Let L be a k-clique edge cover of G of size r. For any $e \in L$ and any subgraph H of G, let $\mathcal{C}_k(H, e) = \{C \in \mathcal{C}_k(H) \colon e \in E(C)\}$. Let $G_1 = G$. Let $e_1 \in L$ such that $|\mathcal{C}_k(G_1, e_1)| = \max\{|\mathcal{C}_k(G_1, e)| \colon e \in L\}$, and let $G_2 = G_1 - e_1$. If $r \geq 2$, then let $e_2 \in L \backslash \{e_1\}$ such that $|\mathcal{C}_k(G_2, e_2)| = \max\{|\mathcal{C}_k(G_2, e)| \colon e \in L \backslash \{e_1\}\}$, and let $G_3 = G_2 - e_2$. If $r \geq 3$, then let $e_3 \in L \backslash \{e_1, e_2\}$ such that $|\mathcal{C}_k(G_3, e_3)| = \max\{|\mathcal{C}_k(G_3, e)| \colon e \in L \backslash \{e_1, e_2\}\}$, and let $G_4 = G_3 - e_3$. Continuing in this way, we obtain $e_1, \dots, e_r$ and $G_1, \dots, G_{r+1}$ such that $L = \{e_1, \dots e_r\}$, $G_{r+1} = G - L$, $\mathcal{C}_k(G_i) \neq \emptyset$ for each $i \in [r]$, and $\mathcal{C}_k(G_{r+1}) = \emptyset$. For each $i \in [r]$, let $\mathcal{A}_i = \mathcal{C}_k(G_i, e_i)$. By construction,

$$|\mathcal{A}_1| \geq \cdots \geq |\mathcal{A}_r|. \tag{4}$$

Note that $\mathcal{A}_1, \dots, \mathcal{A}_r$ partition $\mathcal{C}_k(G)$ (that is, $\mathcal{C}_k(G) = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_r$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$), so $t = \sum_{i=1}^{r} |\mathcal{A}_i|$. For each $i \in [r]$, $1 \leq |\mathcal{A}_i| \leq 2$ as $\mathcal{C}_k(G_i) \neq \emptyset$ and (3) holds.

Let $I_1 = \{i \in [r] \colon |\mathcal{A}_i| = 1\}$ and $I_2 = \{i \in [r] \colon |\mathcal{A}_i| = 2\}$. Let $r_1 = |I_1|$ and $r_2 = |I_2|$. Then, $r_1 + r_2 = r$. Since $\mathcal{C}_k(G) = \mathcal{C}_k^1(G)$, we have $|\mathcal{A}_1| = 2$, so $r_2 \geq 1$. By (4), $I_2 = [r_2]$ and $I_1 = [r] \backslash [r_2]$. We have

$$t = \sum_{i=1}^{r} |\mathcal{A}_i| = \sum_{i \in I_2} |\mathcal{A}_i| + \sum_{i \in I_1} |\mathcal{A}_i| = 2r_2 + r_1.$$

Let $Y = \bigcup_{C \in \mathcal{C}_k(G_{r_2+1})} E(C)$. Let $F = E(G) \backslash Y$. If $r_2 = r$, then $r_1 = 0$ and G_{r_2+1} has no k-cliques. If $r_2 < r$, then we have $|\mathcal{A}_{r_2+1}| = 1$ (as $r_2 + 1 \in I_1$), and hence G_{r_2+1} has no two distinct k-cliques with a common edge. Since $|\mathcal{A}_i| = 1$ for each $i \in [r] \backslash [r_2]$, G_{r_2+1} has exactly r_1 k-cliques. Thus, $|F| = m - r_1\kappa$.

For each $i \in I_2$, let $C_{i,1}$ and $C_{i,2}$ be the two members of $\mathcal{A}_i$. For each pair $(e, i) \in F \times I_2$, let

$$\chi(e, i) = \begin{cases} 1 & \text{if } e \in E(C_{i,1}) \cup E(C_{i,2}); \\ 0 & \text{otherwise.} \end{cases}$$

By (3), $\sum_{i \in I_2} \chi(e, i) \leq 2$ for each $e \in F$. Moreover, by construction, we have $e_1, \dots, e_{r_2} \in F$ and $\sum_{i \in I_2} \chi(e_1, i) = \chi(e_1, 1) = 1$. Thus, we have

$$\sum_{e \in F} \sum_{i \in I_2} \chi(e, i) < \sum_{e \in F} 2 = 2|F| = 2(m - r_1\kappa). \tag{5}$$

Consider any $i \in I_2$. We show that $\sum_{e \in F} \chi(e, i) \geq \kappa$. Let $X_1 = E(C_{i,1}) \cap Y$ and $X_2 = E(C_{i,2}) \cap Y$. If $j \in [2]$ and $X_j = \emptyset$, then $\sum_{e \in F} \chi(e, i) \geq |E(C_{i,j})| = \kappa$. Suppose $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Let $e_{i,1} \in X_1$ and $e_{i,2} \in X_2$. Then, $e_{i,1} \in E(C_{i,1}) \cap E(D_1)$ for some $D_1 \in \mathcal{C}_k(G_{r_2+1})$, and $e_{i,2} \in E(C_{i,2}) \cap E(D_2)$ for some $D_2 \in \mathcal{C}_k(G_{r_2+1})$. Let $\mathcal{D} = \{D \in \mathcal{C}_k(G_{r_2+1}) \colon E(C_{i,2}) \cap E(D) \neq \emptyset\}$. Then, $D_2 \in \mathcal{D}$. Suppose $D' \neq D_1$ for some $D' \in \mathcal{D}$. Let $e' \in E(C_{i,2}) \cap E(D')$. Since the k-cliques of G_{r_2+1} are pairwise edge-disjoint, there exist $j_1, j_2 \in [r] \backslash [r_2]$ such that $j_1 \neq j_2$, $e_{j_1} \in E(D_1)$, and $e_{j_2} \in E(D')$. By the above, we obtain that $(L \backslash \{e_i, e_{j_1}, e_{j_2}\}) \cup \{e_{i,1}, e'\}$ is a k-clique edge cover of

G (that is, we can 'eliminate' $C_{i,1}$, D_1, $C_{i,2}$, and D' by using $e_{i,1}$ and e' instead of e_i, e_{j_1}, and e_{j_2}, and the remaining k-cliques of G are 'eliminated' by the remaining edges in L). This gives us $\lambda_k(G) \leq |L| - 1$, which contradicts $\lambda_k(G) = r = |L|$. Thus, $D_2 = D_1$, $\mathcal{D} = \{D_1\}$, and hence $X_2 \subseteq E(D_1)$. Similarly, $X_1 \subseteq E(D_2)$, so $X_1 \subseteq E(D_1)$ (as $D_2 = D_1$). By (3), $E(C_{i,1}) \cap E(C_{i,2}) \cap E(D_1) = \emptyset$. By Lemma 3.1, $|(E(C_{i,1}) \cup E(C_{i,2}))\backslash E(D_1)| \geq \kappa$. Thus, since

$$\begin{aligned}(E(C_{i,1}) \cup E(C_{i,2}))\backslash E(D_1) &= (E(C_{i,1}) \cup E(C_{i,2}))\backslash (X_1 \cup X_2) \\ &= (E(C_{i,1}) \cup E(C_{i,2}))\backslash Y,\end{aligned}$$

we have $|(E(C_{i,1}) \cup E(C_{i,2}))\backslash Y| \geq \kappa$, and hence $\sum_{e\in F} \chi(e, i) \geq \kappa$, as required.

Therefore, we have

$$r_2\kappa = \sum_{i\in I_2} \kappa \leq \sum_{i\in I_2}\sum_{e\in F} \chi(e, i) = \sum_{e\in F}\sum_{i\in I_2} \chi(e, i) < 2(m - r_1\kappa)$$

(by 5), and hence $2m > r_2\kappa + 2r_1\kappa$. Thus, since $t = 2r_2 + r_1$, we have $2m + \kappa t > 3(r_1 + r_2)\kappa = 3r\kappa$, and hence $r < \frac{2m+\kappa t}{3\kappa}$. □

Proof of Theorem 1.4. We may assume that $G = G(k)$ and $E(G) = [m]$. Let p be a real number satisfying $0 \leq p \leq 1$. We set up an independent random experiment for each edge, and an edge is chosen with probability p. More formally, for $i \in [m]$, let (Ω_i, P_i) be given by $\Omega_i = \{0, 1\}$, $P_i(\{1\}) = p$, and $P_i(\{0\}) = 1 - p$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$, and let $P\colon 2^\Omega \to [0, 1]$ such that $P(\{x\}) = \prod_{i=1}^m P_i(\{x_i\})$ for each $x = (x_1, \ldots, x_m) \in \Omega$, and $P(A) = \sum_{x\in A} P(\{x\})$ for each $A \subseteq \Omega$. Then, (Ω, P) is a probability space.

For each $x = (x_1, \ldots, x_m) \in \Omega$, let $S_x = \{i \in [m]\colon x_i = 1\}$ and $\mathcal{T}_x = \{C \in \mathcal{C}_k(G)\colon E(C) \cap S_x = \emptyset\}$. For each $C \in \mathcal{T}_x$, let $e_C \in E(C)$. Let $T_x = \{e_C\colon C \in \mathcal{T}_x\}$. Let $D_x = S_x \cup T_x$. Then, D_x is a k-clique edge cover of G.

Let $X\colon \Omega \to \mathbb{R}$ be the random variable given by $X(x) = |S_x|$. For each $i \in [m]$, let $X_i\colon \Omega \to \mathbb{R}$ be the indicator random variable for whether edge i is in S_x, that is,

$$X_i(x) = \begin{cases} 1 & \text{if } i \in S_x; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $X = \sum_{i=1}^m X_i$. For each $i \in [m]$, we have $P(X_i = 1) = P_i(\{1\}) = p$.

Let $Y\colon \Omega \to \mathbb{R}$ be the random variable given by $Y(x) = |\mathcal{T}_x|$. For each $C \in \mathcal{C}_k(G)$, let $Y_C\colon \Omega \to \{0, 1\}$ be the random variable given by

$$Y_C(x) = \begin{cases} 1 & \text{if } E(C) \cap S_x = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $Y = \sum_{C\in\mathcal{C}_k(G)} Y_C$. For each $C \in \mathcal{C}_k(G)$, we have $P(Y_C = 1) = (1 - p)^\kappa$.

For any random variable Z, let $\mathrm{E}[Z]$ denote the expected value of Z. By linearity of expectation,

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \sum_{i=1}^m \mathrm{E}[X_i] + \sum_{C\in\mathcal{C}_k(G)} \mathrm{E}[Y_C]$$

$$= \sum_{i=1}^{m} P(X_i = 1) + \sum_{C \in \mathcal{C}_k(G)} P(Y_C = 1) = mp + t(1-p)^{\kappa}.$$

Let $f(p) = mp + t(1-p)^{\kappa}$. By the probabilistic pigeonhole principle, there exists some $x^* \in \Omega$ such that $X(x^*) + Y(x^*) \leq \mathrm{E}[X+Y]$. Thus, $X(x^*) + Y(x^*) \leq f(p)$. We have

$$\lambda_k(G) \leq |D_{x^*}| = |S_{x^*}| + |T_{x^*}| \leq |S_{x^*}| + |\mathcal{T}_{x^*}| = X(x^*) + Y(x^*) \leq f(p).$$

We have shown that $\lambda_k(G) \leq f(p)$ for any real p satisfying $0 \leq p \leq 1$. Let $p^* = 1 - \left(\frac{m}{\kappa t}\right)^{\frac{1}{\kappa-1}}$. Using differentiation, we find that $f(p)$ is a minimum when $p = p^*$. We have $\lambda_k(G) \leq f(p^*) = m\left(1 - (\frac{\kappa-1}{\kappa})(\frac{m}{\kappa t})^{\frac{1}{\kappa-1}}\right)$.

If the k-cliques of G are pairwise edge-disjoint, then $m = \kappa t$ and $\lambda_k(G) = t = \frac{m}{\kappa} = m\left(1 - (\frac{\kappa-1}{\kappa})(\frac{m}{\kappa t})^{\frac{1}{\kappa-1}}\right)$. □

4 Comparison of the main bounds

We conclude this paper with a comparison of the bounds in Theorems 1.3 and 1.4 similar to that in [7]. Let k, t, and m be as in the theorems. For each of the two bounds, we determine ranges of values of m for which the bound is better than the other. Let $\kappa = \binom{k}{2}$. Recall from Section 1 that $m \leq \kappa t$, and that equality holds if and only if the t k-cliques are pairwise edge-disjoint, in which case the two bounds are equal and attained. We now consider $m < \kappa t$. We mention in passing that by the Kruskal–Katona Theorem [15, 17], the minimum value of m in terms of k and t is attained when the vertex sets of the t k-cliques are in colex order (loosely speaking, when they are as close as possible to each other); for further reading, see, for example, [1, 5].

Let $b_1(k,t,m)$ and $b_2(k,t,m)$ be the bound in Theorem 1.3 and the bound in Theorem 1.4, respectively; that is,

$$b_1(k,t,m) = \frac{2m + \kappa t}{3\kappa} \quad \text{and} \quad b_2(k,t,m) = m\left(1 - \left(\frac{\kappa-1}{\kappa}\right)\left(\frac{m}{\kappa t}\right)^{1/(\kappa-1)}\right).$$

We prove the following result, using several well-known facts from real analysis.

Theorem 4.1 *Suppose $k \geq 3$, $t \geq 1$, and $m < \kappa t$. Let $x_1 > 1$ be the real number 6.71... that satisfies $e^{(x_1-1)/3} = x_1$.*

(a) We have

$$b_2(k,t,m) < b_1(k,t,m) \quad \text{if } m \leq \frac{1}{x_1}\kappa t.$$

(b) There exists a unique real number x_0 such that $4 \leq x_0 < x_1$ and $\left(\frac{3\kappa-3}{3\kappa-2-x_0}\right)^{\kappa-1} = x_0$. We have

$$b_1(k,t,m) < b_2(k,t,m) \quad \text{if } m > \frac{1}{x_0}\kappa t.$$

The larger k is, the larger x_0 is. Moreover, for any real $\delta > 0$, $x_0 > x_1 - \delta$ if k is sufficiently large. We have $x_0 = 4$ if $k = 3$, $x_0 = 5.22...$ if $k = 4$, $x_0 = 5.78...$ if $k = 5$, and $x_0 \geq 6.08...$ if $k \geq 6$.

Since m can be at most κt, this result tells us that the range of values of m for which the bound in Theorem 1.3 is better than the bound in Theorem 1.4 is significantly wider than that for which the opposite holds. Also note that the result solves our problem for k sufficiently large in the following sense.

Corollary 4.2 *Let k, t, κ, m, and x_1 be as in Theorem 4.1.*
(a) If $m \leq \kappa t/x_1$, then $b_2(k,t,m) < b_1(k,t,m)$.
(b) If $m > \kappa t/x_1$ and k is sufficiently large, then $b_1(k,t,m) < b_2(k,t,m)$.

We now prove Theorem 4.1. The set of real numbers is denoted by $\mathbb{R}$, and the set of positive real numbers is denoted by $\mathbb{R}^+$. We shall make use of standard notation for real intervals. Let e be the base of the natural logarithm, that is, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828....$

Lemma 4.3 *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by*

$$f(x) = \left(1 + \frac{1}{x}\right)^{x+1}$$

for $x > 0$, then $f(x)$ decreases as x increases, and $\lim_{x\to\infty} f(x) = e$.

Proof. Let $g : (-\frac{1}{2}, \infty) \to \mathbb{R}$ be the function given by

$$g(z) = z - \ln(1+z)$$

for $z > -\frac{1}{2}$. The derivative $\frac{\mathrm{d}g}{\mathrm{d}z}$ is $1 - \frac{1}{1+z}$, which is negative for $-\frac{1}{2} < z < 0$, 0 for $z = 0$, and positive for $z > 0$. Thus, $g(z)$ increases from $g(0) = 0$ as z increases from 0 to infinity, and hence

$$g(z) > 0 \quad \text{for } z > 0. \tag{6}$$

We have $\ln f(x) = (x+1)\ln\left(1+\frac{1}{x}\right)$. Using implicit differentiation, we obtain $\frac{1}{f(x)}\frac{\mathrm{d}f}{\mathrm{d}x} = \ln\left(1+\frac{1}{x}\right) + (x+1)\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}$. Thus, by (6) with $z = \frac{1}{x}$, $-\frac{1}{f(x)}\frac{\mathrm{d}f}{\mathrm{d}x} > 0$, and hence, since $f(x) > 0$, we obtain $\frac{\mathrm{d}f}{\mathrm{d}x} < 0$. Therefore, $f(x)$ decreases as x increases. Now $\lim_{x\to\infty} f(x) = \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right) = e$. □

Lemma 4.4 *Let $A = \{(x,y) \in \mathbb{R}\times\mathbb{R} \colon y \geq 3,\ 1 \leq x < 3y-2\}$. Let $f : A \to \mathbb{R}$ be the function given by*

$$f(x,y) = \left(\frac{3y-3}{3y-2-x}\right)^{y-1} - x$$

for $(x,y) \in A$. For any $y_0 \in [3,\infty)$, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, 3y_0-2)$, and $f(x,y) < 0$ for any $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$ such that $x \neq x_{y_0}$ or $y \neq y_0$.

Moreover, let $x_1 > 1$ *be the real number* 6.71... *that satisfies* $e^{(x_1-1)/3} = x_1$.
(a) If $y_0, y_1 \in [3, \infty)$ *with* $y_0 < y_1$*, then* $x_{y_0} < x_{y_1} < x_1$.
(b) For any real $\delta > 0$*, there exists some* $y_\delta \in [3, \infty)$ *such that* $x_y > x_1 - \delta$ *for any* $y \in (y_\delta, \infty)$.

Proof. Let $g : [1, 3y_0 - 2) \to \mathbb{R}$ such that $g(x) = f(x, y_0)$ for $x \in [1, 3y_0 - 2)$. We have

$$\frac{\mathrm{d}g}{\mathrm{d}x} = (y_0 - 1)\left(\frac{3y_0 - 3}{3y_0 - 2 - x}\right)^{y_0 - 2} \frac{3y_0 - 3}{(3y_0 - 2 - x)^2} - 1 = \frac{1}{3}\left(\frac{3y_0 - 3}{3y_0 - 2 - x}\right)^{y_0} - 1.$$

As x increases from 1 to $3y_0-2$, the value of $\frac{1}{3}\left(\frac{3y_0-3}{3y_0-2-x}\right)^{y_0}$ increases from $\frac{1}{3}$ to ∞, and hence $\frac{\mathrm{d}g}{\mathrm{d}x}$ increases from $-\frac{2}{3}$ to ∞. Thus, there exists a unique $x^* \in (1, 3y_0 - 2)$ such that $\frac{\mathrm{d}g}{\mathrm{d}x}$ is 0 at x^*, and $g(x^*) = \min\{g(x) \colon x \in [1, 3y_0 - 2)\} < g(1) = 0$. Thus, $g(x)$ decreases from $g(1) = 0$ to $g(x^*)$, and then increases from $g(x^*)$ to ∞. Consequently, there exists a unique $x_{y_0} \in (1, 3y_0-2)$ such that $g(x_{y_0}) = 0 = g(1)$ and $g(x) < g(x_{y_0})$ for each $x \in (1, x_{y_0})$.

Now suppose $x \in (1, x_{y_0}]$ and $y \in [y_0, \infty)$. Let $z_0 = \frac{3y_0-2-x}{x-1}$ and $z = \frac{3y-2-x}{x-1}$. Then, $z \geq z_0$. We have

$$\begin{aligned}
f(x, y) + x &= \left(1 + \frac{x - 1}{3y - 2 - x}\right)^{y-1} = \left(1 + \frac{1}{z}\right)^{(z+1)(x-1)/3} \\
&= \left(\left(1 + \frac{1}{z}\right)^{z+1}\right)^{(x-1)/3} \leq \left(\left(1 + \frac{1}{z_0}\right)^{z_0+1}\right)^{(x-1)/3} \qquad \text{(by Lemma 4.3)} \\
&= f(x, y_0) + x. \qquad (7)
\end{aligned}$$

Therefore,

$$f(x, y) \leq f(x, y_0) = g(x) \leq g(x_{y_0}) = 0. \qquad (8)$$

If $x \neq x_{y_0}$, then $x < x_{y_0}$, and hence $g(x) < g(x_{y_0})$. If $y \neq y_0$, then $y > y_0$, $z > z_0$, $\left(1 + \frac{1}{z}\right)^{z+1} < \left(1 + \frac{1}{z_0}\right)^{z_0+1}$ (by Lemma 4.3), and hence $f(x, y) < f(x, y_0)$ by (7). Thus, if $x \neq x_{y_0}$ or $y \neq y_0$, then $g(x) < g(x_{y_0})$ or $f(x, y) < f(x, y_0)$, so $f(x, y) < 0$ by (8).

Let $h : [1, \infty) \to \mathbb{R}$ be the function given by $h(x) = e^{(x-1)/3} - x$ for $x \in [1, \infty)$. Using differentiation, we obtain that the minimum value of h occurs at $x = 1+3\ln 3 < x_1$, and that h has no other turning points. Thus, $h(x)$ decreases from $h(1) = 0$ to $h(1 + 3\ln 3) < 0$ as x increases from 1 to $1 + 3\ln 3$, and $h(x)$ increases to infinity as x increases from $1 + 3\ln 3$. Note that $h(x_1) = 0$. Let $z_0' = \frac{3y_0-2-x_{y_0}}{x_{y_0}-1}$. We have $0 = f(x_{y_0}, y_0) = \left(\left(1 + \frac{1}{z_0'}\right)^{z_0'+1}\right)^{(x_{y_0}-1)/3} - x_{y_0} > e^{(x_{y_0}-1)/3} - x_{y_0}$ by Lemma 4.3. Thus, $h(x_{y_0}) < 0$, and hence $x_{y_0} < x_1$.

Next, suppose $y_0 < y_1$. By the same argument for x_{y_0}, $x_{y_1} < x_1$. Let $p : [1, 3y_1 - 2) \to \mathbb{R}$ such that $p(x) = f(x, y_1)$ for $x \in (1, 3y_1 - 2)$. By the argument above, $p(x) < 0$ for $x \in (1, x_{y_1})$, $p(x_{y_1}) = 0$, and $p(x) > 0$ for $x \in (x_{y_1}, 3y_1 - 2)$. Since $y_1 \in (y_0, \infty)$, we have $p(x) = f(x, y_1) < 0$ for any $x \in (1, x_{y_0}]$, so $x_{y_1} > x_{y_0}$. Thus, (a) is proved.

Finally, let $\delta \in \mathbb{R}^+$. Let $\delta' = \min\{\frac{1}{2}, \delta\}$. Let $x' = x_1 - \delta'$. Since $x' < x_1$, $h(x') < 0$. Let $q : [3, \infty) \to \mathbb{R}$ such that $q(y) = f(x', y)$ for $y \in [3, \infty)$. We have $x' \geq x_1 - \frac{1}{2} > 6$ and $q(3) = \left(\frac{6}{7-x'}\right)^2 - x' > \left(\frac{6}{7-6}\right)^2 - x_1 > 0$. For any $y \in [3, \infty)$, let $z_y = \frac{3y-2-x'}{x'-1}$. As y increases to infinity, z_y increases to infinity. We have $q(y) = \left(\left(1 + \frac{1}{z_y}\right)^{z_y+1}\right)^{(x'-1)/3} - x'$. Thus, by Lemma 4.3, $q(y)$ decreases from $q(3) > 0$ to $h(x') < 0$ as y increases from 3 to infinity. Thus, there exists some $y_\delta \in [3, \infty)$ such that $q(y_\delta) = 0$. We have $f(x', y_\delta) = 0$, so $x' = x_{y_\delta}$. By (a), $x_y > x_{y_\delta}$ for any $y \in (y_\delta, \infty)$. Thus, (b) is proved. $\square$

Proof of Theorem 4.1. Since $k \geq 3$ and $t \geq 1$, $m \geq \kappa \geq 3$. Let $x = \kappa t/m$. Since $m < \kappa t$, $x > 1$. Let $\sim$ be any of the relations $<$, $=$, and $>$. We have

$$\begin{aligned} b_1(k,t,m) \sim b_2(k,t,m) \quad &\Leftrightarrow \quad \frac{2m + \kappa t}{m} \sim 3\kappa \left(1 - \left(\frac{\kappa - 1}{\kappa}\right)\left(\frac{1}{x}\right)^{1/(\kappa-1)}\right) \\ \Leftrightarrow \quad 2 + x \sim 3\kappa - \frac{3(\kappa-1)}{x^{1/(\kappa-1)}} \quad &\Leftrightarrow \quad 3(\kappa - 1) \sim (3\kappa - 2 - x)x^{1/(\kappa-1)}. \end{aligned} \tag{9}$$

Suppose $m \leq \kappa t/(3\kappa - 2)$. Then, $x \geq 3\kappa - 2$. Thus, we have $3(\kappa - 1) > 0 \geq (3\kappa - 2 - x)x^{1/(\kappa-1)}$. By (9), $b_1(k, t, m) > b_2(k, t, m)$.

Now suppose $m > \kappa t/(3\kappa - 2)$. Then, $x < 3\kappa - 2$. By (9),

$$\begin{aligned} b_1(k,t,m) \sim b_2(k,t,m) \quad &\Leftrightarrow \quad (3\kappa-3)^{\kappa-1} \sim (3\kappa - 2 - x)^{\kappa-1}x \\ \Leftrightarrow \left(\frac{3\kappa - 3}{3\kappa - 2 - x}\right)^{\kappa-1} \sim x \quad &\Leftrightarrow \quad \left(1 + \frac{x-1}{3\kappa-2-x}\right)^{\kappa-1} \sim x. \end{aligned} \tag{10}$$

Let $z = (3\kappa - 2 - x)/(x - 1)$. Let f be as in Lemma 4.3. We have

$$\left(1 + \frac{x-1}{3\kappa-2-x}\right)^{\kappa-1} = \left(1 + \frac{1}{z}\right)^{(z+1)(x-1)/3} = (f(z))^{(x-1)/3} > e^{(x-1)/3}, \tag{11}$$

where the last inequality is given by Lemma 4.3.

By the argument for the function h in the proof of Lemma 4.4, $h(y)$ increases from $h(x_1) = 0$ as y increases from x_1, so $h(y) \geq 0$ for $y \geq x_1$. Thus, $e^{(y-1)/3} \geq y$ for $y \geq x_1$. Together with (10) and (11), this gives us that $b_1(k, t, m) > b_2(k, t, m)$ if $x \geq x_1$, that is, if $m \leq \kappa t/x_1$. Thus, (a) is proved.

We now prove (b). Let f now be as in Lemma 4.4. Let $y_0 = \kappa$. By Lemma 4.4, $f(x_{y_0}, y_0) = 0$ for some unique $x_{y_0} \in (1, x_1)$, and the larger y_0 is, the larger x_{y_0} is. Let $x_0 = x_{y_0}$. Thus, the larger k is, the larger x_0 is. Also, by Lemma 4.4 (b), for any real $\delta > 0$, $x_0 > x_1 - \delta$ if k is sufficiently large. It can be checked that $x_0 = 4$ if $k = 3$, $x_0 = 5.22...$ if $k = 4$, $x_0 = 5.78...$ if $k = 5$, and $x_0 = 6.08...$ if $k = 6$. Thus, $x_0 \geq 4$ as $k \geq 3$. Also, $x_0 \geq 6.08...$ if $k \geq 6$.

Suppose $m > \kappa t/x_0$. Then, $x < x_0$. We have $\left(\frac{3\kappa-3}{3\kappa-2-x}\right)^{\kappa-1} - x = f(x, y_0) < 0$ by Lemma 4.4. By (10), $b_1(k,t,m) < b_2(k,t,m)$. $\square$

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