AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **79(1)** (2021), Pages 106–122

Existence of 3-factors in $K_{1,n}$-free graphs with connectivity and edge-connectivity conditions

Keiko Kotani

Department of Mathematics
Tokyo University of Science Shinjuku-ku
Tokyo 162-8601
Japan
kkotani@rs.tus.ac.jp

Shuto Nishida*

Graduate School of Science
Tokyo University of Science Shinjuku-ku
Tokyo 162-8601
Japan
1418702@ed.tus.ac.jp

Abstract

Let t be an integer satisfying $t \geq 5$. We show that if G is a $\lceil (t-1)/3 \rceil$-connected $K_{1,t}$-free graph of even order with minimum degree at least $\lceil (4t-1)/3 \rceil$, then G has a 3-factor, and if G is a $\lceil (4t-4)/3 \rceil$-connected $K_{1,t}$-free graph of even order, then G has a 3-factor. We also show that if G is a 2-edge-connected $K_{1,4}$-free graph of even order with minimum degree at least 6, then G has a 3-factor.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $\deg_G(x)$ denotes the degree of x in G. We let $\delta(G)$ denote the minimum of $\deg_G(x)$ as x ranges over $V(G)$. For an integer $r \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = r$ for all $x \in V(F)$ is called an r-factor of G. The complete bipartite graph $K_{1,t}$ with partite sets of cardinalities 1 and t is called the t-star. We say that G is $K_{1,t}$-free or t-star-free if G does not contain $K_{1,t}$ as an induced subgraph.

* Corresponding author.

ISSN: 2202-3518

The following theorem was proved by Tokuda and Ota in [4].

Theorem A. *Let t, r be integers with $t \geq 3$ and $r \geq 2$. Let G be a connected $K_{1,t}$-free graph, and suppose that*

$$\delta(G) \geq \left(t + \frac{t-1}{r}\right) \left\lceil \frac{t}{2(t-1)} r \right\rceil - \frac{t-1}{r} \left\lceil \frac{t}{2(t-1)} r \right\rceil^2 + t - 3.$$

In the case where r is odd, suppose further that $t \leq r+1$ and $|V(G)|$ is even. Then G has an r-factor.

In the case where $r = 3$, the minimum degree condition in Theorem A takes the following simple form.

Corollary B. *Let t be 3 or 4. Let G be a connected $K_{1,t}$-free graph with $|V(G)|$ even, and suppose that $\delta(G) \geq 5$ or $\delta(G) \geq 7$ according as $t = 3$ or $t = 4$. Then G has a 3-factor.*

The minimum degree condition in Theorem A is best possible, and hence so are those in Corollary B. On the other hand, if we add the assumption that G is 2-connected, then we can relax the minimum degree condition as is shown in the following two results which were proved in [3].

Theorem C. *Let t be 3 or 4. Let G be a 2-connected $K_{1,t}$-free graph with $|V(G)|$ even and suppose that $\delta(G) \geq t+1$. Then G has a 3-factor.*

Theorem D. *Let t be an integer with $5 \leq t \leq 7$. Let G be a 2-connected $K_{1,t}$-free graph with $|V(G)|$ even and suppose that $\delta(G) \geq t+2$. Then G has a 3-factor.*

In Theorems C and D, the conditions on $\delta(G)$ are best possible. However, it is natural to expect that we can weaken the condition on $\delta(G)$ and the condition on t if we replace the assumption that G is 2-connected by a stronger assumption. Along this line, we show the following results.

Theorem 1. *Let t be an integer with $t \geq 5$. Let G be a $\lceil (t-1)/3 \rceil$-connected $K_{1,t}$-free graph with $|V(G)|$ even and suppose that $\delta(G) \geq \lceil (4t-1)/3 \rceil$. Then G has a 3-factor.*

Theorem 2. *Let t be an integer with $t \geq 5$. Let G be a $\lceil (4t-4)/3 \rceil$-connected $K_{1,t}$-free graph with $|V(G)|$ even. Then G has a 3-factor.*

Note that, since $\lceil (t-1)/3 \rceil = 2$ and $t+2 = \lceil (t-1)/3 \rceil$ for each $5 \leq t \leq 7$, Theorem 1 implies Theorem D.

The minimum degree conditions are best possible in Theorems 1 and 2 in the sense that, for each $t \geq 5$, there exist infinitely many $\lceil (4t-7)/3 \rceil$-connected $K_{1,t}$-free graphs G of even order with $\delta(G) \geq \lceil (4t-4)/3 \rceil$ such that G has no 3-factor (see Example 6.1). In Theorem 1, the connectivity condition is best possible in the sense that, for $t \geq 8$, and for any positive integer δ, there exists a $\lceil (t-4)/3 \rceil$-connected $K_{1,t}$-free graph G of even order with $\delta(G) \geq \delta$ such that G has no 3-factor (see Example 6.2). Further, for $K_{1,3}$-free graphs and for $K_{1,4}$-free graphs, results like Theorems 1 and 2 do not hold because there exist infinitely many 3-connected $K_{1,3}$-free graphs of even order with no 3-factor (see Example 6.3) and there exist infinitely many 4-connected $K_{1,4}$-free graphs of even order with no 3-factor (see Example 6.4).

The following result concerning 2-factors with edge-connectivity conditions was proved [1].

Theorem E. *Let t and k be integers with $t \geq 3$ and $k \geq 2$. Let G be a k-edge-connected $K_{1,t}$-free graph such that $\delta(G) \geq t-2+(t-1)/(k-1)$. In the case where $t=3$ and $k=2$, suppose further that $\delta(G) \geq 4$. Then G has a 2-factor.*

We also show the result on 3-factors which correspond to Theorem E concerning $K_{1,4}$-free graphs.

Theorem 3. *Let G be a 2-edge-connected $K_{1,4}$-free graph with $|V(G)|$ even, and suppose that $\delta(G) \geq 6$. Then G has a 3-factor.*

In Theorem 3, the minimum degree condition is best possible in the sense that, there exist infinitely many 2-edge-connected $K_{1,4}$-free graphs G of even order with $\delta(G) \geq 5$ such that G has no 3-factor (see Example 6.5).

Note that, unlike the case of vertex-connectivity, even if we assume that the edge-connectivity is sufficiently large, $K_{1,5}$-free-ness does not imply the existence of a 3-factor; that is to say, for each $k \geq 2$, there exists a k-edge-connected $K_{1,5}$-free graph of even order with no 3-factor (see Example 6.6).

It is natural to expect that we can weaken the condition on $\delta(G)$ in Theorem 3 if we replace the assumption that G is 2-edge-connected by a stronger edge-connectivity condition. This problem is still open, and the result which correspond to Theorem 3 concerning $K_{1,3}$-free graphs is also still open.

Our notation is standard, and is mostly taken from Diestel [2]. Possible exceptions are as follows. Let G be a graph. For $x \in V(G)$, $N(x) = N_G(x)$ denotes the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For $A \subseteq V(G)$, we let $N(A)$ denote the union of $N(x)$ as x ranges over A. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, $E(A,B)$ denotes the set of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the subgraph induced by A in G is denoted by $G[A]$, and the graph obtained from G by deleting all vertices in A together with the edges incident with them is denoted by $G - A$; thus $G - A = G[V(G) - A]$. We often identify a subgraph H of G with its vertex set; for example, we write $N(H)$ for $N(V(H))$. Also a vertex x of G is often identified with the set $\{x\}$; for example, if H is a subgraph with $x \notin V(H)$, we write $E(x,H)$ for $E(\{x\}, V(H))$.

2 Preliminary results

In this section we state preliminary lemmas, which we use in the proof of Theorems 1, 2 and 3.

Let G be a graph. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, define $\theta(S,T)$ by

$$\theta(S,T) = 3|S| + \sum_{y \in T}(\deg_{G-S}(y) - 3) - h(S,T),$$

where $h(S,T)$ denotes the number of those components C of $G - S - T$ such that $|E(T,C)| + |V(C)|$ is odd. The following lemma is a special case of the f-Factor Theorem of Tutte [5].

Lemma 2.1. (i) *The graph G has a 3-factor if and only if $\theta(S,T) \geq 0$ for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.*

(ii) *If $|V(G)|$ is even, then whether G has a 3-factor or not, $\theta(S,T)$ is even for all S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

The following lemma is well-known, and appears as Lemma 2.2 in [3].

Lemma 2.2. *Let $S, T \subseteq V(G)$ be subsets of $V(G)$ with $S \cap T = \emptyset$ for which $\theta(S,T)$ becomes smallest. Then the following hold.*

(i) *Let C be a component of $G - S - T$ such that $|E(T,C)| \leq 1$. Then $|V(C)| \geq 2$.*

(ii) *Suppose that S and T are chosen with $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. Then $deg_{G[T]}(y) \leq 1$ for every $y \in T$.*

3 Notation

Let $t \geq 3$, $l \geq 1$ and $\delta \geq 3$ be integers, and G be an l-connected $K_{1,t}$-free graph of even order with $\delta(G) \geq \delta$. In this section, we fix notation for the proof of Theorems 1,2 and 3.

Let S, T be subsets of $V(G)$ with $S \cap T = \emptyset$ for which $\theta(S,T)$ becomes smallest. We choose $S, T \subseteq V(G)$ so that $|T|$ is as small as possible, subject to the condition that $\theta(S,T)$ is smallest. If $S \cup T = \emptyset$, then since G is connected and has even order, we get $h(S,T) = 0$, and hence $\theta(S,T) = 0$. Thus we may assume $S \cup T \neq \emptyset$.

Let $C_1, \ldots, C_k$ be the components of $G - S - T$. We may assume that there exists an integer a with $0 \leq a \leq k$ such that $|E(T, C_i)| = 0$ for each $0 \leq i \leq a$, and $|E(T, C_i)| \geq 1$ for each $a+1 \leq i \leq k$. We may further assume that there exists an integer b with $0 \leq b \leq k-a$ such that $|E(T, C_i)| = 1$ for each $a+1 \leq i \leq a+b$, and $|E(T, C_i)| \geq 2$ for each $a+b+1 \leq i \leq k$. Note that if $S \neq \emptyset$ and $|T| + k \leq 1$, then $\sum_{y \in T}(3 - \deg_{G-S}(y)) + h(S,T) \leq 3$, and hence $\theta(S,T) \geq 3|S| - 3 \geq 0$. Thus we may assume that if $S \neq \emptyset$, then we have $|T| + k \geq 2$.

Let $a \geq 1$, and let $1 \leq i \leq a$. By Lemma 2.2 (i), $|V(C_i)| \geq 2$. Recall that we have $S \cup T \neq \emptyset$ by the assumption made in the second paragraph. Since G is connected, $\emptyset \neq N(C_i) \cap (S \cup T) = N(C_i) \cap S$; in particular, $S \neq \emptyset$. By the assumption made at the end of the third paragraph in this section, this implies $|T| + k \geq 2$, and hence $G - S \neq C_i$. Since G is l-connected, $|N(C_i) \cap S| \geq l$. Let $x_i^1, x_i^2, \ldots, x_i^l$ be l distinct vertices in $N(C_i) \cap S$ and let e_i^j $(1 \leq j \leq l)$ be an edge joining x_i^j and a vertex u_i^j in $V(C_i)$. Then

$$|\{e_i^j | 1 \leq i \leq a, 1 \leq j \leq l\}| = la. \tag{3.1}$$

For each $x \in S$, let $L(x) = \{u_i^j \,|\, 1 \leq i \leq a, 1 \leq j \leq l, x_i^j = x\}$. Clearly

$$L(x) \subseteq N(x) \text{ and } L(x) \text{ is independent.} \tag{3.2}$$

Also

$$\sum_{x \in S} |L(x)| = la \tag{3.3}$$

by (3.1). If $a = 0$, we let $L(x) = \emptyset$ for each $x \in S$; thus (3.2) and (3.3) hold in this case as well.

We now look at components of $G[T]$. Let $H_1, ..., H_m$ be the components of $G[T]$. Then

$$T = \bigcup_{1 \leq \mu \leq m} V(H_\mu) \text{ (disjoint union).} \tag{3.4}$$

In the remainder of this section, we assign real numbers θ_μ, θ_μ^1, and θ_μ^2 to each H_μ, and show that $\theta(S,T) \geq \sum_{1 \leq \mu \leq m} \theta_\mu$, $\theta(S,T) \geq \sum_{1 \leq \mu \leq m} \theta_\mu^1$, and $\theta(S,T) \geq \sum_{1 \leq \mu \leq m} \theta_\mu^2$. We first prove several claims concerning H_μ. Note that H_μ is a path of order 1 or 2 by Lemma 2.2 (ii). For each $1 \leq \mu \leq m$, set

$$\begin{aligned}
I_\mu^1 &= \{i |\ a+1 \leq i \leq a+b, E(H_\mu, C_i) \neq \emptyset\}, \\
I_\mu^2 &= \{i |\ a+b+1 \leq i \leq k, E(H_\mu, C_i) \neq \emptyset\}, \\
I_\mu &= I_\mu^1 \cup I_\mu^2, \\
I'_\mu &= I_\mu^1 \cup \{i \in I_\mu^2 |\ |E(H_\mu, C_i)| = 1\}, \mathit{and} \\
q_\mu &= \sum_{y \in V(H_\mu)} \deg_{G-S}(y).
\end{aligned}$$

Claim 3.1. *Let* $1 \leq \mu \leq m$.
(i) *If* $|V(H_\mu)| = 1$, *then* $q_\mu \geq 2|I_\mu| - |I'_\mu|$ *and* $|N(H_\mu) \cap S| \geq \max\{\delta - q_\mu, 0\}$.
(ii) *If* $|V(H_\mu)| = 2$, *then* $q_\mu \geq 2|I_\mu| - |I'_\mu| + 2$ *and* $|N(H_\mu) \cap S| \geq \max\{\delta - \lfloor q_\mu/2 \rfloor, 0\}$.

Proof. This immediately follows from the definition of I_μ, I'_μ and q_μ. □

Let $a+1 \le i \le a+b$. Then there exists μ $(1 \le \mu \le m)$ with $|E(H_\mu, C_i)| = 1$, that is to say, there exists exactly one edge joining $V(H_\mu)$ and $V(C_i)$. Let y_iw_i be such an edge ($y_i \in V(H_\mu)$, $w_i \in V(C_i)$). Set

$$J_1 = \{i \,|\, a+1 \le i \le a+b, \text{ there exists an edge joining } S \text{ and } V(C_i) - \{w_i\}\},$$
$$J_1' = \{i \,|\, a+1 \le i \le a+b, i \notin J_1, \text{ there exists an edge joining } S - N(y_i) \text{ and } \{w_i\}\}.$$

For each $j \in J_1$, let x_ju_j be an edge such that $x_j \in S$ and $u_j \in V(C_j) - \{w_j\}$. For each $j \in J_1'$, let x_ju_j be an edge such that $x_j \in S - N(y_j)$ and $u_j = w_j$. Set

$$J_1(x) = \{u_j \,|\, j \in J_1 \cup J_1', x_j = x\}.$$

Set

$$\begin{aligned} J_2' = \{i \,|\; & a+b+1 \le i \le k, |V(C_i)| \ge 2, \text{ there exists } \mu \text{ with } 1 \le \mu \le m \\ & \text{such that } N(C_i) \cap T \subseteq V(H_\mu) \text{ and } |N(H_\mu) \cap V(C_i)| = 1\}, \end{aligned}$$
$$J_2 = \{i \in J_2' \,|\text{ there exists an edge joining } S \text{ and } V(C_i) - N(T)\}.$$

For each $j \in J_2$, let x_ju_j be an edge that $x_j \in S$ and $u_j \in V(C_j) - N(T)$. For each $x \in S$, set

$$J_2(x) = \{u_j \,|\, j \in J_2, x_j = x\}.$$

Clearly $J_1(x) \cup J_2(x) \subseteq N(x)$. Since u and v belong to distinct components of $G - S - T$ for any $u, v \in L(x) \cup J_1(x) \cup J_2(x)$ with $u \ne v$, this together with (3.2) implies

$$L(x) \cup J_1(x) \cup J_2(x) \subseteq N(x) \text{ and } L(x) \cup J_1(x) \cup J_2(x) \text{ is independent.} \tag{3.5}$$

Also

$$|J_1 \cup J_1'| = |\bigcup_{x \in S} J_1(x)| \text{ (disjoint union) and} \tag{3.6}$$
$$|J_2| = |\bigcup_{x \in S} J_2(x)| \text{ (disjoint union).} \tag{3.7}$$

For each $x \in S$, let $\mathcal{N}(x) = \{\mu | \, 1 \le \mu \le m, x \in N(H_\mu)\}$. For each μ $(1 \le \mu \le m)$, set $\mathcal{H}_\mu = G[V(H_\mu) \cup (\bigcup_{i \in I_\mu^1 - J_1 \cup J_1'} V(C_i))]$. Note that if $I_\mu^1 - J_1 \cup J_1' = \emptyset$, then $\mathcal{H}_\mu = H_\mu$. For each $x \in S$ and for each $\mu \in \mathcal{N}(x)$, we let $\mathcal{J}(x,\mu)$ be a maximal independent set of $N(x) \cap V(\mathcal{H}_\mu)$. If $\mu \notin \mathcal{N}(x)$, let $\mathcal{J}(x,\mu) = \emptyset$. Set $\mathcal{J}(x) = \bigcup_{1 \le \mu \le m} \mathcal{J}(x,\mu)$. If $\mu_1 \ne \mu_2$, then $\mathcal{J}(x,\mu_1) \cap \mathcal{J}(x,\mu_2) = \emptyset$ by the definition of $\mathcal{J}(x,\mu)$. Thus

$$|\mathcal{J}(x)| = \sum_{1 \le \mu \le m} |\mathcal{J}(x,\mu)|. \tag{3.8}$$

Since $|\mathcal{J}(x,\mu)| \ge 1$ for each $x \in N(H_\mu) \cap S$,

$$|N(H_\mu) \cap S| \le \sum_{x \in S} |\mathcal{J}(x,\mu)|. \tag{3.9}$$

Claim 3.2. (i) *For each $x \in S$, $\mathcal{J}(x)$ is independent.*
(ii) *Let $x \in S$. Then $E(u, \mathcal{J}(x,\mu)) = \emptyset$ for any $u \in L(x) \cup J_1(x) \cup J_2(x)$ and for any $\mu \in \mathcal{N}(x)$. In particular, for each $x \in S$, we have $E(u, \mathcal{J}(x)) = \emptyset$ for any $u \in L(x) \cup J_1(x) \cup J_2(x)$.*

Proof. By the definition of $\mathcal{J}(x,\mu)$, for each $x \in S$ and for each μ $(1 \le \mu \le m)$, $\mathcal{J}(x,\mu)$ is independent. Since if $\mu_1 \neq \mu_2$, then $E(\mathcal{H}_{\mu_1}, \mathcal{H}_{\mu_2}) = \emptyset$. In particular, for each $x \in S$, we have $E(\mathcal{J}(x,\mu_1), \mathcal{J}(x,\mu_2)) = \emptyset$ for any μ_1, $\mu_2 \in \mathcal{N}(x)$ with $\mu_1 \neq \mu_2$. Thus (i) holds. The statement (ii) immediately follows from the definitions of $\mathcal{J}(x)$, $L(x)$, $J_1(x)$ and $J_2(x)$. □

Claim 3.3. $(t-1)|S| \ge \sum_{1 \le \mu \le m} \sum_{x \in S} |\mathcal{J}(x,\mu)| + la + |J_1 \cup J_1'| + |J_2|$.

Proof. Since G is $K_{1,t}$-free, it follows from (3.5) and Claim 3.2 that $|\mathcal{J}(x)| + |L(x)| + |J_1(x)| + |J_2(x)| \le t-1$ for every $x \in S$. It follows from (3.3), (3.6), (3.7) and (3.8) that

$$\begin{aligned}(t-1)|S| &\ge \sum_{x \in S} \left(\sum_{1 \le \mu \le m} |\mathcal{J}(x,\mu)| + |L(x)| + |J_1(x)| + |J_2(x)| \right) \\ &= \sum_{x \in S} \sum_{1 \le \mu \le m} |\mathcal{J}(x,\mu)| + \sum_{x \in S} |L(x)| + \sum_{x \in S} |J_1(x)| + \sum_{x \in S} |J_2(x)| \\ &= \sum_{1 \le \mu \le m} \sum_{x \in S} |\mathcal{J}(x,\mu)| + la + |J_1 \cup J_1'| + |J_2|,\end{aligned}$$

as desired. □

Claim 3.4. *Suppose that $t \le 3l+1$. If $T = \emptyset$, then $\theta(S,T) \ge 0$.*

Proof. By Claim 3.3, $|S| \ge la/(t-1) \ge a/3$. If $T = \emptyset$, we have $a = k$, and hence $h(S,T) \le k = a$. Hence $\theta(S,T) \ge 3 \cdot a/3 - a \ge 0$. □

In the rest of this section, we suppose that $t \le 3l+1$. In view of Claim 3.4, we may assume $T \neq \emptyset$. For each μ $(1 \le \mu \le m)$ and for each i $(a+1 \le i \le k)$, we set

$$w(H_\mu, C_i) = \begin{cases} 0 & (N(C_i) \cap V(H_\mu) = \emptyset) \\ 1/2 & (N(C_i) \cap V(H_\mu) \neq \emptyset, N(C_i) \cap T \not\subseteq V(H_\mu)) \\ 1 & (N(C_i) \cap V(H_\mu) \neq \emptyset, N(C_i) \cap T \subseteq V(H_\mu)). \end{cases}$$

Then for each i $(a+1 \le i \le k)$, we have

$$\sum_{1 \le \mu \le m} w(H_\mu, C_i) \ge 1, \tag{3.10}$$

and for each μ $(1 \le \mu \le m)$, we have

$$\sum_{i \in I_\mu} w(H_\mu, C_i) \le |I_\mu|. \tag{3.11}$$

We now estimate $\theta(S,T)$ from below. For each $1 \le \mu \le m$, set

$$\theta_\mu = \frac{3}{t-1}\sum_{x\in S}|\mathcal{J}(x,\mu)| + q_\mu - 3|V(H_\mu)| + \frac{3}{t-1}|I_\mu^1 \cap (J_1 \cup J_1')| + \frac{3}{t-1}|I_\mu^2 \cap J_2| - \sum_{i\in I_\mu} w(H_\mu, C_i),$$

$$\theta_\mu^1 = \frac{3}{t-1}|N(H_\mu)\cap S| + q_\mu - 3|V(H_\mu)| + \frac{3}{t-1}|I_\mu \cap (J_1 \cup J_2)| - \sum_{i\in I_\mu} w(H_\mu, C_i), \textit{and}$$

$$\theta_\mu^2 = \sum_{x\in S}|\mathcal{J}(x,\mu)| + q_\mu - 3|V(H_\mu)| + |I_\mu^1 \cap (J_1 \cup J_1')| - \sum_{i\in I_\mu} w(H_\mu, C_i).$$

Claim 3.5. *Suppose that $t \le 3l+1$. Then* (i) *and* (ii) *hold.*
(i) $\theta(S,T) \ge \sum_{1\le\mu\le m}\theta_\mu^1$.
(ii) *In the case where $t=4$, $\theta(S,T) \ge \sum_{1\le\mu\le m}\theta_\mu^2$.*

Proof. Note that

$$k - a \le \sum_{a+1\le i\le k}\sum_{1\le\mu\le m} w(H_\mu, C_i) = \sum_{1\le\mu\le m}\sum_{a+1\le i\le k} w(H_\mu, C_i) = \sum_{1\le\mu\le m}\sum_{i\in I_\mu} w(H_\mu, C_i)$$

by (3.10). Hence $h(S,T) \le k \le a + \sum_{1\le\mu\le m}\sum_{i\in I_\mu} w(H_\mu, C_i)$. By (3.4),

$$\sum_{y\in T}(\deg_{G-S}(y) - 3) = \sum_{1\le\mu\le m}\left(\sum_{y\in V(H_\mu)}\deg_{G-S}(y) - 3|V(H_\mu)|\right).$$

Therefore it follows from Claim 3.3 that

$$\begin{aligned}
\theta(S,T) &= 3|S| + \sum_{y\in T}(\deg_{G-S}(y) - 3) - h(S,T)\\
&\ge \frac{3}{t-1}\Bigg(\sum_{1\le\mu\le m}\sum_{x\in S}|\mathcal{J}(x,\mu)| + la + |J_1\cup J_1'| + |J_2|\Bigg)\\
&\quad + \sum_{1\le\mu\le m}\left(\sum_{y\in V(H_\mu)}\deg_{G-S}(y) - 3|V(H_\mu)|\right) - (a + \sum_{1\le\mu\le m}\sum_{i\in I_\mu} w(H_\mu, C_i))\\
&\ge \sum_{1\le\mu\le m}\Bigg\{\frac{3}{t-1}\Big(\sum_{x\in S}|\mathcal{J}(x,\mu)| + |I_\mu^1\cap(J_1\cup J_1')| + |I_\mu^2\cap J_2|\Big)\\
&\quad + \sum_{y\in V(H_\mu)}\deg_{G-S}(y) - 3|V(H_\mu)| - \sum_{i\in I_\mu} w(H_\mu, C_i)\Bigg\} + \frac{3}{t-1}la - a\\
&\ge \sum_{1\le\mu\le m}\theta_\mu.
\end{aligned}$$

It follows from (3.9), $|I_\mu^1\cap(J_1\cup J_1')| \ge |I_\mu^1\cap J_1|$ and $|I_\mu^1\cap J_1| + |I_\mu^2\cap J_2| = |I_\mu\cap(J_1\cup J_2)|$ that $\theta_\mu \ge \theta_\mu^1$ for each μ $(1\le\mu\le m)$, and hence (i) holds. In the case that $t=4$, we immediately have $\theta_\mu \ge \theta_\mu^2$ for each μ $(1\le\mu\le m)$, and hence (ii) holds. ☐

4 Proofs of Theorems 1 and 2

Let G be an l-connected $K_{1,t}$-free graph with $\delta(G) \geq \delta$. We continue with the notation of the proceeding section with $t \geq 5$ and $l \geq 2$. Thus, in this section, we suppose that the connectivity of G is at least 2. First we prove the following technical claim.

Claim 4.1. *Suppose that $l \geq 2$, and let $1 \leq \mu \leq m$.*
(i) *If $t \geq 7$, then $\sum_{i \in I_\mu} w(H_\mu, C_i) - 3|I_\mu \cap (J_1 \cup J_2)|/(t-1) \leq |I_\mu| - 3|I'_\mu|/(t-1)$.*
(ii) *If $t \leq 6$, then $\sum_{i \in I_\mu} w(H_\mu, C_i) - 3|I_\mu \cap (J_1 \cup J_2)|/(t-1) \leq |I_\mu| - |I'_\mu|/2$.*

Proof. Let $i \in I'_\mu$. First assume that $i \in I^1_\mu$. Then, since $|V(C_i)| \geq 2$ by Lemma 2.2(i) and G is 2-connected, there exists an edge joining S and $V(C_i) - N(H_\mu)$, and hence $i \in J_1$ by the definition of J_1, which implies

$$w(H_\mu, C_i) - \frac{3}{t-1}|\{i\} \cap J_1| \leq 1 - \frac{3}{t-1}. \tag{4.1}$$

Next assume that $i \in \{j \in I^2_\mu \mid |E(H_\mu, C_j)| = 1\}$. Then $N(C_i) \cap T \not\subseteq V(H_\mu)$, and hence $w(H_\mu, C_i) = 1/2$. Therefore

$$\begin{aligned}
&\sum_{i \in I_\mu} w(H_\mu, C_i) - \frac{3}{t-1}|I_\mu \cap (J_1 \cup J_2)| \\
&\leq \sum_{i \in I_\mu - I'_\mu} w(H_\mu, C_i) + \sum_{i \in I'_\mu} w(H_\mu, C_i) - \frac{3}{t-1}|I'_\mu \cap J_1| \\
&= \sum_{i \in I_\mu - I'_\mu} w(H_\mu, C_i) + \sum_{i \in I'_\mu} \left(w(H_\mu, C_i) - \frac{3}{t-1}|\{i\} \cap J_1| \right) \\
&\leq |I_\mu - I'_\mu| + \max\left\{ \left(1 - \frac{3}{t-1}\right), \frac{1}{2} \right\} |I'_\mu| \\
&= |I_\mu| - \min\left\{ \frac{3}{t-1}, \frac{1}{2} \right\} |I'_\mu|,
\end{aligned}$$

which immediately implies (i) and (ii). □

In order to complete the proofs of Theorems 1 and 2, we prove the following three propositions.

Proposition 4.1. *Suppose that $t \geq 5$, $l \geq 2$, $\delta \geq \lceil (4t-4)/3 \rceil$ and $|V(H_\mu)| = 1$. Then $\theta^1_\mu \geq 0$.*

Proof. First we assume $t \geq 7$. It follows from Claims 3.1(i) and 4.1(i) and $|I_\mu| \geq |I'_\mu|$

that

$$\begin{aligned}\theta^1_\mu &\geq \frac{3}{t-1}\left(\left\lceil\frac{4t-4}{3}\right\rceil - q_\mu\right) + q_\mu - 3 - |I_\mu| + \frac{3}{t-1}|I'_\mu| \\ &\geq 1 + \frac{t-4}{t-1}(2|I_\mu| - |I'_\mu|) - |I_\mu| + \frac{3}{t-1}|I'_\mu| \\ &= 1 + \frac{t-7}{t-1}(|I_\mu| - |I'_\mu|) \geq 0.\end{aligned}$$

Next we assume $t = 5$ or 6. It follows from Claims 3.1(i) and 4.1(ii) that

$$\begin{aligned}\theta^1_\mu &\geq \frac{3}{t-1}(\delta - q_\mu) + q_\mu - 3 - |I_\mu| + \frac{1}{2}|I'_\mu| \\ &\geq \frac{3}{t-1}\delta - 3 + \frac{t-4}{t-1}(2|I_\mu| - |I'_\mu|) - |I_\mu| + \frac{1}{2}|I'_\mu| \\ &= \frac{3}{t-1}\delta - 3 - \frac{7-t}{t-1}|I_\mu| + \frac{7-t}{2(t-1)}|I'_\mu|.\end{aligned}$$

Assume for the moment $t = 6$. Then $\delta \geq 7$. Moreover, since G is $K_{1,6}$-free, $|I_\mu| \leq 5$. Hense $\theta^1_\mu \geq (3/5)\cdot 7 - 3 - (1/5)\cdot 5 = 1/5 > 0$. Assume now $t = 5$. Then $\delta \geq 6$. Since G is $K_{1,5}$-free, $|I_\mu| \leq 4$. If $|I_\mu| \leq 3$, then $\theta^1_\mu \geq (3/4)\cdot 6 - 3 - (2/4)\cdot 3 = 0$. If $|I_\mu| = 4$ and $|I'_\mu| \geq 2$, then $\theta^1_\mu \geq (3/4)\cdot 6 - 3 - (2/4)\cdot 4 + (2/8)\cdot 4 = 0$. Thus we may assume that $|I_\mu| = 4$ and $|I'_\mu| \leq 1$. Since $|N(H_\mu) \cap S| \geq 0$,

$$\begin{aligned}\theta^1_\mu &\geq q_\mu - 3 + \frac{3}{4}|I_\mu \cap (J_1 \cup J_2)| - \sum_{i \in I_\mu} w(H_\mu, C_i) \\ &\geq 2|I_\mu| - |I'_\mu| - 3 - |I_\mu| + \frac{1}{2}|I'_\mu| > 0,\end{aligned}$$

which completes the proof of Proposition 4.1. □

Proposition 4.2. *Suppose that* $t \geq 5$, $l \geq 2$, $\delta \geq \lceil (4t-4)/3 \rceil$, $|V(H_\mu)| = 2$ *and* $|I_\mu| \neq 0$. *Then* $\theta^1_\mu \geq 0$.

Proof. First we assume that $t \geq 7$. Assume for the moment that $|I_\mu| \geq 2$. Then, it follows from Claims 3.1(ii) and 4.1(i), and $|I'_\mu| \leq |I_\mu|$ that

$$\begin{aligned}\theta^1_\mu &\geq \frac{3}{t-1}\left(\frac{4t-4}{3} - \left\lfloor\frac{q_\mu}{2}\right\rfloor\right) + q_\mu - 6 - |I_\mu| + \frac{3}{t-1}|I'_\mu| \\ &\geq \frac{2t-5}{2(t-1)}q_\mu - |I_\mu| + \frac{3}{t-1}|I'_\mu| - 2 \\ &\geq \frac{2t-5}{2(t-1)}(2|I_\mu| - |I'_\mu| + 2) - |I_\mu| + \frac{3}{t-1}|I'_\mu| - 2 \\ &\geq -\frac{3}{t-1} + \frac{3}{2(t-1)}|I_\mu| \geq 0.\end{aligned}$$

Assume now $|I_\mu| = 1$. Then $q_\mu \ge 3$. In the case where $|I_\mu| = 1$ and $q_\mu \ge 4$,

$$\begin{aligned}\theta^1_\mu &\ge \frac{3}{t-1}\left(\frac{4t-4}{3} - \left\lfloor \frac{q_\mu}{2} \right\rfloor\right) + q_\mu - 6 - |I_\mu| + \frac{3}{t-1}|I'_\mu| \\ &\ge \frac{2t-5}{2(t-1)} \cdot 4 - 3 \ge 0.\end{aligned}$$

In the case where $|I_\mu| = 1$ and $q_\mu = 3$, since $|I'_\mu| = 1$,

$$\begin{aligned}\theta^1_\mu &\ge \frac{3}{t-1}\left(\frac{4t-4}{3} - \left\lfloor \frac{q_\mu}{2} \right\rfloor\right) + q_\mu - 6 - |I_\mu| + \frac{3}{t-1}|I'_\mu| \\ &\ge \frac{3}{t-1}\left(\frac{4t-4}{3} - 1\right) + 3 - 6 - 1 + \frac{3}{t-1} = 0.\end{aligned}$$

Next we assume $t = 5$ or 6. Note that, if $t = 5$, then $\delta \ge 6$, and if $t = 6$, then $\delta \ge 7$; that is, $\delta \ge t + 1$. Assume for the moment that $|I_\mu| \ge 2$. Then, it follows from Claims 3.1(ii) and 4.1(ii), and $|I'_\mu| \le |I_\mu|$ that

$$\begin{aligned}\theta^1_\mu &\ge \frac{3}{t-1}\left(t + 1 - \left\lfloor \frac{q_\mu}{2} \right\rfloor\right) + q_\mu - 6 - |I_\mu| + \frac{1}{2}|I'_\mu| \\ &\ge \frac{3(t+1)}{t-1} + \frac{2t-5}{2(t-1)}(2|I_\mu| - |I'_\mu| + 2) - 6 - |I_\mu| + \frac{1}{2}|I'_\mu| \\ &\ge -\frac{t-4}{t-1} + \frac{t-4}{2(t-1)}|I_\mu| \ge 0.\end{aligned}$$

Assume now $|I_\mu| = 1$. Then $q_\mu \ge 3$. In the case where $|I_\mu| = 1$ and $q_\mu \ge 4$, it follows from Claim 4.1(ii) that $\theta^1_\mu \ge 3(t+1)/(t-1) + (2t-5)q_\mu/(2t-2) - 6 - |I_\mu| + |I'_\mu|/2 \ge 0$. In the case where $|I_\mu| = 1$ and $q_\mu = 3$, since $|I'_\mu| = 1$, $\theta^1_\mu \ge (7-t)/(2t-2) > 0$, which completes the proof of Proposition 4.2. □

Proposition 4.3. *Suppose that $t \ge 5$, $l \ge 2$, $\delta \ge \lceil (4t-1)/3 \rceil$ and $|V(H_\mu)| = 2$. Then $\theta^1_\mu \ge 0$.*

Proof. Keeping Proposition 4.2 in mind, we may assume $|I_\mu| = 0$, and hence $q_\mu = 2$. It follows from Claim 3.1(ii) that $\theta^1_\mu \ge (3/(t-1)) \cdot ((4t-1)/3 - q_\mu/2) + q_\mu - 6 \ge 0$, which completes the proof of Proposition 4.3. □

We are now in a position to complete the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let t, G be as in Theorem 1; thus $t \ge 5$ and G be a $\lceil (t-1)/3 \rceil$-connected $K_{1,t}$-free graph with $\delta(G) \ge \lceil (4t-1)/3 \rceil$. Let l be the connectivity of G. Then $l \ge \lceil (t-1)/3 \rceil$, and hence $t \le 3l + 1$. If $T = \emptyset$, then $\theta(S,T) \ge 0$ by Claim 3.4. Thus we may assume $T \ne \emptyset$. By Claim 3.5(i), it suffices to show that $\theta^1_\mu \ge 0$ for each $1 \le \mu \le m$. If $|V(H_\mu)| = 1$, $\theta^1_\mu \ge 0$ by Proposition 4.1. If $|V(H_\mu)| = 2$, $\theta^1_\mu \ge 0$ by Proposition 4.3. This completes the proof of Theorem 1 by Lemma 2.2(ii).

Proof of Theorem 2. Let t, G be as in Theorem 2; thus $t \geq 5$ and G be a $\lceil (4t-4)/3 \rceil$-connected $K_{1,t}$-free graph. Thus $\delta(G) \geq \lceil (4t-4)/3 \rceil$. Let l be the connectivity of G. Then $l \geq \lceil (4t-4)/3 \rceil$, and hence $t \leq (3l+4)/4 < 3l+1$. If $T = \emptyset$, then $\theta(S,T) \geq 0$ by Claim 3.4. Thus we may assume $T \neq \emptyset$. By Claim 3.5(i), it suffices to show that $\theta^1_\mu \geq 0$ for each $1 \leq \mu \leq m$. If $|V(H_\mu)| = 1$, $\theta^1_\mu \geq 0$ by Proposition 4.1. If $|V(H_\mu)| = 2$ and $|I_\mu| \neq 0$, $\theta^1_\mu \geq 0$ by Proposition 4.2. Thus we may assume that $|V(H_\mu)| = 2$ and $|I_\mu| = 0$. If $|V(G)| = l+1 \geq 7$, then G is the complete graph, and hence G has a 3-factor. Thus, we may assume that $|V(G)| \geq l+2$. Suppose that $|N(H_\mu) \cap S| < l$. Then $|V(G) - V(H_\mu) - \big(N(H_\mu) \cap S\big)| \geq 1$, and hence $G - \big(N(H_\mu) \cap S\big)$ is disconnected, which contradicts G is l-connected. Hence we have $|N(H_\mu) \cap S| \geq l$. Then

$$\theta^1_\mu = 3|N(H_\mu) \cap S|/(t-1) + 2 - 6 \geq 3l/(t-1) + 2 - 6 \geq 0;$$

this together with Propositions 4.1 and 4.2, completes the proof of Theorem 2.

5 Proof of Theorem 3

Let G be as in Theorem 3; thus G is a 2-edge-connected $K_{1,4}$-free graph with $\delta(G) \geq 6$. We continue with the notation of Section 3 with $t = 4$, $l = 1$, and $\delta = 6$.

Recall that $\theta^2_\mu = \sum_{x \in S} |\mathcal{J}(x,\mu)| + q_\mu - 3|V(H_\mu)| + |I^1_\mu \cap (J_1 \cup J'_1)| - \sum_{i \in I_\mu} w(H_\mu, C_i)$. In view of Claim 3.5(ii), it suffices to show that $\theta^2_\mu \geq 0$ for each $1 \leq \mu \leq m$. We divide the proof into the following two cases.

Case 1. $|V(H_\mu)| = 1$.

Since G is $K_{1,4}$-free, $|I_\mu| \leq 3$, this together with (3.9), (3.11) and Claim 3.1(i) implies $\theta^2_\mu \geq |N(H_\mu) \cap S| + q_\mu - 3 - |I_\mu| \geq 6 - q_\mu + q_\mu - 3 - 3 = 0$.

Case 2. $|V(H_\mu)| = 2$.

Having the definition of I^1_μ in mind, since G is $K_{1,4}$-free,

$$|I^1_\mu| \leq 4. \tag{5.1}$$

By the definition of q_μ, I^1_μ, I^2_μ, and I'_μ, we have

$$q_\mu \geq |I^1_\mu| + 2|I^2_\mu| - |I^2_\mu \cap I'_\mu| + 2. \tag{5.2}$$

By the definition of $w(H_\mu, C_i)$, I^1_μ, I^2_μ, and I'_μ, we also have

$$\sum_{i \in I_\mu} w(H_\mu, C_i) \leq |I^1_\mu| + |I^2_\mu| - \frac{|I^2_\mu \cap I'_\mu|}{2}. \tag{5.3}$$

If $|N(H_\mu) \cap S| \geq 4$, it follows from (3.9), (5.2) and (5.3) that

$$\theta^2_\mu \geq |I^2_\mu| - |I^2_\mu \cap I'_\mu|/2 + |I^1_\mu \cap (J_1 \cup J'_1)| \geq 0.$$

Thus we may assume that

$$|N(H_\mu) \cap S| \leq 3. \tag{5.4}$$

It follows from Claim 3.1(ii), (5.2) and (5.3) that

$$\begin{aligned}
&|N(H_\mu)\cap S| + q_\mu - 3|V(H_\mu)| - w(H_\mu, C_i) \\
&\geq \delta - \left\lfloor \frac{q_\mu}{2} \right\rfloor + q_\mu - 3|V(H_\mu)| - \left(|I_\mu^1| + |I_\mu^2| - \frac{|I_\mu^2 \cap I'_\mu|}{2} \right) \\
&\geq 6 + \frac{|I_\mu^1| + 2|I_\mu^2| - |I_\mu^2 \cap I'_\mu| + 2}{2} - 6 - \left(|I_\mu^1| + |I_\mu^2| - \frac{|I_\mu^2 \cap I'_\mu|}{2} \right) \\
&\geq -\frac{|I_\mu^1|}{2} + 1. \qquad (5.5)
\end{aligned}$$

Suppose that $\sum_{x\in S} |\mathcal{J}(x,\mu)| \geq |N(H_\mu)\cap S| + 1$ or $|I_\mu^1 \cap (J_1 \cup J'_1)| \geq 1$. Then it follows from (5.1) and (5.5) that

$$\begin{aligned}
\theta_\mu^2 &\geq |N(H_\mu)\cap S| + 1 + q_\mu - 3|V(H_\mu)| - w(H_\mu, C_i) \\
&\geq -\frac{|I_\mu^1|}{2} + 2 \geq 0.
\end{aligned}$$

Suppose that $|I_\mu^1| \leq 2$. Then it follows from (3.9) and (5.5) that

$$\begin{aligned}
\theta_\mu^2 &\geq |N(H_\mu)\cap S| + q_\mu - 3|V(H_\mu)| - w(H_\mu, C_i) \\
&\geq -\frac{|I_\mu^1|}{2} + 1 \geq 0.
\end{aligned}$$

Thus we may assume that

$$\sum_{x\in S} |\mathcal{J}(x,\mu)| = |N(H_\mu)\cap S|, \qquad (5.6)$$

$$|I_\mu^1 \cap (J_1 \cup J'_1)| = 0, \quad \text{and} \qquad (5.7)$$

$$|I_\mu^1| = 3 \text{ or } 4. \qquad (5.8)$$

Let $i \in I_\mu^1$. By the definition of I_μ^1, we may write $E(H_\mu, C_i) = \{yz_i\}$ ($y \in V(H_\mu)$, $z_i \in V(C_i)$). Since G is 2-edge-connected and $|I_\mu^1 \cap (J_1 \cup J'_1)| = 0$, there exists $x \in N(H_\mu)\cap S$ such that $x \in N(y)\cap N(z_i)$, say x_i. Since $i \in I_\mu^1$ is arbitrary, $|I_\mu^1| \leq \sum_{x\in N(H_\mu)\cap S} |\mathcal{J}(x,\mu)| = \sum_{x\in S} |\mathcal{J}(x,\mu)|$. Hence it follows from (5.4), (5.6) and (5.8) that $|I'_\mu| = 3$. If $x_i = x_{i'}$ for $i, i' \in I_\mu^1$ ($i \neq i'$) then

$$\begin{aligned}
\sum_{x\in S} |\mathcal{J}(x,\mu)| &= \sum_{x\in S - x_i} |\mathcal{J}(x,\mu)| + |\mathcal{J}(x_i,\mu)| \\
&\geq |N(H_\mu)\cap (S - x_i)| + |\mathcal{J}(x_i,\mu)| \\
&\geq |N(H_\mu)\cap (S - x_i)| + 2 = |N(H_\mu)\cap S| + 1,
\end{aligned}$$

which contradicts (5.6). Thus for each $i, i' \in I_\mu^1$ ($i \neq i'$), $x_i \neq x_{i'}$. Set $I_\mu = \{i_1, i_2, i_3\}$, and $V(H_\mu) = \{y_1, y_2\}$. Then $N(H_\mu)\cap S = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. Since G is $K_{1,4}$-free, we may assume that $|E(y_1, C_{i_1})| = |E(y_1, C_{i_2})| = 1$, $|E(y_1, C_{i_3})| = 0$, $|E(y_2, C_{i_1})| = |E(y_2, C_{i_2})| = 0$ and $|E(y_2, C_{i_3})| = 1$. Let y_1z_1, y_1z_2, $y_2z_3 \in E(G)$ ($z_1 \in V(C_{i_1})$,

$z_2 \in V(C_{i_2})$, $z_3 \in V(C_{i_3})$), and let $x_3 \in N(y_2) \cap N(z_3)$. Since $|E(y_1, C_{i_3})| = 0$, $y_1z_3 \notin E(G)$. Since $N(y_1) - S = \{y_2, z_1, z_2\}$ and $\deg(y_1) \geq \delta = 6$, $|N(y_1) \cap S| \geq 3$; this together with $|N(H_\mu) \cap S| = 3$ implies $x_3 \in N(y_1)$. Hence $|\mathcal{J}(x_3, \mu)| \geq |\{y_1, z_3\}|$. Consequently $\sum_{x\in S} |\mathcal{J}(x, \mu)| \geq \sum_{x\in S-x_3} |\mathcal{J}(x, \mu)| + |\mathcal{J}(x_3, \mu)| = |N(H_\mu) \cap S| + 1$, which contradicts (5.6).

6 Examples

In this section, we construct examples which show that the conditions in Theorems 1, 2 and 3 are best possible.

Example 6.1. Let $t \geq 5$ be an integer. There exist infinitely many $\lceil (4t-7)/3 \rceil$-connected $K_{1,t}$-free graphs G of even order with $\delta(G) \geq \lceil (4t-4)/3 \rceil$ such that G has no 3-factor. Let $m \geq t$ be an arbitrary integer relatively prime to $t-1$. Set $l = \lceil (4t-7)/3 \rceil$. Let $I_1, I_2, \ldots, I_{2m}$ be disjoint copies of the complete graph of order $\lceil l/2 \rceil$, and let $J_1, J_2, \ldots, J_{2m}$ be disjoint copies of the complete graph of order $\lfloor l/2 \rfloor$, and let $H_1, H_2, \ldots, H_{2m(t-1)}$ be disjoint copies of the complete graph of order 2. For each $1 \leq k \leq 2m$, set

$$T_k = \bigcup_{1\leq j\leq t-1} V(H_{(k-1)(t-1)+j}),$$
$$T'_k = \bigcup_{1\leq j\leq t-1} V(H_{(j-1)2m+k}).$$

Now define a graph G by

$$V(G) = \left(\bigcup_{1\leq k\leq 2m} (V(I_k) \cup V(J_k))\right) \cup \left(\bigcup_{1\leq i\leq 2m(t-1)} V(H_i)\right),$$
$$E(G) = \left(\bigcup_{1\leq k\leq 2m} (E(I_k) \cup E(J_k)) \cup \{xy | x \in V(I_k), y \in T_k\} \cup \{xy | x \in V(J_k), y \in T'_k\}\right) \cup \left(\bigcup_{1\leq i\leq 2m(t-1)} E(H_i)\right).$$

Then G is $\lceil (4t-7)/3 \rceil$-connected and $K_{1,t}$-free, and satisfies $\delta(G) = l+1 = \lceil (4t-4)/3 \rceil$. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 with $S = \bigcup_{1\leq k\leq 2m}(V(I_k) \cup V(J_k))$ and $T = \bigcup_{1\leq i\leq 2m(t-1)} V(H_i)$, then we get $\theta(S,T) \leq -2m$).

Example 6.2. Let $t \geq 8$ be an integer. For any positive integer δ, there exists a $\lceil (t-4)/3 \rceil$-connected $K_{1,t}$-free graph G of even order with $\delta(G) \geq \delta$ such that G has no 3-factor. Let $m \geq t$ be an arbitrary integer relatively prime to $t-1$, and set $l = \lceil (t-4)/3 \rceil$. Let $I_1, I_2, \ldots, I_{2m}$ be disjoint copies of the complete graph of order

$\lceil l/2 \rceil$, and let $J_1, J_2, \ldots, J_{2m}$ be disjoint copies of the complete graph of order $\lfloor l/2 \rfloor$. Let p be an odd integer with $p \geq \delta - l + 1$, and let $C_1, \ldots, C_{2m(t-1)}$ be disjoint copies of the complete graph of order p. For each $1 \leq k \leq 2m$, set

$$T_k = \bigcup_{1 \leq j \leq t-1} V(C_{(k-1)(t-1)+j}),$$
$$T'_k = \bigcup_{1 \leq j \leq t-1} V(C_{(j-1)2m+k}).$$

Now define a graph G by

$$V(G) = \left(\bigcup_{1 \leq k \leq 2m} (V(I_k) \cup V(J_k)) \right) \cup \left(\bigcup_{1 \leq i \leq 2m(t-1)} V(C_i) \right),$$
$$E(G) = \left(\bigcup_{1 \leq k \leq 2m} E(I_k) \cup E(J_k) \cup \{xy | x \in V(I_k), y \in T_k\} \cup \{xy | x \in V(J_k), y \in T'_k\} \right)$$
$$\cup \left(\bigcup_{1 \leq i \leq 2m(t-1)} E(C_i) \right).$$

Then G is l-connected and $K_{1,t}$-free, and satisfies $\delta(G) = p - 1 + l \geq \delta$. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 in Section 2 with $S = \bigcup_{1 \leq k \leq 2m}(V(I_k) \cup V(J_k))$ and $T = \emptyset$, then we get $\theta(S,T) \leq -2m$).

Example 6.3. There exist infinitely many 3-connected $K_{1,3}$-free graphs of even order with no 3-factor. Let $m \geq 2$ be an even integer. Let $I_1, I_2, \ldots, I_m$ be disjoint copies of the complete graph of order 1, and set $V(I_k) = \{x_k\}$ $(1 \leq k \leq m)$. Let $H_1, H_2, \ldots, H_{2m}$ disjoint copies of the complete graph of order 2, and set $V(H_i) = \{y_i, y'_i\}$ $(1 \leq i \leq 2m)$. Let L, L' be disjoint copies of the complete graph of order $2m$, and set $V(L) = \{z_1, z_2, \ldots, z_{2m}\}$ and $V(L') = \{z'_1, z'_2, \ldots, z'_{2m}\}$. Now define a graph G of order $9m$ by

$$V(G) = \left(\bigcup_{1 \leq k \leq m} V(I_k) \right) \cup \left(\bigcup_{1 \leq i \leq 2m} V(H_k) \right) \cup V(L) \cup V(L'),$$
$$E(G) = \left(\bigcup_{1 \leq k \leq m} \{x_k y \mid y \in V(H_{2k-1}) \cup V(H_{2k})\} \right)$$
$$\cup \left(\bigcup_{1 \leq i \leq 2m} \{y_i y'_i, y_i z_i, y'_i z'_i\} \right) \cup E(L) \cup E(L').$$

Then G is a 3-connected $K_{1,3}$-free graph of even order. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 in Section 2 with $S = \bigcup_{1 \leq k \leq m} V(I_k)$ and $T = \bigcup_{1 \leq i \leq 2m} V(H_i)$, then we get $\theta(S,T) = -m$).

Example 6.4. There exist infinitely many 4-connected $K_{1,4}$-free graphs of even order with no 3-factor. Let $m \geq 2$ be an arbitrary integer. Let $I_1, I_2, \ldots, I_m$ be disjoint copies of the complete graph of order 2. Let $H_1, H_2, \ldots, H_{3m}$ disjoint copies of the complete graph of order 2, and set $V(H_i) = \{y_i, y_i'\}$ $(1 \leq i \leq 2m)$. Let L, L' be disjoint copies of the complete graph of order $3m+1$, and set $V(L) = \{z_1, z_2, \ldots, z_{3m+1}\}$ and $V(L') = \{z_1', z_2', \ldots, z_{3m+1}'\}$. Now define a graph G of order $14m+2$ by

$$V(G) = \left(\bigcup_{1\leq k\leq m} V(I_k)\right) \cup \left(\bigcup_{1\leq i\leq 3m} V(H_i)\right) \cup V(L) \cup V(L')$$

$$E(G) = \left(\bigcup_{1\leq k\leq m} E(I_k) \cup \{xy \mid x \in V(I_k), y \in V(H_{3k-2}) \cup V(H_{3k-1}) \cup V(H_{3k})\}\right)$$
$$\cup \left(\bigcup_{1\leq i\leq 3m} \{y_i y_i', y_i z_i, y_i' z_i'\}\right) \cup E(L) \cup E(L').$$

Then G is a 4-connected $K_{1,4}$-free graph of even order. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 in Section 2 with $S = \bigcup_{1\leq k\leq m} V(I_k)$ and $T = \bigcup_{1\leq i\leq 3m} V(H_i)$, then we get $\theta(S,T) = -2$).

Example 6.5. There exist infinitely many 2-edge-connected $K_{1,4}$-free graphs of even order satisfies $\delta(G) \geq 5$ with no 3-factor. Let $p_1 \geq 7$ be an odd integer, and let $p_2 \geq 6$ be an even integer. Let C_1, $C_2, \ldots, C_8$ be disjoint copies of the complete graph of order p_1, and let D_1, $D_2, \ldots, D_7$ be disjoint copies of the complete graph of order p_2. For each C_i $(1 \leq i \leq 8)$, take two vertices c_i^1, $c_i^2 \in V(C_i)$. For each D_i $(1 \leq i \leq 7)$, take one vertex $d_i \in V(D_i)$. We define the graph of order $8p_1 + 7p_2 + 8$ by

$$V(G) = \{x_1, x_2\} \cup \{y_1, y_2, y_3, y_4, y_5, y_6\} \cup \bigcup_{1\leq i\leq 8} V(C_i) \cup \bigcup_{1\leq i\leq 7} V(D_i)$$

$$E(G) = \{x_1 y_i, x_1 d_i \mid i = 1, 2, 3\} \cup \{x_2 y_i, x_2 d_i \mid i = 4, 5, 6\}$$
$$\cup \{y_i c_i^1, y_i c_i^2, y_i d_i \mid 1 \leq i \leq 6\} \cup \{y_1 c_7^1, y_4 c_7^2, y_2 c_8^1, y_5 c_8^2\} \cup \{y_3 d_7, y_6 d_7\}$$
$$\cup \bigcup_{1\leq i\leq 8} E(C_i) \cup \bigcup_{1\leq i\leq 7} E(D_i).$$

Then G is a 2-edge-connected $K_{1,4}$-free graph, and satisfies $\delta(G) = 5$. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 in Section 2 with $S = \{x_1, x_2\}$ and $T = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, then we get $\theta(S,T) = -2$).

Example 6.6. For each $k \geq 2$, there exists a k-edge-connected $K_{1,5}$-free graph of even order with no 3-factor. Let $k \geq 2$ and $s \geq \frac{k-1}{2}$ be integers. Let I and J be the complete graphs of order k and 2, respectively. For each $v \in V(I)$, let C_v^1, C_v^2, C_v^3 be disjoint copies of complete graphs of order $2s+1$. For each C_v^i, take k distinct vertices $z_v^i(1)$, $z_v^i(2), \ldots, z_v^i(k)$ from $V(C_v^i)$. Let G be a graph of order $(6s+4)k+2$

by

$$
\begin{aligned}
V(G) =& V(J) \cup V(I) \cup \bigcup_{v\in V(I)} \Big(\bigcup_{i=1}^{3} V(C_v^i)\Big),\\
E(G) =& E(J) \cup E(I) \cup \bigcup_{v\in V(I)} \Big(\bigcup_{i=1}^{3} E(C_v^i)\Big)\\
& \cup \{xy \mid x \in V(J), y \in V(I)\}\\
& \cup \bigcup_{v\in V(I)} \Big(\bigcup_{i=1}^{3} \{vz_v^i(1), vz_v^i(2), \ldots, vz_v^i(k)\}\Big).
\end{aligned}
$$

Then G is a k-edge-connected $K_{1,5}$-free graph of even order. However, we easily see that G does not have a 3-factor (for example, if we apply Lemma 2.1 with $S = V(I)$ and $T = V(J)$, then we get $\theta(S,T) = -4$).

References

[1] R. E. L. Aldred, Y. Egawa, J. Fujisawa, K. Ota and A. Saito, The existence of a 2-factor in $K_{1,n}$-free graphs with large connectivity and large edge-connectivity, *J. Graph Theory* **68** (2011), 77–89.

[2] R. Diestel, “Graph Theory”, fifth ed., Graduate Texts in Mathematics, vol. 173, Springer (2010).

[3] Y. Egawa, K. Kotani and T. Yashima, Existence of 3-Factors in 2-connected Star-free Graphs, *Far East J. Appl. Math.* **79(2)** (2013), 127–150.

[4] K. Ota and T. Tokuda, A degree condition for the existence of regular factors in $K_{1,n}$-free graphs, *J. Graph Theory* **22** (1996), 59–64.

[5] W. T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952), 314–328.

(Received 19 Mar 2020; revised 18 Oct 2020)