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A note on coloring of $\frac{3}{3}$-power of subquartic graphs

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Abstract

For any $k \in \mathbb{N}$, the k-subdivision of a graph G is a simple graph $G^{\frac{1}{k}}$, which is constructed by replacing each edge of G with a path of length k. In [M.N. Iradmusa, *Discrete Math.* 310 (10-11) (2010), 1551–1556], the mth power of the n-subdivision of G was introduced as a fractional power of G, denoted by $G^{\frac{m}{n}}$. F. Wang and X. Liu in [*Discrete Math. Algorithms Appl.* 10 (3), (2018), 1850041] showed that $\chi(G^{\frac{3}{3}}) \leq 7$ for any subcubic graph G. In this note, we prove that the $\frac{3}{3}$-power of every subquartic graph admits a proper coloring with at most nine colors. We conjecture that $\chi(G^{\frac{3}{3}}) \leq 2\Delta(G) + 1$ for any graph G with maximum degree $\Delta(G) \geq 2$.

1 Introduction

All graphs we consider in this note are simple, finite and undirected. We mention some of the definitions that are referred to throughout this note, and for other necessary definitions and notation we refer the reader to a standard text-book [2]. Minimum degree, maximum degree and maximum size of cliques of a graph G are denoted by $\delta(G)$, $\Delta(G)$ and $\omega(G)$, respectively. For each vertex $v \in V(G)$, the neighbors of v are the vertices adjacent to v in G. The neighborhood $N_G(v)$ of v is the set of all neighbors of v in G. Any vertex of degree k is called a *k-vertex* and any path of length k is called a *k-path.* Also, a path $P : v_1, \ldots, v_k$ is a *simple path* if the degrees of all vertices $v_2, \ldots, v_{k-1}$ are 2, and a cycle C is called a *loop cycle* if all of its vertices are 2-vertices except one that is a 3-vertex.

Let G be a graph and k be a positive integer. The *k-power of G*, denoted by G^k, is defined on the vertex set $V(G)$ by adding edges joining any two distinct vertices x and y with distance at most k. Also the *k-subdivision of G*, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge xy of G with a path of length k, say P_{xy}. These k-paths are called *superedges.* We denote a vertex by $(xy)_l$ if it belongs to P_{xy} and has distance l from the vertex x, where $l \in \{0, 1, 2, \ldots, k\}$. Note that

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$(xy)_l = (yx)_{k-l}$, $x = (xy)_0 = (yx)_k$ and $y = (yx)_0 = (xy)_k$. Any vertex $(xy)_0$ of $G^{\frac{1}{k}}$ is called a *terminal vertex* (or briefly t-vertex) and any of the remaining vertices is called an *internal vertex* (or briefly i-vertex). The fractional power of graphs was first introduced in [5] as follows.

Definition 1.1 Let G be a graph and $m, n \in \mathbb{N}$. The graph $G^{\frac{m}{n}}$ is defined to be the m-power of the n-subdivision of G. In other words $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$.

We denote the set of terminal vertices of $G^{\frac{m}{n}}$ by $V_t(G^{\frac{m}{n}})$ and the set of internal vertices by $V_i(G^{\frac{m}{n}})$. It is worth noting that $G^{\frac{1}{1}} = G$ and $G^{\frac{2}{2}} = T(G)$, where $T(G)$, the total graph of G, is the the graph whose vertex set is $V(G) \cup E(G)$, in which two vertices are adjacent if and only if they are adjacent or incident in G [1].

As usual, a proper k-coloring of G is a mapping from $V(G)$ to $\{1, \dots, k\}$, where any two adjacent vertices have distinct colors. The chromatic number of a graph G is the minimum integer k for which G has a proper k-coloring, and is denoted by $\chi(G)$. By the definition of a total graph, $\chi''(G) := \chi(T(G)) = \chi(G^{\frac{2}{2}})$. In 1965, Behzad [1] conjectured that $\chi''(G)$ never exceeds $\Delta(G) + 2$. By virtue of Definition 1.1, one can show that $\omega(G^{\frac{2}{2}}) = \Delta(G) + 1$ and the Total Coloring Conjecture can be reformulated as follows.

Conjecture 1.2 *For any simple graph G,* $\chi(G^{\frac{2}{2}}) \leq \omega(G^{\frac{2}{2}}) + 1$.

There is a relation between an incidence coloring of a graph G and a vertex coloring of $G^{\frac{3}{3}}$, which is one of the motivations of this note. The concept of incidence coloring was introduced by Brualdi and Massey in 1993.

Definition 1.3 [3] Let $G = (V, E)$ be a multigraph. An incidence of G is a pair (v, e) where $v \in V(G)$, $e \in E(G)$ and e is incident with v. Let I be the set of incidences of G. An incidence graph of G, denoted by $I(G)$, has its vertex set $V(I(G)) = I$ such that two incidences (v, e) and (w, f) are adjacent in $I(G)$ if one of the following holds:

(1) $v = w$,

(2) $e = f$,

(3) the end-vertices of e or f are v and w.

Definition 1.4 [3] An incidence coloring σ of G is a map from I to the color set C such that adjacent incidence pairs are assigned different colors. If $\sigma : I \longrightarrow C$ is an incidence coloring with $|C| = k$, then we say that σ is a k-incidence coloring of G. The incidence chromatic number of G, denoted $\chi_i(G)$, is the smallest k for which there exists a k-incidence coloring of G.

In [3], it is proved that for a graph G with maximum degree Δ, $\chi_i(G) \leq 2\Delta$. Also, by definition of the fractional power of a graph, the incidence graph $I(G)$ is the subgraph of $G^{\frac{3}{3}}$ induced by the set of internal vertices. So we have $\chi_i(G) = \chi(G^{\frac{3}{3}}[V_i(G^{\frac{3}{3}})]) \leq \chi(G^{\frac{3}{3}})$. In addition, the partition $\{V_t(G^{\frac{3}{3}}), V_i(G^{\frac{3}{3}})\}$ of the vertices of $G^{\frac{3}{3}}$ implies that

$$\chi(G^{\frac{3}{3}}) \leq \chi(G^{\frac{3}{3}}[V_t(G^{\frac{3}{3}})]) + \chi(G^{\frac{3}{3}}[V_i(G^{\frac{3}{3}})]) = \chi(G) + \chi_i(G).$$

Also, in [6] it was proved that if $\Delta(G) \geq 3$, then $\chi(G^{\frac{3}{3}}) \leq \chi(G) + \chi_i(G) - 1$.

In this note, we are investigating the chromatic number of $G^{\frac{3}{3}}$. When $\Delta(G) = 1$, one can easily show that $\chi(G^{\frac{3}{3}}) = 4$, and by applying the following theorem, which was proved in [5], we can prove that $\chi(G^{\frac{3}{3}}) \leq 5$ for any graph G with $\Delta(G) = 2$.

Theorem 1.5 *If* $m, n, k \in \mathbb{N}$ *and* $k \geq 3$, *then*
(i) $\chi(C_k^{\frac{m}{n}}) = \begin{cases} nk & m \geq \frac{nk}{2} \\ \lceil \frac{nk}{\lfloor \frac{nk}{m+1} \rfloor} \rceil & m < \frac{nk}{2}, \end{cases}$
(ii) $\chi(P_k^{\frac{m}{n}}) = \min\{m+1, (k-1)n+1\}$.

In [8], Wang and Liu proved that $\chi(G^{\frac{3}{3}}) \leq 7$ for any subcubic graph G. Recall that a graph G is subcubic if $\Delta(G) \leq 3$. Recently, a simple proof of this result was given in [6] by using the following theorem about the 5-colorability of the incidences of any subcubic graph.

Theorem 1.6 [7] *For any subcubic graph G, we have* $\chi_i(G) \leq 5$.

The graph G is called *subquartic* if $\Delta(G) \leq 4$. Also any 4-regular graph is known as a *quartic* graph. The main theorem of this note is stated as follows.

Theorem 1.7 *Let G be a subquartic graph. Then* $\chi(G^{\frac{3}{3}}) \leq 9$.

Remark 1.8 In [3], it was conjectured that $\chi_i(G) \leq \Delta(G) + 2$ for every graph G. This was disproved by Guiduli in [4] who showed that Paley graphs with sufficiently large maximum degree have incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. However, this conjecture seems to hold for graphs with small maximum degree. Theorem 1.6 shows that the conjecture is true for cubic graphs. It remains an open problem whether the conjecture is true for quartic graphs. But if the conjecture holds for all quartic graphs, then easily we can prove Theorem 1.7 by use of the inequality $\chi(G^{\frac{3}{3}}) \leq \chi(G) + \chi_i(G) - 1$. So Theorem 1.7 provides evidence that the conjecture may be true for all quartic graphs.

Considering the results for cycles, paths, subcubic and subquartic graphs, we conjecture that $2\Delta(G) + 1$ colors suffice for the proper coloring of $G^{\frac{3}{3}}$ when G is a graph with maximum degree at least two.

Conjecture 1.9 *Let G be a graph with* $\Delta(G) \geq 2$. *Then* $\chi(G^{\frac{3}{3}}) \leq 2\Delta(G) + 1$.

The clique number of the fractional power of a graph was obtained in [5] for powers less than one. As mentioned in [8], one can easily show that $\omega(G^{\frac{3}{3}}) = \Delta(G)+2$ when $\Delta(G) \geq 2$, and $\omega(G^{\frac{3}{3}}) = 4$ when $\Delta(G) = 1$. Therefore, if Conjecture 1.2 holds, we conclude that $\chi''(G) = \chi(G^{\frac{2}{2}}) \leq \chi(G^{\frac{3}{3}})$. So, the following conjecture seems strongly true.

Conjecture 1.10 *For any graph G, $\chi(G^{\frac{2}{2}}) \leq \chi(G^{\frac{3}{3}})$.*

2 Proof of Theorem 1.7

For convenience, we need some notation and preliminaries.

Let G be a graph and consider $G^{\frac{3}{3}}$. On each superedge P_{uv} there are two internal vertices $(uv)_1$ and $(uv)_2$ which correspond to the incidences of the edge uv. We denote $(uv)_1$ and $(uv)_2$ by (u,v) and (v,u), respectively.

We need to establish the following lemmas before the proof of Theorem 1.7.

Lemma 2.1 *Let $P_n : v_1, v_2, \ldots, v_n$ be a path of order $n \geq 5$, $n \neq 6, 7$ and the vertices v_1, v_2, v_{n-1} and v_n are respectively colored with the colors a, b, c and d from the set $C = \{1, 2, \ldots, 5\}$ that are all distinct except possibly a and d. Then we can extend this partial coloring to a proper coloring of P_n^3 with colors from C.*

Proof To extend this partial coloring to a proper coloring of P_n^3, we consider 8 cases:

(1) $a = d$, $n \equiv 1 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abec)(abec)(abec)\cdots$ where $e \in C \setminus \{a, b, c\}$.

(2) $a = d$, $n \equiv 2 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abefc)(abec)(abec)\cdots$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

(3) $a = d$, $n \equiv 3 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abefc)(abefc)(abec)(abec)\cdots$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

(4) $a = d$, $n \equiv 0 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abce)(abce)\cdots(abce)(fbca)$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

(5) $a \neq d$, $n \equiv 1 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abecd)(abcd)(abcd)\cdots$ where $e \in C \setminus \{a, b, c, d\}$.

(6) $a \neq d$, $n \equiv 2 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abecd)(abecd)(abcd)(abcd)\cdots$ where $e \in C \setminus \{a, b, c, d\}$.

(7) $a \neq d$, $n \equiv 3 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abde)(abde)\cdots(abde)(acd)$ where $e \in C \setminus \{a, b, c, d\}$.

(8) $a \neq d$, $n \equiv 0 \pmod 4$: Starting from v_1, we color the vertices of P_n^3 sequentially by colors $(abcd)(abcd)(abcd)\cdots$.

Each of these colorings preserves the colors of v_1, v_2, v_{n-1} and v_n and it can be easily seen that the given coloring is a proper coloring of P_n^3 with color set C. □

Lemma 2.2 *Let G be a subcubic graph with $\delta(G) \geq 2$ and*

$$V_3 = \{v \in V(G) \mid d_G(v) = 3\}.$$

Then $\chi(G^{\frac{3}{3}} \setminus V_3) \leq 5$.

Proof Each connected component of G is a cycle or a subcubic graph in which any 2-vertex belongs to a simple path or a loop cycle. Let H be a connected component of G. If H is a cycle, then by Theorem 1.5, we can color the vertices of $H^{\frac{3}{3}} \setminus V_3 = H^{\frac{3}{3}}$ with the colors $C = \{1, 2, 3, 4, 5\}$. Now let H be a component of the second type in G. To find a proper coloring for $H^{\frac{3}{3}} \setminus V_3$, at first we identify all 2-vertices lying in simple paths and remove all 2-vertices of the loop cycles in H. Let H_1 be the resulting graph. By Theorem 1.6, $\chi(H_1^{\frac{3}{3}}[V_i(H_1^{\frac{3}{3}})]) = \chi_i(H_1) \leq 5$. Suppose that $c : V_i(H_1^{\frac{3}{3}}) \longrightarrow C$ is a proper coloring of $H_1^{\frac{3}{3}}[V_i(H_1^{\frac{3}{3}})]$. Suppose the simple path $P : v_1, v_2, \ldots, v_n$ in H. The vertices $v_2, \ldots, v_{n-1}$ of P contracted to a single vertex v^* in H_1. The subgraph of $H^{\frac{3}{3}} \setminus V_3$ induced by

$$V_P = \{(v_1, v_2), (v_2, v_1), v_2, (v_2, v_3), (v_3, v_2), v_3, \ldots, v_{n-1}, (v_{n-1}, v_n), (v_n, v_{n-1})\}$$

is isomorphic to P^3_{3n-4}. Now we color the first two vertices and the last two vertices of V_P as follows:

$$c(v_1, v_2) = c(v_1, v^*), c(v_2, v_1) = c(v^*, v_1),$$
$$c(v_n, v_{n-1}) = c(v_n, v^*), c(v_{n-1}, v_n) = c(v^*, v_n).$$

Because $5 \leq 3n - 4 \neq 6, 7$, by Lemma 2.1 we can extend c to the other vertices of $P^{\frac{3}{3}}$ except v_1 and v_n and similarly, to all vertices of the other simple paths. We denote by c' this extension of c.

Finally, we color the vertices of the loop cycles. Let $L : v_1, v_2, \ldots, v_n, v_1$ be a loop cycle of G such that $d_G(v_1) = 3$ and $N_G(v_1) = \{v_0, v_2, v_n\}$. Suppose that $c'((v_0, v_1)) = a$ and $c'((v_1, v_0)) = b$. Because $L^{\frac{3}{3}}$ is isomorphic to $C_n^{\frac{3}{3}}$, by Theorem 1.5, we have $\chi(L^{\frac{3}{3}}) \leq 5$. Let c_L be a proper coloring of $L^{\frac{3}{3}}$ such that $c_L(v_1) = b$, $c_L((v_1, v_n)) = c$ and $c_L((v_1, v_2)) = d$ such that $\{c, d\} \subset C \setminus \{a, b\}$. Therefore, $c_L((v_1, v_2)) \neq a = c'((v_0, v_1)) \neq c_L((v_1, v_n))$. Because c_L is a proper coloring, $c_L((v_2, v_1)) \neq b = c'((v_1, v_0)) \neq c_L((v_n, v_1))$. Now we delete the color of v_1 and add this coloring of $\frac{3}{3}$-power of L to c'. By repeating this method for each loop cycle, c' gives rise to a proper coloring of $G^{\frac{3}{3}} \setminus V_3$. □

Proof of Theorem 1.7. Since each subquartic graph is a subgraph of a quartic graph, we only prove the theorem for quartic graphs. Let G be a quartic graph. If G is a complete graph, then $G = K_5$. In Figure 1, a proper 7-coloring of ${K_5}^{\frac{3}{3}}$ is shown. Some edges of ${K_5}^{\frac{3}{3}}$ were removed in the figure for simplicity. Now, suppose that G

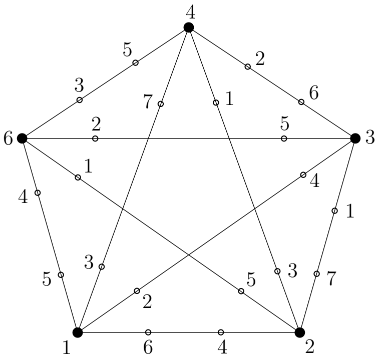

Figure 1: 7-proper coloring of $K_5^{\frac{3}{3}}$.

is not a complete graph and $M_1 = \{e_1, \ldots, e_k\}$ is a maximum matching. Since M_1 is a maximum matching of G, $A = V(G) \setminus V(M_1)$ is an independent set of G and so $N_G(v) \subseteq V(M_1)$ for each $v \in A$.

Consider the bipartite subgraph H with bipartition $(A, V(M_1))$ that contains all edges between A and $V(M_1)$. Since $d_H(v) = 4 > d_H(u)$ for any vertex $v \in A$ and any vertex $u \in V(M_1)$, by Hall's Theorem, H has a matching, named M_2, which covers all the vertices of A. Without loss of generality, suppose that $M_2 = \{f_1, \ldots, f_{k'}\}$. Note that none of the edges in M_1 is adjacent to two edges of M_2; otherwise, M_1 is not a maximum matching of G.

Consider the subgraph F of G induced by $M_1 \cup M_2$ and let $B_1 = V(F^{\frac{3}{3}}) \setminus (V(M_1) \cap V(M_2))$ and $B_2 = V(G^{\frac{3}{3}}) \setminus B_1$. Now consider the partition $\{B_1, B_2\}$ of $V(G^{\frac{3}{3}})$. We prove that $\chi(G^{\frac{3}{3}}[B_1]) \leq 4$ and $\chi(G^{\frac{3}{3}}[B_2]) \leq 5$ and then we conclude that $\chi(G^{\frac{3}{3}}) \leq \chi(G^{\frac{3}{3}}[B_1]) + \chi(G^{\frac{3}{3}}[B_2]) \leq 9$.

Since G is neither a complete graph nor an odd cycle, by Brooks' Theorem, $\chi(G) \leq 4$. In addition, the subgraph of $G^{\frac{3}{3}}$ induced by the t-vertices is isomorphic to G. Therefore, there exists a proper coloring c of the t-vertices of $G^{\frac{3}{3}}[B_1]$ with colors in $\{1, 2, 3, 4\}$. Now, we color the i-vertices of $G^{\frac{3}{3}}[B_1]$ with colors in $\{1, 2, 3, 4\}$. For any edge $e = uv \in M_1$ that is not adjacent with any edge of M_2, color i-vertices (u, v) and (v, u) with two different colors of $\{1, 2, 3, 4\} \setminus \{c(u), c(v)\}$. Now, suppose that $e = uv \in M_1$ is adjacent with an edge $f = vw \in M_2$. If $c(u) \neq c(w)$, then color the i-vertices (u, v) and (w, v) with a same color from $\{1, 2, 3, 4\} \setminus \{c(u), c(w)\}$. Also, assign colors $c(u)$ and $c(w)$ to i-vertices (v, w) and (v, u), respectively. If $c(u) = c(w)$, color the i-vertices (u, v) and (w, v) with a same color from $\{1, 2, 3, 4\} \setminus \{c(u)\}$ and then assign two different colors in $\{1, 2, 3, 4\} \setminus \{c(u), c((u, v))\}$ to the i-vertices (v, w) and (v, u). Therefore $\chi(G^{\frac{3}{3}}[B_1]) \leq 4$.

To find a proper 5-coloring of $G^{\frac{3}{3}}[B_2]$, we apply Lemma 2.2. Let G_1 be the spanning subgraph of G with edge set $E(G_1) = E(G) \setminus (M_1 \cup M_2)$ and V_3 be the set of 3-vertices of G_1. Because $d_{G_1}(v) = 2$ for any vertex $v \in V(M_1) \cap V(M_2)$ and $d_{G_1}(v) = 3$ for any vertex $v \notin V(M_1) \cap V(M_2)$, we have $\delta(G_1) \geq 2$, $\Delta(G_1) = 3$ and $G_1^{\frac{3}{3}} \setminus V_3 = G^{\frac{3}{3}}[B_2]$. Therefore, by Lemma 2.2, we have $\chi(G^{\frac{3}{3}}[B_2]) = \chi(G_1^{\frac{3}{3}} \setminus V_3) \leq 5$, which completes the proof. □

Problem 1 We did not find a subquartic graph G with $\chi(G^{\frac{3}{3}}) = 9$. Therefore, it remains an open problem as to whether the upper bound in Theorem 1.7 can be decreased to 8 colors.

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