2-perfect closed m-trail systems of the complete directed graph with loops

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Dedicated to the memory of Alan Rahilly, 1947–1992

Abstract. Certain decompositions of complete directed graphs with loops into collections of closed trails which partition the edge set of the graph give rise to, and arise from, quasigroups. Such decompositions are said to be 2-perfect. The existence of these 2-perfect decompositions in which the closed trails are all of the same length m is examined. In particular, the set of values of n for which the order n complete directed graph with loops can be decomposed into 2-perfect closed trails of length m is determined (with four possible exceptions; two in the case $m = 5$ and two in the case $m = 14$) for all $m \leq 15$.

1. Introduction

A *quasigroup* $(Q, *)$ is usually defined as a set Q together with a binary operation $*$ (called *multiplication*) which satisfy the conditions that for any $a, b \in Q$, the equations

$$a * x = b \quad \text{and} \quad y * a = b$$

have unique solutions for x and y in Q. It is well known and easy to show that the multiplication table of a quasigroup is a *latin square* (each row and each column contains every element exactly once).

It is sometimes useful to introduce two further binary operations, *right division* $/$, and *left division* $\backslash$. These are defined by letting b/a denote the unique y satisfying $y * a = b$, and $a \backslash b$ denote the unique x satisfying $a * x = b$. The three binary operations satisfy the following identities:

(1) $(x * y)/y = x$;
(2) $(x/y) * y = x$;
(3) $y\backslash(y * x) = x$;
(4) $y * (y\backslash x) = x$.

Conversely, if an algebra $\mathbf{Q} = \langle Q, *, /, \backslash\rangle$ satisfies the above identities, then $(Q, *)$ is a quasigroup. Identities (2) and (4) give existence of solutions of the equations, and identities (1) and (3) give uniqueness of these solutions (see [4]). In this paper we will be looking at a class of graph decompositions which can be used to construct, or be constructed from, quasigroups.

A *trail* in a graph is a walk with no repeated edges, and a *cycle* is a closed trail with no repeated vertices; see [**2**]. A great deal of work has been done on decompositions of complete undirected graphs into cycles so that the edge set of the graph is partitioned; see [**5**]. Analogous decompositions of directed graphs (sometimes referred to as Mendelsohn designs) have also attracted much attention, see [**1**] for example. More recently, decompositions of complete directed graphs with loops into collections of closed trails which partition the edge set of the graph have been studied; see [**3**]. The presence of loops necessitates decomposing into closed trails instead of cycles, because repeated vertices cannot be avoided.

Definition 1.1. A *closed trail system* of a graph G is a pair (V, T) where V is the vertex set of the graph and T is a set of closed trails with the property that each edge of G occurs exactly once in T. If all the trails in a closed trail system of G are of length m, the system is called a *closed m-trail system of G*.

A closed m-trail which contains the edges $x_1x_2, x_2x_3, \ldots, x_mx_1$ is written as a cyclically ordered m-tuple $(x_1, x_2, \ldots, x_m)$ of vertices. □

It has been shown (see [**3**]) that there exists a closed m-trail system of L_n (the complete directed graph with loops on each of its n vertices) if and only if m divides n^2. In [**3**] it was also shown that closed trail systems of L_n are in one-one correspondence with groupoids of order n whose multiplication table is a column-latin square. The correspondence is obtained as follows:

(1) Given a closed trail system (V, T) of L_n we define $*$ on V by $a * b = c$ if and only if the edge ab is immediately followed by the edge bc in T.
(2) Given a groupoid $(G, *)$ of order n whose multiplication table is a column-latin square we construct a closed trail system (G, T) of L_n by stipulating that in T the edge ab is followed by the edge $b(a * b)$.

It is not difficult to see that these groupoids are quasigroups if and only if the closed trail systems have the property described in the following definition.

Definition 1.2. Let (V,T) be a closed trail system of a graph G, and let $T(2)$ be the set of trails obtained by replacing each closed trail of T by its corresponding distance 2 graph. This can be done by replacing the trail $(x_1, x_2, \ldots, x_m)$ by $(x_1, x_3, \ldots, x_m, x_2, x_4, \ldots, x_{m-1})$ if m is odd, or by the trails $(x_1, x_3, \ldots, x_{m-1})$ and $(x_2, x_4, \ldots, x_m)$ if m is even. If $(V, T(2))$ is again a closed trail system of G, then (V,T) is said to be *2-perfect*. □

In the remainder of the paper we examine 2-perfect closed trail systems of L_n. We will determine precisely (with four possible exceptions; two in the case $m = 5$ and two in the case $m = 14$) for all $m \leq 15$, the set $S(m)$ of values of n for which there exist 2-perfect closed m-trail systems of L_n.

2. Some Examples

First, we introduce some notation for the graphs we will be considering.

Definition 2.1.

(1) Let the graph with vertex set $V = \{1,2,\ldots,9\} \times \{1,2,\ldots,k\}$ and edge set

$$E = \{(x_1,y_1)(x_2,y_2) | (x_1,y_1),(x_2,y_2) \in V \text{ and } x_1 \neq x_2\}$$

be denoted by $M_{9,k}$.

(2) Let V and H be sets with $H \subseteq V$, $|V| = n_1$ and $|H| = n_2$, then the graph with vertex set V and edge set

$$E = \{xy | x,y \in V \text{ and } \{x,y\} \not\subseteq H\}$$

is denoted by $L_{n_1} \setminus L_{n_2}$. □

Next, we present a table outlining the graphs for which we require 2-perfect closed m-trail systems. These systems are needed in Section 4, and either appear in the Appendix, or exist by Lemma 2.4.

Table 2.2.

Case	*Graphs*
m=5	L_5, L_{15}, L_{30}
m=6	L_6
m=7	L_7, L_{14}
m=8	L_4, L_8
m=9	L_6, L_9, L_{15}, L_{21}
	$L_{12} \setminus L_3$, $L_{15} \setminus L_6$
m=10	L_{10}, L_{20}
m=11	L_{11}, L_{22}
m=12	L_6
m=13	L_{13}, L_{26}
m=14	L_{14}
m=15	L_{15}

Definition 2.3. Let $\mathbf{G}_n$ be the groupoid obtained by defining a binary operation $*$ on the set of elements of $\mathbb{Z}_n$ by $a * b = 2b - a + 1 \pmod{n}$. □

Lemma 2.4. *If n is odd then $\mathbf{G}_n$ corresponds to a 2-perfect closed n-trail system of L_n.*

Proof. It is convenient to introduce the following notation. Given a groupoid $(V, *)$ whose multiplication table is a column-latin square we define a sequence of words $W_i(x, y)$ by

$$W_0(x,y) = x$$
$$W_1(x,y) = y$$
$$W_2(x,y) = x * y$$
$$W_3(x,y) = y * (x * y)$$

and inductively define

$$W_i(x,y) = W_{i-2}(x,y) * W_{i-1}(x,y).$$

Let (V, T) be the closed trail system corresponding to $(V, *)$, $a, b \in V$, and let τ be the trail of T which contains the edge ab. Then, starting at the vertex a of the edge ab, and counting t vertices to the right in τ, leads to the vertex $W_t(a, b)$. Clearly, if t is such that $W_t(a,b) = a$ and $W_{t+1}(a,b) = b$, then the length of the trail containing the edge ab divides t and is equal to the smallest such t ($t > 0$).

Let $a, b \in \mathbb{Z}_n$. Let $x = \frac{b+a-1}{2}$ then

$$a * x = 2\frac{b+a-1}{2} - a + 1 = b + a - 1 - a + 1 = b.$$

Hence x is a solution to the equation $a * x = b$. If x_1 and x_2 are any two solutions to the equation then $2x_1 - a + 1 = 2x_2 - a + 1 \pmod{n}$, and so $2x_1 = 2x_2 \pmod{n}$. Thus, since n is odd, $x_1 = x_2 \pmod{n}$. Hence x is a unique solution.

Let $y = 2a - b + 1$ then

$$y * a = 2a - (2a - b + 1) + 1 = b.$$

Hence y is a solution to the equation $y * a = b$. If y_1 and y_2 are any two solutions to the equation then $2a - y_1 + 1 = 2a - y_2 + 1 \pmod{n}$, so $y_1 = y_2 \pmod{n}$. Hence y is a unique solution and $\mathbf{G}_n$ is a quasigroup of order n.

Now, it is easy to show (by induction) that for any x, y in $\mathbf{G}_n$ and any non-negative integer s, $W_s(x, y) = x + s(y - x) + \frac{s(s-1)}{2} \pmod{n}$.

Now, let $\mathbf{t} \in T$ where $(\{0, 1, 2, \ldots n-1\}, T)$ is the 2-perfect closed trail system corresponding to $\mathbf{G}_n$, let ab be any edge in $\mathbf{t}$ and suppose $\mathbf{t}$ has length l. Then $W_l(a, b) = a$ and $W_{l+1}(a, b) = b$. Since $W_l(a, b) = a$,

$$a + l(b - a) + \frac{l(l-1)}{2} = a \pmod{n},$$

so

$$l(b - a) + \frac{l(l-1)}{2} = 0 \pmod{n}$$

and then

$$2l(b - a) + l(l-1) = 0 \pmod{2n}.$$

Since $W_{l+1}(a, b) = b$,

$$a + (l+1)(b - a) + \frac{l(l+1)}{2} = b \pmod{n}$$

so

$$b + l(b - a) + \frac{l(l+1)}{2} = b \pmod{n},$$

$$l(b - a) + \frac{l(l+1)}{2} = 0 \pmod{n},$$

and thus

$$2l(b - a) + l(l+1) = 0 \pmod{2n}.$$

Hence, $l(l-1) = l(l+1) \pmod{2n}$. That is, $2l = 0 \pmod{2n}$, and so $l = 0 \pmod{n}$.

Now, for any a and b,

$$W_n(a, b) = a + n(b - a) + \frac{n(n-1)}{2} = a + \frac{n(n-1)}{2} \pmod{n}$$

and

$$W_{n+1}(a, b) = a + (n+1)(b - a) + \frac{(n+1)n}{2} = b + \frac{(n+1)n}{2} \pmod{n}$$

Hence, if n is odd then for any a and b, $W_n(a, b) = a$ and $W_{n+1}(a, b) = b$ so $\mathbf{G}_n$ corresponds to a 2-perfect closed n-trail system of L_n. □

3. The kn Construction

In what follows we denote the entry in row i and column j of a rectangular array A by $a(i,j)$.

Definition 3.1. Two columns j_1 and j_2 in a k^2 by m rectangular array with entries chosen from $K = \{1,2,\ldots,k\}$ are *orthogonal* if

$$\{(a(i,j_1),a(i,j_2)) \mid 1 \le i \le k^2\} = \{(a(i,j_2),a(i,j_1)) \mid 1 \le i \le k^2\} = K \times K.$$

□

Definition 3.2. In a k^2 by m rectangular array, columns j and $(j+1)$ for $j = 1,2,\ldots,m-1$, as well as columns 1 and m, are said to be *adjacent*. Two columns are *near adjacent* if they are distinct and they have a common adjacent column. □

Definition 3.3. A (k,m)-*[1,2]-Orthogonal Array* (or (k,m)-[1,2]-OA) is a k^2 by m rectangular array with entries chosen from $K = \{1,2,\ldots k\}$ and with the property that any pair of adjacent or near adjacent columns is orthogonal. □

We state a few well-known results concerning the number of mutually orthogonal latin squares of side k.

(1) For each positive integer k, there exists a latin square of side k.
(2) For each positive integer k, except $k = 2$ and 6, there exists a pair of orthogonal latin squares of side k, see [**6**], for example.
(3) For each positive integer k, except $k = 2,3,6$ and 10, there exists a set of 3 mutually orthogonal latin squares of side k; see [**7,8**].

Lemma 3.4. *For $k \ne 2,3,6$ or 10, there exists a $(k,5)$-[1,2]-OA.*

Proof. Since $k \ne 2,3,6$ or 10, there exists a set of 3 mutually orthogonal latin squares of side k, and hence a set of three mutually orthogonal quasigroups $(K,*_1)$, $(K,*_2)$ and $(K,*_3)$. The required array is obtained as follows:

(1) let columns 1 and 2 be any pair of orthogonal columns;
(2) for $i = 1,2,\ldots,k^2$, let $a(i,3) = a(i,1) *_1 a(i,2)$;
(3) for $i = 1,2,\ldots,k^2$, let $a(i,4) = a(i,1) *_2 a(i,2)$;
(4) for $i = 1,2,\ldots,k^2$, let $a(i,5) = a(i,1) *_3 a(i,2)$.

□

Lemma 3.5. *For all positive integers k and all $m \equiv 0 \pmod 3$, there exists a (k,m)-[1,2]-OA.*

Proof. Let $(K, *)$ be a quasigroup. The required array is obtained as follows:

(1) let columns 1 and 2 be any pair of orthogonal columns;
(2) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 1 \pmod 3$ with $j \leq m-2$ let $a(i,j) = a(i,1)$;
(3) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 2 \pmod 3$ with $j \leq m-1$ let $a(i,j) = a(i,2)$;
(4) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 0 \pmod 3$ with $j \leq m$ let $a(i,j) = a(i,1) * a(i,2)$.

□

Lemma 3.6. *For all $k \neq 2$ or 6 and for all $m \equiv 1 \pmod 3$ $(m \geq 4)$ there exists a (k,m)-[1,2]-OA.*

Proof. Since $k \neq 2$ or 6, there exists a pair of orthogonal latin squares of side k and hence a pair of orthogonal quasigroups $(K, *_1)$ and $(K, *_2)$. The required array is obtained as follows:

(1) let columns 1 and 2 be any pair of orthogonal columns;
(2) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 1 \pmod 3$ with $j \leq m-3$ let $a(i,j) = a(i,1)$;
(3) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 2 \pmod 3$ with $j \leq m-2$ let $a(i,j) = a(i,2)$;
(4) for $i = 1, 2, \ldots, k^2$ and for all $j \equiv 0 \pmod 3$ with $j \leq m-1$ let $a(i,j) = a(i,1) *_1 a(i,2)$;
(5) for $i = 1, 2, \ldots k^2$ let $a(i,m) = a(i,1) *_2 a(i,2)$.

□

Lemma 3.7. *For all $k \neq 2$ or 6 and for all $m \equiv 2 \pmod 3$ $(m \geq 8)$ there exists a (k,m)-[1,2]-OA.*

Proof. Since $k \neq 2$ or 6, there exists a pair of orthogonal latin squares of side k and hence a pair of orthogonal quasigroups $(K, *_1)$ and $(K, *_2)$. The required array is obtained as follows:

(1) let columns 1 and 2 be any pair of orthogonal columns;
(2) for $i = 1, 2, \ldots, k^2$ for $j = m-3$ and for all $j \equiv 1 \pmod 3$ with $j \leq m-7$ let $a(i,j) = a(i,1)$;
(3) for $i = 1, 2, \ldots, k^2$ for $j = m-2$ and for all $j \equiv 2 \pmod 3$ with $j \leq m-6$ let $a(i,j) = a(i,2)$;
(4) for $i = 1, 2, \ldots, k^2$ for $j = m-1$ and for all $j \equiv 0 \pmod 3$ with $j \leq m-5$ let $a(i,j) = a(i,1) *_1 a(i,2)$;
(5) for $i = 1, 2, \ldots k^2$ let $a(i,m) = a(i,m-4) = a(i,1) *_2 a(i,2)$.

Theorem 3.8. *If $n \in S(m)$ and there exists a (k,m)-[1,2]-OA then $kn \in S(m)$.*

Proof. Let (V,T) be a 2-perfect closed m-trail system of L_n and $K = \{1,2,\ldots,k\}$. We construct a 2-perfect closed m-trail system $(V \times K, T')$ of L_{kn} as follows. For each $(x_1, x_2, \ldots, x_m) \in T$ and each row $[y_1, y_2, \ldots, y_m]$ of a (k,m)-[1,2]-OA let

$$((x_1,y_1),(x_2,y_2),\ldots,(x_m,y_m)) \in T'.$$

□

4. The Main Results

A necessary condition for the existence of a 2-perfect closed m-trail system of L_n is that m divides n^2, since the number of edges in L_n is n^2 and each m-circuit contains m edges. Clearly $S(1) = \{1\}$. There can't be a closed 2-trail containing a loop precisely once, as the closed trail (a,a) contains the loop aa twice, and trails (a,a,b, ...) have length at least 3. Hence $S(2) = \emptyset$.

Any closed 3-trail (V,T) of L_n is necessarily 2-perfect since for any $a,b \in V$, the edge ab occurs in the distance 2 graph of the closed 3-trail in T which contains the edge ba. Hence, $S(3) = \{n | n \equiv 0(\text{mod } 3)$. If a loop aa is contained in the distance 2 graph of a closed 4-trail then it immediately occurs twice , and hence $S(4) = \emptyset$.

Summarising the above two paragraphs gives:

(1) $S(1) = \{1\}$;
(2) $S(2) = \emptyset$;
(3) $S(3) = \{n | n \equiv 0(\text{mod } 3)\}$;
(4) $S(4) = \emptyset$.

Theorem 4.1. *$S(5) = \{n | n \equiv 0(\mathit{mod}\, 5)\}$, with the possible exceptions $n = 10$ and $n = 50$.*

Proof. There are 2-perfect closed 5-trail systems of L_5, L_{15} and L_{30}, and there is a $(k,5)$-[1,2]-OA for all positive integers $k \neq 2,3,6$ and 10 (see Lemma 3.4). Hence, by Theorem 3.8, there is a 2-perfect closed 5-trail system of L_{5k} for all $k \neq 2,3,6$ or 10. Hence the only possible exceptions are $n = 10$ and $n = 50$. No 2-perfect closed 5-trail systems have been found in these cases. □

Remark. The existence of a 2-perfect closed 5-trail system of L_{10} implies the existence of a set of 3 mutually orthogonal latin squares of side 10.

Theorem 4.2. *$S(6) = \{n | n \equiv 0(\mathit{mod}\, 6)\}$.*

Proof. There is a 2-perfect closed 6-trail system of L_6, and there is a $(k,6)$-[1,2]-OA for all positive integers k (see Lemma 3.5). Hence, by Theorem 3.8, there is a 2-perfect closed 6-trail system of L_{6k} for all positive integers k. □

Theorem 4.3. $S(7) = \{n | n \equiv 0 (mod\ 7)\}$.

Proof. There is a 2-perfect closed 7-trail system of L_7, and there is a $(k,7)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.6). Hence, by Theorem 3.8, there is a 2-perfect closed 7-trail system of L_{7k} for all positive integers $k \notin \{2,6\}$. There is a 2-perfect closed 7-trail system of L_{14}, and Lemma 3.6 and Theorem 3.8 hence give a closed 7-trail system of L_{42}. □

Theorem 4.4. $S(8) = \{n | n \equiv 0 (mod\ 4)\}$.

Proof. There is a 2-perfect closed 8-trail system of L_4, and there is a $(k,8)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.7). Hence, by Theorem 3.8, there is a 2-perfect closed 8-trail system of L_{4k} for all positive integers $k \notin \{2,6\}$. There is a 2-perfect closed 8-trail system of L_8, and Lemma 3.7 and Theorem 3.8 hence give a closed 8-trail system of L_{24}. □

Lemma 4.5. *There is no 2-perfect closed 9-trail system of L_3.*

Proof. A 9-quasigroup of order 3 would consist of a single 9-circuit. The loops 11, 22 and 33 must be separated in this circuit. That is, $(\ldots a, a, b, b \ldots)$ is not allowed, since its distance 2 graph contains the edge ab twice. Hence, without loss of generality we may assume the circuit looks like $(1,1,.,2,2,.,3,3,.)$. From here it is clear that the only possibility is $(1,1,3,2,2,1,3,3,2)$, and this is not allowed since (among other things) the edge 13 occurrs twice. □

The non-existence of a 2-perfect closed 9-trail system of L_3 means that a different technique is needed to determine $S(9)$.

Lemma 4.6. $\{n | n \equiv 0 (mod\ 9)\} \subseteq S(9)$.

Proof. There is a 2-perfect closed 9-trail system of L_9, and there is a $(k,9)$-[1,2]-OA for all positive integers k (see Lemma 3.5). Hence, by Theorem 3.8, there is a 2-perfect closed 9-trail system of L_{9k} for all positive integers k. □

Lemma 4.7. *For all positive integers $k \neq 2$ there is a 2-perfect closed 9-trail system of $M_{9,k}$.*

Proof. Define a $(k,9)$-[1,2]-OA as in Lemma 3.5 but ensure that the quasigroup $(K,*)$ is idempotent (there exist idempotent quasigroups of order k for all positive integers $k \neq 2$, see [**9**]). Then construct a (k^2-k) by m rectangular array B from this array by removing the k rows in which all the entries are the same.

Let $(\{1,2,\ldots,9\},T)$ be a 2-perfect closed 9-trail system of L_9 and for each $(x_1,x_2,\ldots,x_9) \in T$ and each row $[y_1,y_2,\ldots,y_9]$ of the array B let

$$((x_1,y_1),(x_2,y_2),\ldots,(x_9,y_9)) \in T'.$$

Then $(V \times K, T')$ is the required system. □

Lemma 4.8. $\{n|n \neq 3 \text{ and } n \equiv 3(mod\, 9)\} \subseteq S(9)$.

Proof. Since $6 \in S(9)$, Lemma 3.5 and Theorem 3.8 give $12 \in S(9)$. Also, $21 \in S(9)$. Let $n = 9k+3$ where $k \geq 3$. Let $(V \times K, T')$ be a 2-perfect closed 9-trail system of $M_{9,k}$. Introduce three new vertices v_1, v_2 and v_3. Let $((V \times \{1\}) \cup \{v_1,v_2,v_3\}, T_1)$ be a 2-perfect closed 9-trail system of L_{12} and for $i = 2,3,\ldots,k$ let $((V \times \{i\}) \cup \{v_1,v_2,v_3\}, T_i)$ be a 2-perfect closed 9-trail system of $L_{12} \setminus L_3$ (the elements in H being v_1, v_2 and v_3).

Then $((V \times K) \cup \{v_1,v_2,v_3\}, T' \cup T_1 \cup T_2 \cup T_3 \cup \cdots \cup T_k)$ is a 2-perfect closed 9-trail system of L_{9k+3}. □

Lemma 4.9. $\{n|n \equiv 6(mod\, 9)\} \subseteq S(9)$.

Proof. There are 2-perfect closed 9-trail systems of L_6 and L_{15}. Since $6 \in S(9)$, $24 \in S(9)$ by Lemma 3.5 and Theorem 3.8. Let $n = 9k+6$, where $k \geq 3$. Let $(V \times K, T')$ be a 2-perfect closed 9-trail system of $M_{9,k}$. Introduce six new vertices v_1,v_2,v_3,v_4,v_5 and v_6. Let $((V \times \{1\}) \cup \{v_1,v_2,v_3,v_4,v_5,v_6\}, T_1)$ be a 2-perfect closed 9-trail system of L_{15} and for $i = 2,3,\ldots k$ let $((V \times \{i\}) \cup \{v_1,v_2,v_3,v_4,v_5,v_6\}, T_i)$ be a 2-perfect closed 9-trail system of $L_{15} \setminus L_6$ (the elements in H being v_1,v_2,v_3,v_4,v_5 and v_6).

Then $((V \times K) \cup \{v_1,v_2,v_3,v_4,v_5,v_6\}, T' \cup T_1 \cup T_2 \cup T_3 \cup \cdots \cup T_k)$ is a 2-perfect closed 9-trail system of L_{9k+6}. □

Theorem 4.10 follows from Lemmas 4.5,4.6,4.8 and 4.9.

Theorem 4.10. $S(9) = \{n|n \neq 3 \text{ and } n \equiv 0(mod\, 3)\}$.

Theorem 4.11. $S(10) = \{n|n \equiv 0(mod\, 10)\}$.

Proof. There is a 2-perfect closed 10-trail system of L_{10}, and there is a $(k,10)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.6). Hence, by Theorem 3.8, there is a 2-perfect closed 10-trail system of L_{10k} for all positive integers $k \notin \{2,6\}$. There is a 2-perfect closed 10-trail system of L_{20}, and Lemma 3.6 and Theorem 3.8 hence give a closed 10-trail system of L_{60}. □

Theorem 4.12. $S(11) = \{n | n \equiv 0 (mod\ 11)\}$.

Proof. There is a 2-perfect closed 11-trail system of L_{11}, and there is a $(k,11)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.7). Hence, by Theorem 3.8, there is a 2-perfect closed 11-trail system of L_{11k} for all positive integers $k \notin \{2,6\}$. There is a 2-perfect closed 11-trail system of L_{22}, and Lemma 3.7 and Theorem 3.8 hence give a closed 11-trail system of L_{66}. □

Theorem 4.13. $S(12) = \{n | n \equiv 0 (mod\ 6)\}$.

Proof. There is a 2-perfect closed 12-trail system of L_6, and there is a $(k,12)$-[1,2]-OA for all positive integers k (see Lemma 3.5). Hence, by Theorem 3.8, there is a 2-perfect closed 12-trail system of L_{6k} for all positive integers k. □

Theorem 4.14. $S(13) = \{n | n \equiv 0 (mod\ 13)\}$.

Proof. There is a 2-perfect closed 13-trail system of L_{13}, and there is a $(k,13)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.6). Hence, by Theorem 3.8, there is a 2-perfect closed 13-trail system of L_{13k} for all positive integers $k \notin \{2,6\}$. There is a 2-perfect closed 13-trail system of L_{26}, and Lemma 3.6 and Theorem 3.8 hence give a closed 13-trail system of L_{78}. □

Theorem 4.15. $S(14) = \{n | n \equiv 0 (mod\ 14)\}$, *with the possible exceptions* $n = 28$ *and* $n = 84$.

Proof. There is a 2-perfect closed 14-trail system of L_{14} and there is a $(k,14)$-[1,2]-OA for all positive integers $k \notin \{2,6\}$ (see Lemma 3.7). Hence, by Theorem 3.8, there is a 2-perfect closed 14-trail system of L_{14k} for all $k \notin \{2,6\}$. Hence the only possible exceptions are $n = 28$ and $n = 84$. No 2-perfect closed 14-trail systems have been found in these cases. □

Theorem 4.16. $S(15) = \{n | n \equiv 0 (mod\ 15)\}$.

Proof. There is a 2-perfect closed 15-trail system of L_{15} and there is a $(k,15)$-[1,2]-OA for all positive integers k (see Lemma 3.5). Hence, by Theorem 3.8, there is a 2-perfect closed 15-trail system of L_{15k} for all positive integers k. □

5. Concluding Remarks

In this paper, we have determined the set $S(m)$ of values of n for which there exist 2-perfect closed m-trail systems of L_n, for all $m \leq 15$, with four possible exceptions. The results are summarised in the following table.

m	S(m)	Undecided Values
1	$\{1\}$	
2	$\emptyset$	
3	$\{n \mid n \equiv 0 (\text{mod } 3)\}$	
4	$\emptyset$	
5	$\{n \mid n \equiv 0 (\text{mod } 5)\}$	$n = 10, n = 50$
6	$\{n \mid n \equiv 0 (\text{mod } 6)\}$	
7	$\{n \mid n \equiv 0 (\text{mod } 7)\}$	
8	$\{n \mid n \equiv 0 (\text{mod } 4)\}$	
9	$\{n \mid n \neq 3, n \equiv 0 (\text{mod } 3)\}$	
10	$\{n \mid n \equiv 0 (\text{mod } 10)\}$	
11	$\{n \mid n \equiv 0 (\text{mod } 11)\}$	
12	$\{n \mid n \equiv 0 (\text{mod } 6)\}$	
13	$\{n \mid n \equiv 0 (\text{mod } 13)\}$	
14	$\{n \mid n \equiv 0 (\text{mod } 14)\}$	$n = 28, n = 84$
15	$\{n \mid n \equiv 0 (\text{mod } 15)\}$	

Acknowledgement. We wish to thank Dr. Sheila Oates-Williams for her advice and assistance in the preparation of this paper. We would also like to acknowledge the support of the A.R.C. for helping to fund this research.

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6. Appendix

Within this appendix, each 2-perfect closed m-trail system of a graph G is given as (V,T), where V is the vertex set of G, and T is the collection of m-trails.

$$\boxed{m=5}$$

$\boxed{L_{15}}$ $V=\{i_j \mid 0 \leqslant i \leqslant 4;\ j=1,2,3\}$. T as follows, with i cycled modulo 5:

$$\begin{array}{lll}(0_1,1_1,1_2,4_1,1_1), & (0_1,3_1,0_2,1_2,0_3), & (0_1,1_2,4_3,2_2,2_3),\\ (0_1,3_2,2_1,4_3,4_2), & (0_1,4_2,1_3,1_3,3_3), & (0_1,0_3,4_1,0_3,4_3),\\ (0_1,3_3,4_2,3_2,0_1), & (0_1,4_3,2_3,3_3,0_2), & (0_2,2_2,3_3,2_2,0_2).\end{array}$$

$\boxed{L_{30}}$ $V=\{i_j \mid 0 \leqslant i \leqslant 14;\ j=1,2\}$. T as follows, with i cycled modulo 15:

$$\begin{array}{lll}(0_1,8_2,8_1,12_2,11_2), & (0_1,2_2,12_1,13_2,12_1), & (0_1,0_1,1_1,5_1,4_1),\\ (0_1,0_2,9_1,1_1,13_1), & (0_1,10_1,7_2,8_1,13_2), & (0_1,14_2,4_2,9_1,2_1),\\ (0_1,5_1,11_2,0_2,12_2), & (0_1,6_1,9_2,4_2,7_2), & (0_1,9_1,1_2,8_2,4_2),\\ (0_1,11_2,2_1,11_2,2_2), & (0_1,10_2,8_2,9_2,3_2), & (0_1,13_2,6_2,6_2,8_2).\end{array}$$

$$\boxed{m=6}$$

$\boxed{L_6}$ $V=\{i_j \mid 0 \leqslant i \leqslant 2;\ j=1,2\}$. T as follows, with i cycled modulo 3:

$$(0_1,1_1,0_1,0_1,0_2,2_2),\quad (0_1,1_2,1_2,2_2,2_1,1_2),$$

$$\boxed{m=7}$$

$\boxed{L_{14}}$ $V=\{i_j \mid 0 \leqslant i \leqslant 6;\ j=1,2\}$. T as follows, with i cycled modulo 7:

$$\begin{array}{ll}(0_1,0_2,0_2,0_1,1_2,2_1,5_2), & (0_1,6_1,0_1,5_2,3_1,6_1,1_2),\\ (0_1,0_1,4_1,6_1,5_2,3_2,4_2), & (0_1,5_1,2_2,1_2,5_2,1_2,3_2).\end{array}$$

$\boxed{m=8}$

$\boxed{L_4}$ $V=\mathbb{Z}_4$. T as follows, uncycled:

$$(0,1,0,0,2,3,2,2),\quad (0,3,1,1,2,1,3,3).$$

$\boxed{L_8}$ $V=\mathbb{Z}_8$. T as follows, uncycled:

$$\begin{array}{llll}(0,0,1,3,6,1,2,7), & (0,2,0,3,4,2,1,1), & (0,4,5,2,3,2,5,5), & (0,5,6,2,4,7,4,4),\\ (0,6,7,3,1,4,3,3), & (0,7,2,2,6,4,6,6), & (1,5,4,1,7,1,6,5), & (3,5,3,7,7,5,7,6).\end{array}$$

$\boxed{m=9}$

$\boxed{L_6}$ $V=\mathbb{Z}_6$. T as follows, uncycled:

$$\begin{array}{ll}(0,1,0,0,2,1,2,2,3), & (0,3,3,4,3,5,2,5,4),\\ (0,4,4,2,4,1,1,4,5), & (0,5,5,1,3,1,5,3,2).\end{array}$$

$\boxed{L_{15}}$ $V=\{i_j \mid 0\leqslant i\leqslant 4;\ \ j=1,2,3\}$. T as follows, with i cycled modulo 5:

$$\begin{array}{ll}(0_1,0_2,0_2,0_1,1_1,3_2,2_2,0_1,3_2), & (0_1,4_2,3_1,4_2,0_1,2_1,1_1,0_3,0_3),\\ (0_1,3_3,2_3,0_2,0_3,1_3,3_2,1_2,3_3), & (0_1,0_1,1_3,4_1,2_1,4_3,4_2,3_3,1_3),\\ (0_1,0_3,4_2,1_2,2_2,0_3,1_2,2_3,4_3). & \end{array}$$

$\boxed{L_{21}}$ $V=\{i_j \mid 0\leqslant i\leqslant 6;\ \ j=1,2,3\}$. T as follows, with i cycled modulo 7:

$$\begin{array}{ll}(0_1,0_1,0_2,0_1,1_1,3_1,0_2,0_2,1_2), & (0_1,3_1,0_3,2_3,3_1,2_3,5_2,3_1,2_1),\\ (0_1,5_2,6_1,0_3,4_1,0_2,2_2,5_3,1_3), & (0_1,2_2,1_2,5_2,2_3,1_3,1_1,5_1,4_2),\\ (0_1,1_2,2_3,4_1,4_3,1_3,4_1,2_3,2_3), & (0_1,2_3,3_3,3_2,1_3,0_2,0_3,2_2,5_2),\\ (0_1,3_3,1_3,6_2,5_3,6_2,1_3,5_2,3_2). & \end{array}$$

$\boxed{L_{12}\setminus L_3}$ $V=\{i_j \mid 0\leqslant i\leqslant 2;\ \ j=1,2,3\}\cup\{A,B,C\}$. T follows, with i cycled modulo 3:

$$\begin{array}{ll}(A,0_1,0_1,B,0_2,0_3,C,0_3,0_2), & (A,0_2,1_3,B,0_3,1_1,C,0_1,1_3),\\ (A,0_3,2_2,B,1_1,0_2,C,1_2,0_1), & (0_1,1_1,0_1,0_2,1_1,2_2,2_2,1_3,0_3),\\ (0_1,0_3,1_3,0_1,2_3,2_3,0_2,2_2,0_2). & \end{array}$$

$\boxed{L_{15}\setminus L_6}$ $V=\{i_j \mid 0\leqslant i\leqslant 2;\ \ j=1,2,3\}\cup\{A,B,C,D,E,F\}$.

T as follows, with i cycled modulo 3:

$$\begin{array}{ll}(A,0_1,0_1,A,0_2,0_2,A,0_3,0_3), & (B,0_1,1_1,B,0_2,1_2,B,0_3,1_3),\\ (C,0_1,2_1,C,0_2,2_2,C,0_3,2_3), & (D,0_1,0_2,D,0_2,0_3,D,0_3,0_1),\\ (E,0_1,1_3,E,0_2,1_1,E,0_3,1_2), & (F,0_1,2_3,F,0_2,2_1,F,0_3,2_2),\\ (0_1,1_2,1_1,0_2,1_3,1_2,0_3,1_1,1_3). & \end{array}$$

$$\boxed{m = 10}$$

$\boxed{L_{10}}$ $V = \{i_j \mid 0 \leqslant i \leqslant 4;\ \ j = 1,2\}$. T as follows, with i cycled modulo 5:

$$(0_1, 0_1, 0_2, 1_2, 1_1, 4_2, 1_1, 0_2, 4_1, 3_1),\quad (0_1, 1_1, 4_1, 0_2, 2_2, 0_1, 2_2, 2_2, 1_2, 4_2).$$

$\boxed{L_{20}}$ $V = \{i_j \mid 0 \leqslant i \leqslant 4;\ \ j = 1,2,3,4\}$. T as follows, with i cycled modulo 5:

$$\begin{array}{ll}
(0_1, 0_1, 0_2, 0_3, 0_3, 0_4, 0_2, 0_1, 1_3, 3_3), & (0_1, 2_1, 1_2, 0_2, 4_1, 4_4, 4_4, 0_1, 3_1, 2_3),\\
(0_1, 0_3, 0_1, 1_1, 4_3, 0_2, 2_1, 4_3, 4_2, 0_4), & (0_1, 4_1, 1_4, 0_3, 2_4, 4_2, 4_4, 0_2, 1_1, 2_2),\\
(0_1, 2_2, 4_4, 4_3, 2_4, 0_2, 0_2, 3_4, 2_4, 3_4), & (0_1, 3_2, 2_4, 1_2, 2_2, 0_2, 2_2, 3_3, 2_1, 1_4),\\
(0_1, 3_4, 1_1, 2_4, 4_4, 0_3, 3_3, 4_3, 1_2, 4_3), & (0_2, 4_3, 0_4, 2_3, 1_2, 3_3, 2_4, 0_4, 3_3, 2_3).
\end{array}$$

$$\boxed{m = 11}$$

$\boxed{L_{22}}$ $V = \{i_j \mid 0 \leqslant i \leqslant 10;\ \ j = 1,2\}$. T as follows, with i cycled modulo 11:

$$\begin{array}{ll}
(0_1, 1_1, 0_1, 0_1, 2_1, 5_1, 0_1, 8_1, 1_1, 0_2, 0_2), & (0_1, 5_1, 0_2, 1_1, 8_1, 2_2, 1_1, 1_2, 7_1, 4_2, 2_2),\\
(0_1, 9_1, 2_2, 6_1, 4_2, 3_2, 0_2, 2_1, 4_2, 9_2, 4_2), & (0_1, 1_2, 5_2, 10_1, 6_2, 8_2, 0_2, 3_1, 6_2, 7_2, 3_2).
\end{array}$$

$$\boxed{m = 12}$$

$\boxed{L_6}$ $V = \mathbb{Z}_6$. T as follows, uncycled:

$$\begin{array}{ll}
(0, 1, 0, 0, 2, 1, 1, 2, 3, 2, 4, 3), & (0, 3, 3, 5, 2, 5, 4, 2, 2, 0, 5, 5),\\
(0, 4, 4, 1, 5, 1, 4, 5, 3, 1, 3, 4). &
\end{array}$$

$$\boxed{m = 13}$$

$\boxed{L_{26}}$ $V = \{i_j \mid 0 \leqslant i \leqslant 12;\ \ j = 1,2\}$. T as follows, with i cycled modulo 13:

$$\begin{array}{l}
(0_1, 1_1, 0_2, 8_1, 8_1, 10_1, 0_2, 9_1, 6_1, 1_1, 3_2, 0_2, 1_1),\\
(0_1, 3_1, 1_1, 6_1, 10_1, 3_1, 0_2, 7_1, 1_1, 1_2, 7_1, 1_2, 0_2),\\
(0_1, 9_1, 1_2, 4_1, 0_2, 2_1, 6_2, 0_2, 10_1, 8_2, 12_2, 4_2, 2_2),\\
(0_1, 1_2, 4_2, 3_1, 11_2, 12_2, 12_2, 4_1, 10_2, 12_2, 7_2, 3_2, 9_2).
\end{array}$$

$$\boxed{m = 14}$$

$\boxed{L_{14}}$ $V = \{i_j \mid 0 \leqslant i \leqslant 6;\ \ j = 1,2\}$. T as follows, with i cycled modulo 7:

$$\begin{array}{l}
(0_1, 1_1, 0_1, 0_1, 2_1, 5_1, 3_1, 0_2, 0_1, 4_1, 0_2, 0_2, 1_2, 3_2),\\
(0_1, 0_2, 4_2, 0_1, 1_2, 6_2, 4_1, 6_2, 0_1, 5_2, 1_2, 0_1, 6_2, 5_2).
\end{array}$$

(Received 22/7/92; revised 13/5/93)