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6-Cycle decompositions of complete 3-uniform hypergraphs

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Abstract

A complete 3-uniform hypergraph of order n has vertex set V with $|V| = n$ and the set of all 3-subsets of V as its edge set. A 6-cycle in this hypergraph is $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_1$ where $v_1, v_2, v_3, v_4, v_5, v_6$ are distinct vertices and $e_1, e_2, e_3, e_4, e_5, e_6$ are distinct edges such that $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, 3, 4, 5\}$ and $v_6, v_1 \in e_6$. A decomposition of a hypergraph is a partition of its edge set into disjoint subsets. In this paper we give necessary and sufficient conditions for a decomposition of the complete 3-uniform hypergraph of order n into 6-cycles.

1 Introduction

A *hypergraph* $\mathcal{H}$ consists of a finite nonempty set V of *vertices* and a set $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ of *edges* where each $e_i \subseteq V$ with $|e_i| > 0$ for $i \in \{1, 2, \ldots, m\}$. If $|e_i| = h$, then we call e_i an *h-edge*. If every edge of $\mathcal{H}$ is an h-edge for some h, then we say that $\mathcal{H}$ is *h-uniform*. The *complete h-uniform hypergraph* $K_n^{(h)}$ is the hypergraph with vertex set V, where $|V| = n$, in which every h-subset of V determines an h-edge. It then follows that $K_n^{(h)}$ has $\binom{n}{h}$ edges. When $h = 2$, $K_n^{(2)} = K_n$, the complete graph on n vertices.

A *decomposition* of a hypergraph $\mathcal{H}$ is a set $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ of *subhypergraphs* of $\mathcal{H}$ such that $\mathcal{E}(\mathcal{F}_1) \cup \mathcal{E}(\mathcal{F}_2) \cup \cdots \cup \mathcal{E}(\mathcal{F}_k) = \mathcal{E}(\mathcal{H})$ and $\mathcal{E}(\mathcal{F}_i) \cap \mathcal{E}(\mathcal{F}_j) = \emptyset$ for all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. We denote this by $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$. If $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$ is a decomposition such that $\mathcal{F}_1 \cong \mathcal{F}_2 \cong \cdots \cong \mathcal{F}_k \cong \mathcal{G}$, where $\mathcal{G}$ is a fixed hypergraph, then $\mathcal{F}$ is called a *$\mathcal{G}$-decomposition* of $\mathcal{H}$.

A *cycle of length k* in a hypergraph $\mathcal{H}$ is a sequence of the form $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$, where $v_1, v_2, \ldots, v_k$ are distinct vertices and $e_1, e_2, \ldots, e_k$ are distinct edges satisfying $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, \ldots, k-1\}$ and $v_k, v_1 \in e_k$.

Decompositions of $K_n^{(3)}$ into Hamilton cycles were considered in [1, 2] and the proof of their existence was given in [10]. Decompositions of $K_n^{(h)}$ into Hamilton

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cycles were considered in [5, 6], a complete solution for $h \geq 4$ and $n \geq 30$ was given in [5], and cyclic decompositions were considered in [6]. In [3], necessary and sufficient conditions were given for a $\mathcal{G}$-decomposition of $K_n^{(3)}$, where $\mathcal{G}$ is any 3-uniform hypergraph with at most three edges and at most six vertices. In [4], decompositions of $K_n^{(3)}$ into 4-cycles were considered and their existence was established.

In this paper, we are interested in 6-cycle decompositions of $K_n^{(3)}$. For convenience, we will often write the edge $\{v_a, v_b, v_c\}$ as v_a-v_b-v_c and the cycle $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_1$ as $(v_1\text{-}y_1\text{-}v_2, v_2\text{-}y_2\text{-}v_3, v_3\text{-}y_3\text{-}v_4, v_4\text{-}y_4\text{-}v_5, v_5\text{-}y_5\text{-}v_6, v_6\text{-}y_6\text{-}v_1)$, where $e_i = v_i\text{-}y_i\text{-}v_{i+1}$ for $i \in \{1, 2, 3, 4, 5\}$ and $e_6 = v_6\text{-}y_6\text{-}v_1$. A necessary condition for the existence of a 6-cycle decomposition of $K_n^{(3)}$ is: 6 divides the number of edges in $K_n^{(3)}$, that is, $6|\binom{n}{3}$. Clearly, if n is even and $6|\binom{n}{3}$, then $n \equiv 0$, 2 or 10 (mod 18) and if n is odd and $6|\binom{n}{3}$, then $n \equiv 1$, 9 or 29 (mod 36). Thus we have:

Lemma 1.1. *For $n \geq 6$, if there exists a 6-cycle decomposition of $K_n^{(3)}$, then $n \equiv 0 \pmod{18}$, $2 \pmod{18}$, $10 \pmod{18}$, $1 \pmod{36}$, $9 \pmod{36}$ or $29 \pmod{36}$.*

In Sections 3 through 8, we prove sufficiency. To prove it, we need the following theorems.

Theorem 1.1. (Šajna [7]) *Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The complete graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

Theorem 1.2. (Tarsi [9]) *Let t and n be positive integers. There exists a P_{t+1}-decomposition of the complete graph K_n if and only if $n \geq t+1$ and $n(n-1) \equiv 0 \pmod{2t}$, where P_{t+1} is the path of length t.*

Theorem 1.3. (Sotteau [8]) *The complete bipartite graph $K_{m,n}$ can be decomposed into $2k$-cycles if and only if m and n are even, $m \geq k$, $n \geq k$, and $2k$ divides mn.*

2 Preliminary lemmas

We assume the vertex set of $K_n^{(3)}$ is $\{v_i : i \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n$ is the set of integers modulo n. For non-negative integers i and j with $i < j$, we denote the set $\{v_i, v_{i+1}, \ldots, v_j\}$ by $[v_i, v_j]$, and the set $\{i, i+1, \ldots, j\}$ by $[i, j]$.

2.1 The hypergraph $\mathcal{H}_m^{'}$

Define the 3-uniform hypergraph $\mathcal{H}_m^{'}$ of order $3m$ as follows. Let $V(\mathcal{H}_m^{'})$ be $\{v_i : i \in \mathbb{Z}_{3m}\}$, and let $\mathcal{E}(\mathcal{H}_m^{'})$ be the set of all 3-edges v_a-v_b-v_c such that $a \in [0, m-1]$, $b \in [m, 2m-1]$ and $c \in [2m, 3m-1]$. Note that $\left|\mathcal{E}(\mathcal{H}_m^{'})\right| = m^3$.

A necessary condition for the existence of a 6-cycle decomposition of $\mathcal{H}_m^{'}$ is: $6|m^3$, i.e., $m \equiv 0 \pmod{6}$. Our aim is to decompose $\mathcal{H}_m^{'}$ into $\frac{m^3}{6}$ edge-disjoint 6-cycles whenever $m \equiv 0 \pmod{6}$.

By Theorem 1.3, the complete bipartite graph $K_{m,m}$ with partite sets $[v_0, v_{m-1}]$ and $[v_m, v_{2m-1}]$ can be decomposed into 6-cycles if and only if $m \equiv 0 \pmod 6$. Let $\mathscr{F}$ be a decomposition of $K_{m,m}$ into 6-cycles. For each 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_1)$ of $\mathscr{F}$, construct m edge-disjoint 6-cycles $(x_1\text{-}v_i\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}v_i\text{-}x_1)$ of $\mathcal{H}_m^{'}$ where $i \in [2m, 3m-1]$. Thus, we have

Lemma 2.1. *For $m \equiv 0 \pmod 6$, $\mathcal{H}_m^{'}$ decomposes into 6-cycles.*

2.2 The hypergraph $\mathcal{H}_m^{''}$

Define the hypergraph $\mathcal{H}_m^{''}$ of order $2m+1$ as follows: let $V(\mathcal{H}_m^{''}) = \{\infty\} \cup \{v_i : i \in \mathbb{Z}_{2m}\}$ and let $\mathcal{E}(\mathcal{H}_m^{''})$ be the set of all 3-edges $\infty\text{-}v_b\text{-}v_c$ where $b \in [0, m-1]$ and $c \in [m, 2m-1]$. Note that $\left|\mathcal{E}(\mathcal{H}_m^{''})\right| = m^2$.

A necessary condition for the existence of a 6-cycle decomposition of $\mathcal{H}_m^{''}$ is that $6|m^2$, i.e., $m \equiv 0 \pmod 6$. Our aim is to decompose $\mathcal{H}_m^{''}$ into $\frac{m^2}{6}$ edge-disjoint 6-cycles whenever $m \equiv 0 \pmod 6$.

By Theorem 1.3, the complete bipartite graph $K_{m,m}$ with partite sets $[v_0, v_{m-1}]$ and $[v_m, v_{2m-1}]$ can be decomposed into 6-cycles if and only if $m \equiv 0 \pmod 6$. Let $\mathscr{F}$ be a decomposition of $K_{m,m}$ into 6-cycles. For each 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_1)$ of $\mathscr{F}$, construct the 6-cycle $(x_1\text{-}\infty\text{-}x_2, x_2\text{-}\infty\text{-}x_3, x_3\text{-}\infty\text{-}x_4, x_4\text{-}\infty\text{-}x_5, x_5\text{-}\infty\text{-}x_6, x_6\text{-}\infty\text{-}x_1)$ of $\mathcal{H}_m^{''}$. Thus, we have

Lemma 2.2. *For $m \equiv 0 \pmod 6$, $\mathcal{H}_m^{''}$ decomposes into 6-cycles.*

2.3 The hypergraph $\mathcal{H}_m$

Define the 3-uniform hypergraph $\mathcal{H}_m$ of order $2m$ as follows: let $V(\mathcal{H}_m) = \{v_i : i \in \mathbb{Z}_{2m}\}$ grouped as $G_0 = [v_0, v_{m-1}]$ and $G_1 = [v_m, v_{2m-1}]$. Let $\mathcal{E}(\mathcal{H}_m)$ be the set of all 3-edges $v_a\text{-}v_b\text{-}v_c$ such that v_a, v_b and v_c are not all from the same group, that is, at least one of v_a, v_b, v_c is an element of G_0 and at least one of v_a, v_b, v_c is an element of G_1. Note that $|\mathcal{E}(\mathcal{H}_m)| = m^2(m-1)$.

A necessary condition for the existence of a 6-cycle decomposition of $\mathcal{H}_m$ is that $6|m^2(m-1)$, i.e., $m \equiv 0$, 1, 3 or 4 (mod 6). For required m, our aim is to decompose $\mathcal{H}_m$ into $\frac{m^2(m-1)}{6}$ edge-disjoint 6-cycles.

By Theorem 1.1, if m is odd and $12|m(m-1)$, i.e., $m \equiv 1$ or $9 \pmod{12}$, then K_m with vertex set G_0 and K_m with vertex set G_1 are decomposable into 6-cycles. Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be decompositions of K_m into 6-cycles with vertex sets G_0 and G_1, respectively. For each 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_1)$ of $\mathscr{F}_0$, construct m edge-disjoint 6-cycles $(x_1\text{-}v_i\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}v_i\text{-}x_1)$, where $v_i \in G_1$ and for each 6-cycle $(y_1, y_2, y_3, y_4, y_5, y_6, y_1)$ of $\mathscr{F}_1$, construct m edge-disjoint 6-cycles $(y_1\text{-}v_j\text{-}y_2, y_2\text{-}v_j\text{-}y_3, y_3\text{-}v_j\text{-}y_4, y_4\text{-}v_j\text{-}y_5, y_5\text{-}v_j\text{-}y_6, y_6\text{-}v_j\text{-}y_1)$, where $v_j \in G_0$. The collection of all these 6-cycles yields a decomposition of $\mathcal{H}_m$. Thus, we have:

Lemma 2.3. *Let $m \equiv 1$ or $9 \pmod{12}$. If $m \neq 1$, then $\mathcal{H}_m$ decomposes into 6-cycles.*

Lemma 2.4. *$\mathcal{H}_6$ decomposes into 6-cycles.*

Proof. The 6-cycle decomposition of $\mathcal{H}_6$ is as follows:

For $v_i \in [v_6, v_{11}]$,

$(v_i\text{-}v_0\text{-}v_1, v_1\text{-}v_i\text{-}v_5, v_5\text{-}v_i\text{-}v_2, v_2\text{-}v_i\text{-}v_4, v_4\text{-}v_i\text{-}v_3, v_3\text{-}v_2\text{-}v_i)$ and

$(v_i\text{-}v_1\text{-}v_2, v_2\text{-}v_i\text{-}v_0, v_0\text{-}v_i\text{-}v_3, v_3\text{-}v_i\text{-}v_5, v_5\text{-}v_i\text{-}v_4, v_4\text{-}v_1\text{-}v_i)$;

for $v_j \in [v_0, v_5]$,

$(v_j\text{-}v_6\text{-}v_7, v_7\text{-}v_j\text{-}v_{11}, v_{11}\text{-}v_j\text{-}v_8, v_8\text{-}v_j\text{-}v_{10}, v_{10}\text{-}v_j\text{-}v_9, v_9\text{-}v_8\text{-}v_j)$ and

$(v_j\text{-}v_7\text{-}v_8, v_8\text{-}v_j\text{-}v_6, v_6\text{-}v_j\text{-}v_9, v_9\text{-}v_j\text{-}v_{11}, v_{11}\text{-}v_j\text{-}v_{10}, v_{10}\text{-}v_7\text{-}v_j)$;

for $(k, \ell) \in \{(6, 7), (8, 9), (10, 11)\}$,

$(v_\ell\text{-}v_3\text{-}v_1, v_1\text{-}v_3\text{-}v_k, v_k\text{-}v_0\text{-}v_4, v_4\text{-}v_\ell\text{-}v_0, v_0\text{-}v_k\text{-}v_5, v_5\text{-}v_0\text{-}v_\ell)$;

and for $(k, \ell) \in \{(0, 1), (2, 3), (4, 5)\}$,

$(v_\ell\text{-}v_9\text{-}v_7, v_7\text{-}v_9\text{-}v_k, v_k\text{-}v_6\text{-}v_{10}, v_{10}\text{-}v_\ell\text{-}v_6, v_6\text{-}v_k\text{-}v_{11}, v_{11}\text{-}v_6\text{-}v_\ell)$. □

Lemma 2.5. *If $m \equiv 0 \pmod{18}$, then $\mathcal{H}_m$ decomposes into 6-cycles.*

Proof. Let $m = 18k$, where k is a positive integer, $G_0 = A_1 \cup A_2 \cup \cdots \cup A_{3k}$ and $G_1 = B_1 \cup B_2 \cup \cdots \cup B_{3k}$, where $A_i = [v_{6i-6}, v_{6i-1}]$ and $B_j = [v_{18k+6j-6}, v_{18k+6j-1}]$.

For $i, j \in \{1, 2, \dots, 3k\}$, let $\mathcal{H}_{i,j} \cong \mathcal{H}_6$ be the hypergraph with vertex set grouped A_i and B_j. By Lemma 2.4, $\mathcal{H}_6$ is 6-cycle decomposable.

For $i, j, k \in \{1, 2, \dots, 3k\}$ with $j < k$, let $\mathcal{H}'_{i;j,k} \cong \mathcal{H}'_6$ be the hypergraph with vertex set $A_i \cup B_j \cup B_k$ and edge set $\{E : |E \cap A_i| = |E \cap B_j| = |E \cap B_k| = 1\}$. For $i, j, k \in \{1, 2, \dots, 3k\}$ with $i < j$, let $\mathcal{H}''_{i,j;k} \cong \mathcal{H}'_6$ be the hypergraph with vertex set $A_i \cup A_j \cup B_k$ and edge set $\{E : |E \cap A_i| = |E \cap A_j| = |E \cap B_k| = 1\}$. By Lemma 2.1, $\mathcal{H}'_6$ is 6-cycle decomposable.

Since $\mathcal{H}_m = \mathcal{H}_{18k} = 9k^2\mathcal{H}_6 \oplus 9k^2(3k-1)\mathcal{H}'_6$, the lemma follows. □

Lemma 2.6. *$\mathcal{H}_{10}$ decomposes into 6-cycles.*

Proof. Note that $V(\mathcal{H}_{10}) = \{v_i : i \in \mathbb{Z}_{20}\}$, $G_0 = [v_0, v_9]$ and $G_1 = [v_{10}, v_{19}]$.

The complete graph K_{10} with vertex set $[v_0, v_9]$ is Hamilton-path decomposable by Theorem 1.2. Decompose each Hamilton-path P_{10} in the decomposition into a P_7 and a P_4. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the resulting decomposition of K_{10}, $(v_i\text{-}x_1\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}x_7\text{-}v_i)$, where $i \in [10, 19]$, is a 6-cycle in $\mathcal{H}_{10}$. For each $P_4 : (y_1, y_2, y_3, y_4)$ in the resulting decomposition of K_{10}, $(v_k\text{-}y_2\text{-}y_1, y_1\text{-}v_\ell\text{-}y_2, y_2\text{-}v_k\text{-}y_3, y_3\text{-}y_2\text{-}v_\ell, v_\ell\text{-}y_3\text{-}y_4, y_4\text{-}y_3\text{-}v_k)$, where $(k, \ell) \in \{(10, 11), (12, 13), (14, 15), (16, 17), (18, 19)\}$ is a 6-cycle in $\mathcal{H}_{10}$.

Similarly, the complete graph K_{10} with vertex set $[v_{10}, v_{19}]$ is Hamilton-path decomposable. Decompose each Hamilton-path P_{10} in the decomposition into a P_7 and a P_4. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the resulting decomposition of K_{10}, $(v_j\text{-}x_1\text{-}x_2, x_2\text{-}v_j\text{-}x_3, x_3\text{-}v_j\text{-}x_4, x_4\text{-}v_j\text{-}x_5, x_5\text{-}v_j\text{-}x_6, x_6\text{-}x_7\text{-}v_j)$, where $j \in [0, 9]$, is a 6-cycle in $\mathcal{H}_{10}$. For each $P_4 : (y_1, y_2, y_3, y_4)$ in the resulting decomposition of K_{10}, $(v_k\text{-}y_2\text{-}y_1, y_1\text{-}v_\ell\text{-}y_2, y_2\text{-}v_k\text{-}y_3, y_3\text{-}y_2\text{-}v_\ell, v_\ell\text{-}y_3\text{-}y_4, y_4\text{-}y_3\text{-}v_k)$, where $(k, \ell) \in \{(0, 1), (2, 3), (4, 5), (6, 7), (8, 9)\}$, is a 6-cycle in $\mathcal{H}_{10}$.

The collection of all these 6-cycles yields a decomposition of $\mathcal{H}_{10}$ into 6-cycles. □

2.4 The hypergraph $K^{(3)}_{m,n}$

Define the 3-uniform hypergraph $K^{(3)}_{m,n}$ of order $m+n$ as follows. Let $V(K^{(3)}_{m,n}) = \{v_i : i \in \mathbb{Z}_{m+n}\}$ be grouped as $G_0 = [v_0, v_{m-1}]$ and $G_1 = [v_m, v_{m+n-1}]$. Let $\mathcal{E}(K^{(3)}_{m,n})$ be the set of all 3-edges v_a-v_b-v_c such that v_a, v_b and v_c are not all from the same group, that is, at least one of v_a, v_b, v_c is an element of G_0 and at least one of v_a, v_b, v_c is an element of G_1. Note that $\left|\mathcal{E}(K^{(3)}_{m,n})\right| = \frac{mn(m+n-2)}{2}$ and $K^{(3)}_{m,m} = \mathcal{H}_m$. A necessary condition for the existence of a 6-cycle decomposition of $K^{(3)}_{m,n}$ is that $12|mn(m+n-2)$.

Lemma 2.7. *If* $m \equiv 1$ *or* $9 \pmod{12}$, $n \equiv 0,\ 1,\ 4$ *or* $9 \pmod{12}$ *and* $n \geq 7$, *then* $K^{(3)}_{m,n}$ *decomposes into* 6*-cycles.*

Proof. By Theorem 1.1, K_m with vertex set $[v_0, v_{m-1}]$ is 6-cycle decomposable. For each 6-cycle $(x_1, x_2, x_3, x_4, x_5, x_6, x_1)$ in the C_6-decomposition of K_m, the 6-cycle (v_j-x_1-x_2, x_2-v_j-x_3, x_3-v_j-x_4, x_4-v_j-x_5, x_5-v_j-x_6, x_6-x_1-v_j), where $j \in [m, m+n-1]$ is a 6-cycle in $K^{(3)}_{m,n}$. By Theorem 1.2, K_n with vertex set $[v_m, v_{m+n-1}]$ is P_7-decomposable. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the P_7-decomposition of K_n, (v_i-x_1-x_2, x_2-v_i-x_3, x_3-v_i-x_4, x_4-v_i-x_5, x_5-v_i-x_6, x_6-x_7-v_i), where $i \in [0, m-1]$, is a 6-cycle in $K^{(3)}_{m,n}$. The collection of all these 6-cycles yields a 6-cycle decomposition of $K^{(3)}_{m,n}$. □

Lemma 2.8. $K^{(3)}_{10,18}$ *decomposes into* 6*-cycles.*

Proof. The 6-cycle decomposition of $K^{(3)}_{10,18}$ is as follows.

The complete graph K_{10} with vertex set $[v_0, v_9]$ is Hamilton-path decomposable. Decompose each Hamilton-path P_{10} in the decomposition into a P_7 and a P_4. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the resulting decomposition of K_{10}, (v_i-x_1-x_2, x_2-v_i-x_3, x_3-v_i-x_4, x_4-v_i-x_5, x_5-v_i-x_6, x_6-x_7-v_i), where $i \in [10, 27]$, is a 6-cycle in $K^{(3)}_{10,18}$. For each $P_4 : (y_1, y_2, y_3, y_4)$ in the resulting decomposition of K_{10}, (v_k-y_2-y_1, y_1-v_ℓ-y_2, y_2-v_k-y_3, y_3-y_2-v_ℓ, v_ℓ-y_3-y_4, y_4-y_3-v_k), where $(k, \ell) \in \{(10, 11), (12, 13), \ldots, (26, 27)\}$, is a 6-cycle in $K^{(3)}_{10,18}$.

For convenience, relabel the vertices in $[v_{10}, v_{27}]$ by $[u_0, u_{17}]$. The complete graph K_{18} with vertex set $[u_0, u_{17}]$ is decomposable into 25 P_7's, one P_3 and one P_2. To see this, for $i \in \{0, 1, \ldots, 8\}$, let

$$H_i = u_i u_{i+1} u_{i+17} u_{i+2} u_{i+16} u_{i+3} u_{i+15} u_{i+4} u_{i+14} u_{i+5} u_{i+13} u_{i+6} u_{i+12} u_{i+7} u_{i+11} u_{i+8} u_{i+10} u_{i+9}$$

be a Hamilton path decomposition of K_{18}, where subscripts are reduced modulo 18. For $i \in \{0, 1, \ldots, 7\}$, decompose H_i into

$$\begin{aligned} u_i u_{i+1} u_{i+17} u_{i+2} u_{i+16} u_{i+3} \ &\oplus\ u_{i+3} u_{i+15} u_{i+4} u_{i+14} u_{i+5} u_{i+13} u_{i+6} \\ &\oplus\ u_{i+6} u_{i+12} u_{i+7} u_{i+11} u_{i+8} u_{i+10} u_{i+9}, \end{aligned}$$

a P_6 and two copies of P_7. Decompose H_8 into $u_8u_9u_7u_{10}u_6u_{11}u_5 \oplus u_5u_{12}u_4u_{13}u_3u_{14}u_2 \oplus u_2u_{15}u_1 \oplus u_1u_{16} \oplus u_{16}u_0 \oplus u_0u_{17}$, two copies of P_7, one P_3 and three P_2's. Now decompose (eight P_6's and two P_2's) $\{u_iu_{i+1}u_{i+17}u_{i+2}u_{i+16}u_{i+3} : i \in \{0, 1, \ldots, 7\}\} \cup$

$\{u_1u_{16}, u_0u_{17}\}$ into (seven P_7's) $\{u_{17}u_0u_1u_{17}u_2u_{16}u_3,\ u_{16}u_1u_2u_0u_3u_{17}u_4,\ u_2u_3u_1u_4u_0u_5u_{10},\ u_3u_4u_2u_5u_1u_6u_9,\ u_4u_5u_3u_6u_2u_7u_8,\ u_5u_6u_4u_7u_3u_8u_6,\ u_6u_7u_5u_8u_4u_9u_5\}$. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the resulting decomposition of K_{18}, $(v_i\text{-}x_1\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}x_7\text{-}v_i)$, where $i \in [0, 9]$, is a 6-cycle in $K^{(3)}_{10,18}$. Obtain from $P_3 \cup P_2$: $u_2u_{15}u_1 \cup u_0u_{16}, (v_k\text{-}u_{15}\text{-}u_2, u_2\text{-}v_\ell\text{-}u_{15}, u_{15}\text{-}v_k\text{-}u_1, u_1\text{-}u_{15}\text{-}v_\ell, v_\ell\text{-}u_0\text{-}u_{16}, u_{16}\text{-}u_0\text{-}v_k)$, where $(k, \ell) \in \{(0, 1), (2, 3), \ldots, (8, 9)\}$, a 6-cycle in $K^{(3)}_{10,18}$.

The collection of all these 6-cycles yields a decomposition of $K^{(3)}_{10,18}$ into 6-cycles. □

Lemma 2.9. $K^{(3)}_{29,36}$ *decomposes into* 6*-cycles.*

Proof. The complete graph K_{29} with vertex set $[v_0, v_{28}]$ is Hamilton-cycle decomposable. Decompose each Hamilton-cycle C_{29} in the decomposition into four P_7, one P_4 and one P_3. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the resulting decomposition of K_{29}, $(v_i\text{-}x_1\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}x_7\text{-}v_i)$, where $i \in [29, 64]$, is a 6-cycle in $K^{(3)}_{29,36}$. For each $P_4 : (y_1, y_2, y_3, y_4)$ in the resulting decomposition of K_{29}, $\{v_k\text{-}y_2\text{-}y_1, y_1\text{-}v_\ell\text{-}y_2, y_2\text{-}v_k\text{-}y_3, y_3\text{-}y_2\text{-}v_\ell, v_\ell\text{-}y_3\text{-}y_4, y_4\text{-}y_3\text{-}v_k\}$, where $(k, \ell) \in \{(29, 30), (31, 32), \ldots, (63, 64)\}$, is a 6-cycle in $K^{(3)}_{29,36}$. For each $P_3 : (z_1, z_2, z_3)$ in the resulting decomposition of K_{29}, $\{z_2\text{-}z_3\text{-}v_k, v_k\text{-}z_2\text{-}z_1, z_1\text{-}z_2\text{-}v_\ell, v_\ell\text{-}z_2\text{-}z_3, z_3\text{-}z_2\text{-}v_m, v_m\text{-}z_1\text{-}z_2\}$, where $(k, \ell, m) \in \{(29, 30, 31), (32, 33, 34), \ldots, (62, 63, 64)\}$, is a 6-cycle in $K^{(3)}_{29,36}$.

By Theorem 1.2, the complete graph K_{36} with vertex set $[v_{29}, v_{64}]$ is P_7-decomposable. For each $P_7 : (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in the P_7-decomposition of K_{36}, $(v_i\text{-}x_1\text{-}x_2, x_2\text{-}v_i\text{-}x_3, x_3\text{-}v_i\text{-}x_4, x_4\text{-}v_i\text{-}x_5, x_5\text{-}v_i\text{-}x_6, x_6\text{-}x_7\text{-}v_i)$, where $i \in [0, 28]$, is a 6-cycle in $K^{(3)}_{29,36}$.

The collection of all these 6-cycles yields a decomposition of $K^{(3)}_{29,36}$ into 6-cycles. □

2.5 $K^{(3)}_n$ to $K^{(3)}_{n+1}$

Lemma 2.10. *If* $n \geq 7$, $n \equiv 0$, 1, 4, *or* 9 (mod 12) *and the hypergraph* $K^{(3)}_n$ *has a* 6*-cycle decomposition, then the hypergraph* $K^{(3)}_{n+1}$ *has a* 6*-cycle decomposition.*

Proof. Let $V(K^{(3)}_{n+1}) = \{\infty\} \cup \{v_i : i \in \mathbb{Z}_n\}$ and $\mathcal{E}(K^{(3)}_{n+1}) = \mathcal{E}(K^{(3)}_n) \cup \{(\infty\text{-}v_i\text{-}v_j) \mid i, j \in [0, n-1]\}$. By hypothesis, $K^{(3)}_n$ has a 6-cycle decomposition. It is enough to prove that the remaining 3-uniform hypergraph $\{\infty\text{-}v_i\text{-}v_j \mid i, j \in [0, n-1]\}$ admits a 6-cycle decomposition. By Theorem 1.2, the complete graph K_n has a P_7-decomposition. Let $\mathcal{P}$ be the set of all paths of length 6 in the decomposition of K_n. If $P_7 = (v_0, v_1, \ldots, v_6) \in \mathcal{P}$, then $(\infty\text{-}v_0\text{-}v_1, v_1\text{-}\infty\text{-}v_2, \ldots, v_4\text{-}\infty\text{-}v_5, v_5\text{-}v_6\text{-}\infty)$ is a 6-cycle in $K^{(3)}_{n+1}$. Applying the method to each path $P_7 \in \mathcal{P}$, we get a 6-cycle decomposition of $K^{(3)}_{n+1}$. □

3 $n \equiv 0 \pmod{18}$

Lemma 3.1. *$K_9^{(3)}$ decomposes into 6-cycles.*

Proof. A 6-cycle decomposition of $K_9^{(3)}$ is as follows:

$$\begin{aligned}
&(v_0\text{-}v_1\text{-}v_2, v_2\text{-}v_3\text{-}v_4, v_4\text{-}v_5\text{-}v_6, v_6\text{-}v_7\text{-}v_8, v_8\text{-}v_4\text{-}v_3, v_3\text{-}v_2\text{-}v_0),\\
&(v_0\text{-}v_1\text{-}v_3, v_3\text{-}v_2\text{-}v_5, v_5\text{-}v_7\text{-}v_4, v_4\text{-}v_6\text{-}v_8, v_8\text{-}v_5\text{-}v_7, v_7\text{-}v_8\text{-}v_0),\\
&(v_0\text{-}v_2\text{-}v_4, v_4\text{-}v_1\text{-}v_3, v_3\text{-}v_0\text{-}v_5, v_5\text{-}v_2\text{-}v_1, v_1\text{-}v_4\text{-}v_8, v_8\text{-}v_5\text{-}v_0),\\
&(v_0\text{-}v_4\text{-}v_1, v_1\text{-}v_5\text{-}v_6, v_6\text{-}v_8\text{-}v_2, v_2\text{-}v_5\text{-}v_7, v_7\text{-}v_0\text{-}v_4, v_4\text{-}v_5\text{-}v_0),\\
&(v_0\text{-}v_8\text{-}v_3, v_3\text{-}v_5\text{-}v_1, v_1\text{-}v_7\text{-}v_4, v_4\text{-}v_3\text{-}v_7, v_7\text{-}v_4\text{-}v_6, v_6\text{-}v_8\text{-}v_0),\\
&(v_0\text{-}v_8\text{-}v_1, v_1\text{-}v_3\text{-}v_8, v_8\text{-}v_5\text{-}v_3, v_3\text{-}v_7\text{-}v_6, v_6\text{-}v_4\text{-}v_2, v_2\text{-}v_6\text{-}v_0),\\
&(v_1\text{-}v_0\text{-}v_6, v_6\text{-}v_4\text{-}v_0, v_0\text{-}v_6\text{-}v_7, v_7\text{-}v_2\text{-}v_3, v_3\text{-}v_4\text{-}v_5, v_5\text{-}v_4\text{-}v_1),\\
&(v_4\text{-}v_3\text{-}v_0, v_0\text{-}v_5\text{-}v_6, v_6\text{-}v_8\text{-}v_1, v_1\text{-}v_6\text{-}v_7, v_7\text{-}v_1\text{-}v_8, v_8\text{-}v_0\text{-}v_4),\\
&(v_5\text{-}v_6\text{-}v_7, v_7\text{-}v_4\text{-}v_8, v_8\text{-}v_3\text{-}v_6, v_6\text{-}v_5\text{-}v_2, v_2\text{-}v_8\text{-}v_4, v_4\text{-}v_2\text{-}v_5),\\
&(v_5\text{-}v_1\text{-}v_0, v_0\text{-}v_7\text{-}v_3, v_3\text{-}v_1\text{-}v_7, v_7\text{-}v_0\text{-}v_2, v_2\text{-}v_7\text{-}v_6, v_6\text{-}v_3\text{-}v_5),\\
&(v_6\text{-}v_5\text{-}v_8, v_8\text{-}v_2\text{-}v_3, v_3\text{-}v_6\text{-}v_2, v_2\text{-}v_3\text{-}v_1, v_1\text{-}v_2\text{-}v_4, v_4\text{-}v_3\text{-}v_6),\\
&(v_6\text{-}v_3\text{-}v_0, v_0\text{-}v_5\text{-}v_2, v_2\text{-}v_8\text{-}v_5, v_5\text{-}v_3\text{-}v_7, v_7\text{-}v_5\text{-}v_1, v_1\text{-}v_2\text{-}v_6),\\
&(v_7\text{-}v_8\text{-}v_2, v_2\text{-}v_0\text{-}v_8, v_8\text{-}v_7\text{-}v_3, v_3\text{-}v_1\text{-}v_6, v_6\text{-}v_4\text{-}v_1, v_1\text{-}v_2\text{-}v_7),\\
&(v_8\text{-}v_5\text{-}v_4, v_4\text{-}v_7\text{-}v_2, v_2\text{-}v_8\text{-}v_1, v_1\text{-}v_0\text{-}v_7, v_7\text{-}v_0\text{-}v_5, v_5\text{-}v_1\text{-}v_8).
\end{aligned}$$

□

Lemma 3.2. *$K_{18}^{(3)}$ decomposes into 6-cycles.*

Proof. By Lemmas 3.1 and 2.3, $K_9^{(3)}$ and $\mathcal{H}_9$ are, respectively, 6-cycle decomposable, and so is $K_{18}^{(3)} = 2K_9^{(3)} \oplus \mathcal{H}_9$. □

Lemma 3.3. *For each positive integer $n \geq 36$, with $n \equiv 0 \pmod{18}$, $K_n^{(3)}$ decomposes into 6-cycles.*

Proof. Let $n = 18k$ where $k \geq 2$ is a positive integer. We may think of $K_n^{(3)}$ as k copies of $K_{18}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{18}$ and $k(k-1)(k-2)/6$ copies of $\mathcal{H}'_{18}$. That is: for $k = 2$, $K_{36}^{(3)} = 2K_{18}^{(3)} \oplus \mathcal{H}_{18}$; and for $k \geq 3$, $K_{18k}^{(3)} = kK_{18}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{18} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}'_{18}$. As each of the hypergraphs $K_{18}^{(3)}$, $\mathcal{H}_{18}$ and $\mathcal{H}'_{18}$ is decomposable into 6-cycles by Lemmas 3.2, 2.5 and 2.1, respectively, we have the required decomposition. □

4 $n \equiv 2 \pmod{18}$

Lemma 4.1. *$K_{20}^{(3)}$ decomposes into 6-cycles.*

Proof. By Lemmas 2.10 and 2.6, $K_{10}^{(3)}$ and $\mathcal{H}_{10}$ are, respectively, 6-cycle decomposable and so is $K_{20}^{(3)} = 2K_{10}^{(3)} \oplus \mathcal{H}_{10}$. □

Lemma 4.2. *For each positive integer $n \geq 38$, with $n \equiv 2 \pmod{18}$, $K_n^{(3)}$ decomposes into 6-cycles.*

Proof. Let $n = 18k + 2$ where $k \geq 2$ is a positive integer. We may think of $K_n^{(3)}$ as k copies of $K_{20}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{18}$, $k(k-1)(k-2)/6$ copies of $\mathcal{H}_{18}^{'}$ and $k(k-1)$ copies of $\mathcal{H}_{18}^{''}$. That is: for $k \geq 2$, $K_{38}^{(3)} = 2K_{20}^{(3)} \oplus \mathcal{H}_{18} \oplus 2\mathcal{H}_{18}^{''}$; and for $k \geq 3$, $K_{18k+2}^{(3)} = kK_{20}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{18} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}_{18}^{'} \oplus k(k-1)\mathcal{H}_{18}^{''}$. As each of the hypergraphs $K_{20}^{(3)}$, $\mathcal{H}_{18}$, $\mathcal{H}_{18}^{'}$ and $\mathcal{H}_{18}^{''}$ is decomposable into 6-cycles by Lemmas 4.1, 2.5, 2.1 and 2.2, respectively, we have the required decomposition. □

5 $n \equiv 1 \pmod{36}$

Lemma 5.1. *For each positive integer* $n \geq 37$*, with* $n \equiv 1 \pmod{36}$*,* $K_n^{(3)}$ *decomposes into* 6*-cycles.*

Proof. By Lemma 3.3, $K_{36}^{(3)}$ is decomposable into 6-cycles, and therefore by Lemma 2.10, $K_{37}^{(3)}$ is decomposable into 6-cycles.

Let $n = 36k + 1$, where $k \geq 2$ is a positive integer. We may think of $K_n^{(3)}$ as k copies of $K_{36}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{36}$, $k(k-1)(k-2)/6$ copies of $\mathcal{H}_{36}^{'}$ and $k(k-1)/2$ copies of $\mathcal{H}_{36}^{''}$. That is: for $k = 2$, $K_{73}^{(3)} = 2K_{37}^{(3)} \oplus \mathcal{H}_{36} \oplus \mathcal{H}_{36}^{''}$; and for $k \geq 3$, $K_{36k+1}^{(3)} = kK_{37}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{36} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}_{36}^{'} \oplus \frac{k(k-1)}{2}\mathcal{H}_{36}^{''}$. As each of the hypergraphs $K_{37}^{(3)}$, $\mathcal{H}_{36}$, $\mathcal{H}_{36}^{'}$ and $\mathcal{H}_{36}^{''}$ is decomposable into 6-cycles by above and by Lemmas 2.5, 2.1 and 2.2, respectively, we have the required decomposition. □

6 $n \equiv 10 \pmod{18}$

Lemma 6.1. $K_{10}^{(3)}$ *decomposes into* 6*-cycles.*

Proof. By Lemma 3.1, $K_9^{(3)}$ is decomposable into 6-cycles, and therefore by Lemma 2.10, $K_{10}^{(3)}$ is decomposable into 6-cycles. □

Lemma 6.2. $K_{28}^{(3)}$ *decomposes into* 6*-cycles.*

Proof. By Lemmas 6.1, 3.2 and 2.8, $K_{10}^{(3)}$, $K_{18}^{(3)}$ and $K_{10,18}^{(3)}$ are, respectively, 6-cycle decomposable, and so is $K_{28}^{(3)} = K_{10}^{(3)} \oplus K_{18}^{(3)} \oplus K_{10,18}^{(3)}$. □

Lemma 6.3. *For each positive integer* $n \geq 46$*, with* $n \equiv 10 \pmod{18}$*,* $K_n^{(3)}$ *decomposes into* 6*-cycles.*

Proof. Let $n = 18k + 10$, where $k \geq 2$ is a positive integer. We may think of $K_n^{(3)}$ as an edge-disjoint union of a copy of $K_{10}^{(3)}$, k copies of $K_{18}^{(3)}$, k copies of $K_{10,18}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{18}$, $k(k-1)(k-2)/6$ copies of $\mathcal{H}_{18}^{'}$ and $5k(k-1)$ copies of $\mathcal{H}_{18}^{''}$. That is: for $k = 2$, $K_{46}^{(3)} = K_{10}^{(3)} \oplus 2K_{18}^{(3)} \oplus 2K_{10,18}^{(3)} \oplus \mathcal{H}_{18} \oplus 10\mathcal{H}_{18}^{''}$; and for $k \geq 3$, $K_{18k+10}^{(3)} = K_{10}^{(3)} \oplus kK_{18}^{(3)} \oplus kK_{10,18}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{18} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}_{18}^{'} \oplus 5k(k-1)\mathcal{H}_{18}^{''}$. As each of the hypergraphs $K_{10}^{(3)}$, $K_{18}^{(3)}$, $K_{10,18}^{(3)}$, $\mathcal{H}_{18}$, $\mathcal{H}_{18}^{'}$ and $\mathcal{H}_{18}^{''}$ is decomposable into 6-cycles by Lemmas 6.1, 3.2, 2.8, 2.5, 2.1 and 2.2, respectively, we have the required decomposition. □

7 $n \equiv 9 \pmod{36}$

Lemma 7.1. *For each positive integer $n \geq 45$, with $n \equiv 9 \pmod{36}$, $K_n^{(3)}$ decomposes into 6-cycles.*

Proof. Let $n = 36k+9$, where k is a positive integer. We may think of $K_n^{(3)}$ as an edge-disjoint union of a copy of $K_9^{(3)}$, k copies of $K_{36}^{(3)}$, k copies of $K_{9,36}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{36}$, $k(k-1)(k-2)/6$ copies of $\mathcal{H}_{36}^{'}$ and $9k(k-1)/2$ copies of $\mathcal{H}_{36}^{''}$. That is: for $k = 1$, $K_{45}^{(3)} = K_9^{(3)} \oplus K_{36}^{(3)} \oplus K_{9,36}^{(3)}$; for $k = 2$, $K_{81}^{(3)} = K_9^{(3)} \oplus 2K_{36}^{(3)} \oplus 2K_{9,36}^{(3)} \oplus \mathcal{H}_{36} \oplus 9\mathcal{H}_{36}^{''}$; and for $k \geq 3$, $K_{36k+9}^{(3)} = K_9^{(3)} \oplus kK_{36}^{(3)} \oplus kK_{9,36}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{36} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}_{36}^{'} \oplus 9\frac{k(k-1)}{2}\mathcal{H}_{36}^{''}$. As each of the hypergraphs $K_9^{(3)}$, $K_{36}^{(3)}$, $K_{9,36}^{(3)}$, $\mathcal{H}_{36}$, $\mathcal{H}_{36}^{'}$ and $\mathcal{H}_{36}^{''}$ is decomposable into 6-cycles by Lemmas 3.1, 3.3, 2.7, 2.5, 2.1 and 2.2, respectively, we have the required decomposition. □

8 $n \equiv 29 \pmod{36}$

Lemma 8.1. *$K_{29}^{(3)}$ decomposes into 6-cycles.*

Proof. By Lemma 6.2, $K_{28}^{(3)}$ is decomposable into 6-cycles, and therefore by Lemma 2.10, $K_{29}^{(3)}$ is decomposable into 6-cycles. □

Lemma 8.2. *For each positive integer $n \geq 65$, with $n \equiv 29 \pmod{36}$, $K_n^{(3)}$ decomposes into 6-cycles.*

Proof. Let $n = 36k + 29$, where k is a positive integer. We may think of $K_n^{(3)}$ as an edge-disjoint union of a copy of $K_{29}^{(3)}$, k copies of $K_{36}^{(3)}$, k copies of $K_{29,36}^{(3)}$, $k(k-1)/2$ copies of $\mathcal{H}_{36}$, $k(k-1)(k-2)/6$ copies of $\mathcal{H}_{36}^{'}$ and $29k(k-1)/2$ copies of $\mathcal{H}_{36}^{''}$. That is: for $k = 1$, $K_{65}^{(3)} = K_{29}^{(3)} \oplus K_{36}^{(3)} \oplus K_{29,36}^{(3)}$; for $k = 2$, $K_{101}^{(3)} = K_{29}^{(3)} \oplus 2K_{36}^{(3)} \oplus 2K_{29,36}^{(3)} \oplus \mathcal{H}_{36} \oplus 29\mathcal{H}_{36}^{''}$; and for $k \geq 3$, $K_{36k+29}^{(3)} = K_{29}^{(3)} \oplus kK_{36}^{(3)} \oplus kK_{29,36}^{(3)} \oplus \frac{k(k-1)}{2}\mathcal{H}_{36} \oplus \frac{k(k-1)(k-2)}{6}\mathcal{H}_{36}^{'} \oplus 29\frac{k(k-1)}{2}\mathcal{H}_{36}^{''}$. As each of the hypergraphs $K_{29}^{(3)}$, $K_{36}^{(3)}$, $K_{29,36}^{(3)}$, $\mathcal{H}_{36}$, $\mathcal{H}_{36}^{'}$ and $\mathcal{H}_{36}^{''}$ is decomposable into 6-cycles by Lemmas 8.1, 3.3, 2.9, 2.5, 2.1 and 2.2, respectively, we have the required decomposition. □

9 Main result

Theorem 9.1. *For $n \geq 6$, the complete 3-uniform hypergraphs $K_n^{(3)}$ has a 6-cycle decomposition if and only if $n \equiv 0 \pmod{18}$, $2 \pmod{18}$, $10 \pmod{18}$, $1 \pmod{36}$, $9 \pmod{36}$ or $29 \pmod{36}$.*

Proof. This follows from Lemmas 1.1, 3.2, 3.3, 4.1, 4.2, 5.1, 6.1, 6.2, 6.3, 3.1, 7.1, 8.1 and 8.2. □

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