# On the Wiener index of Cohen-Macaulay and very well-covered graphs 

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#### Abstract

In this paper, we obtain some lower bounds for the Wiener index of Cohen-Macaulay graphs. We also give a lower bound for the Wiener index of very well-covered graphs.


## 1 Introduction

In a molecule, if we represent atoms by vertices and bonds by edges, we obtain a molecular graph $[16,18]$. Graph theoretic invariants of molecular graphs, which predict properties of the corresponding molecule, are known as topological indices. The oldest topological index is the Wiener index [26], which was introduced in 1947 as the path number.

At first the Wiener index was used for predicting the boiling points of paraffins [26], but later a strong correlation between the Wiener index and the chemical properties of a compound was found. Nowadays this index is a tool used for preliminary screening of drug molecules [1]. The Wiener index also predicts binding energy of protein-ligand complex at a preliminary stage. A great deal of knowledge on the Wiener index is accumulated in several survey papers $[5,6,11]$.

In this paper, a graph is assumed to be finite and simple. Denote by $G=$ $(V(G), E(G))$ the graph with vertex set $V(G)$ and edge set $E(G)$. The distance between the vertices $u$ and $v$ of $G$ is denoted by $d_{G}(u, v)$ or $d(u, v)$ which is defined as the length of a shortest path between $u$ and $v$ in $G$. The Wiener index of a graph $G$, denoted by $W(G)$, is the sum of the distances between all (unordered) pairs of vertices of $G$,

$$
W(G)=\sum_{\{u, v\} \subseteq V(G)} d(u, v) .
$$

Let $\mathbb{K}$ be a field and let $S=\mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ be the polynomial ring in $n$ variables over $\mathbb{K}$ with each $x_{i}$ of degree 1 . Let $I \subset S$ be a monomial ideal and $G(I)$ its unique minimal monomial generators.

We consider the polynomial ring $\mathbb{K}[V(G)]$ whose variables are $x_{v}, v \in V(G)$. The ideal of $\mathbb{K}[V(G)]$ generated by quadratic squarefree monomial ideals $x_{u} x_{v},\{u, v\} \in$ $E(G)$ is called the edge ideal of $G$ and is denoted by $I(G)$. A graph $G$ is called Cohen-Macaulay over the field $\mathbb{K}$ if $\mathbb{K}[V(G)] / I(G)$ is a Cohen-Macaulay ring (see [2, 24]). A subset $F$ of $V(G)$ is a stable set or independent set if $e \nsubseteq F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. Here $G$ is called well-covered if every maximal stable set has the same cardinality. On the other hand, a subset $D$ of $V(G)$ is a vertex cover of $G$ if $D \cap e \neq \emptyset$ for every $e \in E(G)$. The number of vertices in a minimum vertex cover of $G$ is called the covering number of $G$ and is denoted by $\alpha(G)$. This number coincides with height $(I(G))$, the height of $I(G)$. If the minimal vertex covers have the same cardinality, then $G$ is called an unmixed graph. The Stanley-Reisner complex of $I(G)$, denoted by $\Delta_{G}$, is the simplicial complex whose faces are the stable sets of $G$. Recall that a simplicial complex $\Delta$ is called pure if every facet has the same number of elements. Thus, $\Delta_{G}$ is pure if and only if $G$ is well-covered.

Some properties of $G, \Delta_{G}$ and $I(G)$ allow an interaction between commutative algebra and combinatorial theory. Examples of these properties are: CohenMacaulayness, shellability, vertex decomposability and well-coveredness. These properties have been studied in $[2,7,12,20,22,23,24]$.

The present paper is organized as follows. In Section 1 we connect the Wiener index with some homological and algebraic invariants, such as projective dimension and regularity. In Theorem 2.4, we give a lower bound for the Wiener index of general Cohen-Macaulay graphs in terms of projective dimension of the graphs. We use this result to obtain a lower bound for the Wiener index of a Cohen-Macaulay graph $G$ by using the regularity of $J(G)=I(G)^{\vee}$, see Corollary 2.6.

In Section 2 we consider a class of graphs $G$ such that the height of the edge ideal $I(G)$ is half of the number $|V(G)|$ of the vertices. Such a class of graphs is rich, because it includes all the unmixed bipartite graphs and all the grafted graphs. In Theorem 3.2, we obtain a lower bound for the Wiener index of very well-covered graphs. In the last section we give a lower bound for the Wiener index of Boolean graphs, see Theorem 4.1.

## 2 Cohen-Macaulay graphs

In this section we give some lower bounds for the Wiener index of Cohen-Macaulay graphs. Since calculating the Wiener index of a graph can be computationally expensive, it is of some interest to know the extreme values of the Wiener index.

Let $S=\mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ be the polynomial ring over a field $\mathbb{K}$ in the variables $x_{1}, \ldots, x_{n}$, and let $I \subset S$ be a monomial ideal. Let $G$ be a finite simple graph on the vertex set $V(G)$ with edge set $E(G)$. In other words, $|V(G)|<\infty$ and $E(G) \subset V(G) \times V(G) \backslash\{\{v, v\}: v \in V(G)\}$. All graphs considered in this paper are finite simple graphs, which henceforth will simply be called graphs. Let $\mathbb{K}[V(G)]$ be the polynomial ring over a field $\mathbb{K}$ whose variables are vertices of $G$. The empty
graph on $n$ vertices is denoted by $E_{n}$.
A vertex cover of a graph $G$ on $V(G)$ is a subset $D \subset V(G)$ such that $\{u, v\} \cap D \neq$ $\emptyset$ for all $\{u, v\} \in E(G)$. A vertex cover $D$ is called minimal if $D$ is a vertex of $G$, and no proper subset of $D$ is a vertex cover of $G$. We denote by $\alpha(G)$ the set of minimal vertex covers of $G$.

An independent set of $G$ is a set $F \subset V(G)$ such that $\{u, v\} \notin E(G)$ for all $u, v \in F$. Obviously, $F$ is an independent set of $G$ if and only if $V(G) \backslash F$ is a vertex cover of $G$. Thus the maximal independent sets of $G$ correspond to the minimal vertex covers of $G$. The vertex independence number, denoted by $\beta(G)$, is the number of vertices in any largest independent set of vertices.

Consider the minimal free graded resolution of $M=\mathbb{K}[V(G)] / I(G)$ as a $\mathbb{K}[V(G)]$ module.

$$
0 \rightarrow \oplus_{j \in \mathbf{Z}} \mathbb{K}[V(G)](-j)^{\beta_{p j}}(M) \rightarrow \cdots \rightarrow \oplus_{j \in \mathbf{Z}} \mathbb{K}[V(G)](-j)^{\beta_{0 j}}(M) \rightarrow M \rightarrow 0
$$

The Castelnuovo-Mumford regularity (or simply the regularity) of $M=$ $\mathbb{K}[V(G)] / I(G)$ is defined as

$$
\operatorname{reg}(\mathbb{K}[V(G)] / I(G)):=\max \left\{j-i: \beta_{i, j} \neq 0\right\}
$$

Also, the projective dimension of $M$ is defined as

$$
\operatorname{pd}(M):=\max \left\{i: \beta_{i, j} \neq 0 \text { for some } j\right\} .
$$

We define $\operatorname{pd}(G):=\operatorname{pd}(\mathbb{K}[V(G)] / I(G))$.
$G$ is said to be a Cohen-Macaulay graph over $\mathbb{K}$ if

$$
\operatorname{depth}(\mathbb{K}[V(G)] / I(G))=\operatorname{dim}(\mathbb{K}[V(G)] / I(G))
$$

Remark 2.1. The edge ideal $I(G)$ is Cohen-Macaulay if and only if $\mathbb{K}[V(G)] / I(G)$ is Cohen-Macaulay, that is,

$$
\operatorname{depth}(\mathbb{K}[V(G)] / I(G))=\operatorname{dim}(\mathbb{K}[V(G)] / I(G))
$$

A finite graph $G$ is called unmixed if all minimal vertex covers of $G$ have the same cardinality.

Let $G_{1}$ and $G_{2}$ be graphs on the vertex sets $V\left(G_{1}\right)=\left\{x_{1}, \ldots, x_{n}\right\}$ and $V\left(G_{2}\right)=$ $\left\{y_{1}, \ldots, y_{m}\right\}$, respectively. Then the join of $G_{1}$ and $G_{2}$, denoted by $G_{1} * G_{2}$, is a graph on the vertex set $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and edge set

$$
E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{\left\{x_{i}, y_{j}\right\}: 1 \leq i \leq n, 1 \leq j \leq m\right\} .
$$

Example 2.2. Let $E_{m}$ be the empty graph on the vertex set $\left\{y_{1}, \ldots, y_{m}\right\}$ and let $K_{n} * E_{m}$ be the join of complete graph $K_{n}$ and $E_{m}$. It is easy to see that $K_{1} * E_{n} \cong S_{n}$ and $K_{n} * E_{1} \cong K_{n}$, where $S_{n}$ is the star graph on $n+1$ vertices.

In the following, we compute the minimal prime ideals of joins of graphs.
Proposition 2.3. Let $G_{1}$ and $G_{2}$ be graphs on the vertex sets $V\left(G_{1}\right)=\left\{x_{1}, \ldots, x_{n}\right\}$ and $V\left(G_{2}\right)=\left\{y_{1}, \ldots, y_{m}\right\}$, respectively. Suppose that $R=\mathbb{K}\left[V\left(G_{1} * G_{2}\right)\right]$.
i) If $n+\operatorname{height}\left(I\left(G_{2}\right)\right)=m+\operatorname{height}\left(I\left(G_{1}\right)\right)$ then

$$
\left|\operatorname{Min}\left(I\left(G_{1} * G_{2}\right)\right)\right|=\left|\operatorname{Min}\left(I\left(G_{1}\right)\right)\right|+\left|\operatorname{Min}\left(I\left(G_{2}\right)\right)\right| .
$$

ii) If $n+\operatorname{height}\left(I\left(G_{2}\right)\right)>m+\operatorname{height}\left(I\left(G_{1}\right)\right)$ then

$$
\left|\operatorname{Min}\left(I\left(G_{1} * G_{2}\right)\right)\right|=\left|\operatorname{Min}\left(I\left(G_{1}\right)\right)\right| .
$$

iii) If $n+\operatorname{height}\left(I\left(G_{2}\right)\right)<m+\operatorname{height}\left(I\left(G_{1}\right)\right)$ then

$$
\left|\operatorname{Min}\left(I\left(G_{1} * G_{2}\right)\right)\right|=\left|\operatorname{Min}\left(I\left(G_{2}\right)\right)\right| .
$$

Proof. For a graph $G$, it is well-known that the minimal prime ideals of $I(G)$ correspond to the minimal vertex covers of $G$. Note that if $\mathbf{p} \in \operatorname{Min}\left(I\left(G_{1} * G_{2}\right)\right)$, then either $\mathbf{p}=p_{1}+\left(y_{1}, \ldots, y_{m}\right)$ or $\mathbf{p}=p_{2}+\left(x_{1}, \ldots, x_{n}\right)$, where $p_{1} \in \operatorname{Min}\left(I\left(G_{1}\right)\right)$ and $p_{2} \in \operatorname{Min}\left(I\left(G_{2}\right)\right)$. Hence the assertion follows.

A simplicial complex $\Delta$ on the vertex set $V=\left\{x_{1}, \ldots, x_{n}\right\}$ is a collection of subsets of $V$ such that (i) $x_{i} \in \Delta$ for all $x_{i} \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a face of $\Delta$. For $F \subset V$, we define the dimension of $F$ by $\operatorname{dim} F=|F|-1$. Let $d=\max \{|F|: F \in \Delta\}$ and define the dimension of $\Delta$ to be $\operatorname{dim} \Delta=d-1$. A maximal face of $\Delta$ with respect to inclusion is called a facet of $\Delta$. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called pure.

If $I$ is an ideal of $S$ generated by squarefree monomials, the Stanley-Reisner simplicial complex $\Delta_{I}$ associated to $I$ has vertex set $V=\left\{x_{i} \mid x_{i} \notin I\right\}$ and its faces are defined by

$$
\Delta_{I}=\left\{\left\{x_{i_{1}}, \ldots, x_{i_{k}}\right\} \mid i_{1}<\cdots<i_{k}, x_{i_{1}} \cdots x_{i_{k}} \notin I\right\} .
$$

Conversely if $\Delta$ is a simplicial complex with vertex set $V$ contained in $\left\{x_{1}, \ldots, x_{n}\right\}$, the Stanley-Reisner ideal $I_{\Delta}$ is defined as

$$
I_{\Delta}=\left(\left\{x_{i_{1}} \cdots x_{i_{r}} \mid i_{1}<\cdots<i_{r},\left\{x_{i_{1}}, \ldots, x_{i_{r}}\right\} \notin \Delta\right\}\right),
$$

and its Stanley-Reisner ring $\mathbb{K}[\Delta]$ is defined as the quotient ring $S / I_{\Delta}$.
A simplicial complex $\Delta$ is said to be Cohen-Macaulay over $\mathbb{K}$ if the StanleyReisner ring $\mathbb{K}[\Delta]$ is a Cohen-Macaulay ring.

Note that the Stanley-Reisner complex of $I(G)$ is given by $\Delta_{I(G)}=\Delta_{G}$, where $\Delta_{G}$ is the simplicial complex whose faces are the independent vertex sets of $G$. Thus

$$
\mathbb{K}\left[\Delta_{G}\right]=\mathbb{K}[V(G)] / I(G),
$$

where $\mathbb{K}\left[\Delta_{G}\right]$ is the Stanley-Reisner ring of $\Delta_{G}$. The simplicial complex $\Delta_{G}$ whose faces are the independent vertex sets of $G$ is called the independence complex of $G$.

Let $I \subset S$ be a monomial ideal and $G(I)$ its unique minimal monomial generators. The Auslander and Buchsbaum Theorem says that depth $S / I=\operatorname{dim} S-\operatorname{pd} S / I$, see for example [2].

Since height $(I)=\operatorname{dim} S-\operatorname{dim} S / I$, and since $S / I$ is Cohen-Macaulay if and only if $\operatorname{dim} S / I=\operatorname{depth} S / I$, it follows that

$$
\begin{equation*}
S / I \text { is Cohen-Macaulay } \Longleftrightarrow \operatorname{height}(I)=\operatorname{pd} S / I \tag{1}
\end{equation*}
$$

As the main result of this section we have the following.
Theorem 2.4. Let $G$ be a Cohen-Macaulay graph on $N$ vertices. Then

$$
W(G) \geq N^{2}+(1+\operatorname{pd}(G))\left(\frac{1}{2} \operatorname{pd}(G)-N\right)
$$

and the equality holds if and only if $\operatorname{pd}(G)=N-1$.
Proof. Let $G$ be a Cohen-Macaulay graph on the vertex set $V(G)$. We set $|V(G)|=$ $N$. Then [12, Lemma 9.1.10] implies that $G$ is unmixed, i.e., all of its associated primes have the same height. According to [12, Lemma 9.1.4] we have that the associated primes of an edge ideal correspond to the minimal vertex covers of $G$. Then [25, Corollary 7.2.5] implies that

$$
\text { height } I(G)=\operatorname{dim}(\mathbb{K}[V(G)])-\operatorname{dim}(\mathbb{K}[V(G)] / I(G))=N-\operatorname{dim}(\mathbb{K}[V(G)] / I(G)) \text {. }
$$

The complement of a vertex cover is an independent set, that is, a face of $\Delta_{G}$. It follows from [19, Theorem 1.3] that

$$
\operatorname{dim}(\mathbb{K}[V(G)] / I(G))=\operatorname{dim}\left(\mathbb{K}[V(G)] / I_{\Delta_{G}}\right)=\operatorname{dim} \Delta_{G}+1=\beta(G)-1+1=\beta(G)
$$

Since all the minimal vertex covers have the same cardinality, so do the facets of $\Delta_{G}$, that is, $\Delta_{G}$ is pure and $\operatorname{dim}\left(\Delta_{G}\right)=N-\operatorname{height}(I(G))-1$. Hence, by applying [2, Theorem 1.3.3] we obtain:

$$
\operatorname{pd}(G)=\operatorname{dim}(\mathbb{K}[V(G)])-\operatorname{depth}(\mathbb{K}[V(G)] / I(G))=N-\operatorname{dim}(\mathbb{K}[V(G)] / I(G))
$$

Suppose that $F \in \Delta_{G}$ is a maximum independent set of $G$. Thus [25, Corollary 6.3.5], [25, Corollary 7.2.5] together with (1) yield

$$
\begin{aligned}
W(G) & =\sum_{u, v \in G \backslash F} d(u, v)+\sum_{u, v \in F} d(u, v)+\sum_{u \in G \backslash F, v \in F} d(u, v) \\
& \geq\binom{ N-\beta(G)}{2}+2\binom{\beta(G)}{2}+\beta(G)(N-\beta(G)) \\
& =\frac{1}{2}(N-\beta(G))(N-\beta(G)-1)+\beta(G)(N-1) \\
& =\frac{1}{2}\left(N-\operatorname{dim} \Delta_{G}-1\right)\left(N-\operatorname{dim} \Delta_{G}-2\right)+\left(\operatorname{dim} \Delta_{G}+1\right)(N-1) \\
& =\frac{1}{2} \operatorname{pd}(G)(\operatorname{pd}(G)-1)+(N-\operatorname{pd}(G))(N-1) \\
& =N^{2}+(1+\operatorname{pd}(G))\left(\frac{1}{2} \operatorname{pd}(G)-N\right) .
\end{aligned}
$$

If the equality holds, then for every $u, v \in V(G) \backslash F$ the edge $\{u, v\} \in E(G)$ and every vertex of $F$ is adjacent to all vertices of $V(G) \backslash F$. This implies

$$
G \cong K_{\text {height }(I(G))} * E_{N-\operatorname{height}(I(G))} .
$$

One can see that if $C$ is a minimal vertex cover of $K_{\text {height }(I(G))} * E_{N-\operatorname{height}(I(G))}$, then either $C=A \cup V\left(E_{N-\operatorname{height}(I(G))}\right)$ or $C=V\left(K_{\text {height }(I(G))}\right) \cup B$, where $A$ and $B$ are minimal vertex covers of $K_{\text {height }(I(G))}$ and $E_{N-\operatorname{height}(I(G))}$, respectively. Then by [2, Proposition 1.2.9] we have

$$
\operatorname{depth}\left(\mathbb{K}\left[V\left(K_{\text {height }(I(G))} * E_{N-\operatorname{height}(I(G))}\right)\right] / I\left(K_{\operatorname{height}(I(G))} * E_{N-\operatorname{height}(I(G))}\right)\right)=1
$$

Since $G$ is a Cohen-Macaulay graph, by Lemma 2.3 together with (1) we have

$$
N-\operatorname{pd}\left(K_{\operatorname{pd}(G)} * E_{N-\operatorname{pd}(G)}\right)=1
$$

Thus $\operatorname{pd}(G)=N-1$.
Conversely, suppose that $\operatorname{pd}(G)=N-1$; then it is obvious that

$$
W(G)=N^{2}+(1+\operatorname{pd}(G))\left(\frac{1}{2} \operatorname{pd}(G)-N\right)
$$

and the proof is complete.
For a monomial ideal $I=\left(x_{11} \ldots x_{1 n_{1}}, \ldots, x_{t 1} \ldots x_{t n_{t}}\right)$ of the polynomial ring $S$, the Alexander dual ideal of $I$, denoted by $I^{\vee}$, is defined as

$$
I^{\vee}:=\left(x_{11}, \ldots, x_{1 n_{1}}\right) \cap \cdots \cap\left(x_{t 1}, \ldots, x_{t n_{t}}\right) .
$$

The cover ideal associated to a graph $G$ is the monomial ideal

$$
J(G):=I(G)^{\vee}=\bigcap_{\{i, j\} \in E(G)}\left(x_{i}, x_{j}\right)
$$

The following theorem, which was proved in [21], is one of our main tools in the study of the regularity of the ring $\mathbb{K}[V(G)] / I(G)$.

Theorem 2.5. [21, Theorem 2.1] Let $I \subset S=\mathbb{K}\left[x_{1}, \ldots, x_{n}\right]$ be a squarefree monomial ideal. Then $\operatorname{pd}\left(I^{\vee}\right)=\operatorname{reg}(S / I)$.

By using Theorems 2.4 and 2.5, we have the following corollary.
Corollary 2.6. Let $G$ be a Cohen-Macaulay graph on $N$ vertices. Then

$$
W(G) \geq N^{2}+(1+\operatorname{reg}(J(G)))\left(\frac{1}{2} \operatorname{reg}(J(G))-N\right)
$$

and the equality holds if and only if $\operatorname{reg}(J(G))=N-1$.

Example 2.7. Consider the class $\mathcal{S Q C}$ of well-covered graphs from [17]. A vertex $v$ of a graph $G$ is said to be simplicial if the induced subgraph of $G$ on the set $N[v]$ is a complete graph and we say this complete graph is a simplex of $G$.

A 5 -cycle $C_{5}$ of a graph $G$ is called basic if $C_{5}$ does not contain two adjacent vertices of degree 3 or more in $G$; a 4 -cycle is called basic if it contains two adjacent vertices of degree 2, and the remaining two vertices belong to a complete subgraph or a basic 5-cycle of $G$.

A graph is in the class $\mathcal{S Q C}$ if there are simplicial vertices $x_{1}, \ldots, x_{m}$; basic 5 -cycles $C^{1}, \ldots, C^{s}$; and basic 4 -cycles $Q^{1}, \ldots, Q^{t}$ such that

$$
V(G)=\bigcup_{j=1}^{m} N\left[x_{j}\right] \cup \bigcup_{j=1}^{s} V\left(C_{j}\right) \cup \bigcup_{j=1}^{t} B\left(Q^{j}\right)
$$

and this forms a partition of $V(G)$, where $B\left(Q^{j}\right)$ is the set of two vertices of degree 2 of the basic 4 -cycle $Q^{j}$. Such a graph is Cohen-Macaulay [13, Theorem 2.3]. Therefore by Theorem 2.4 and [17, Theorem 3.1], we have

$$
\begin{aligned}
W(G) & \geq|V(G)|^{2}+(|V(G)|-m-2 s-t+1)\left(\frac{1}{2}(|V(G)|-m-2 s-t)-|V(G)|\right) \\
& =|V(G)|^{2}+(|V(G)|-m-2 s-t+1)\left(-\frac{1}{2}|V(G)|-m-2 s-t\right) \\
& =\frac{1}{2}(|V(G)|)(|V(G)|-1)+(-m-2 s-t)\left(\frac{1}{2}|V(G)|-m-2 s-t+1\right)
\end{aligned}
$$

and the equality holds if and only if $m+2 s+t=1$.

## 3 Wiener index of very well-covered graphs

In [9], Gitler and Valencia proved that if $G$ is a well-covered graph without isolated vertices, then $h(I(G)) \geq \frac{|V(G)|}{2}$.

A graph $G$ is called very well-covered if it is unmixed without isolated vertices and with $h(I(G))=\frac{|V(G)|}{2}$. Since the class of very well-covered graphs contains unmixed bipartite graphs, whiskered graphs and grafted graphs (see [4, 8]), it is interesting in the algebraic sense as well.

The main goal of this section is to study the Wiener index of very well-covered graphs. The following is a useful result on very well-covered graphs that allows us to assume a certain order on their vertices and edges.

Lemma 3.1. [10, Corollary 3.2] Let $G$ be a very well-covered graph with $2 n$ vertices. Then there is a relabeling of vertices $V(G)=\left\{x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right\}$ such that the following two conditions hold:
(1) $X=\left\{x_{1}, \ldots, x_{n}\right\}$ is a minimal vertex cover of $G$ and $Y=\left\{y_{1}, \ldots, y_{n}\right\}$ is a maximal independent set of $G$;
(2) for all $1 \leq i \leq n,\left\{x_{i}, y_{i}\right\} \in E(G)$.

Theorem 3.2. Let $G$ be a very well-covered graph with $2 n$ vertices. Then

$$
W(G) \geq \frac{1}{2} n(5 n-3)
$$

and the equality holds if and only if $G \cong K_{n} * E_{n}$.
Proof. Let $G$ be a very well-covered graph with $2 n$ vertices, and let $\Delta_{G}$ be its independence complex. By [25, Corollary 6.3.5] together with [25, Corollary 7.2.5] it follows that

$$
\begin{aligned}
2 n=\operatorname{height}(I(G))+\beta(G) & =\operatorname{height}(I(G))+\operatorname{dim}(\mathbb{K}[V(G)] / I(G)) \\
& =\operatorname{height}(I(G))+\operatorname{dim}\left(\mathbb{K}[V(G)] / I_{\Delta_{G}}\right) \\
& =\operatorname{height}(I(G))+\operatorname{dim} \Delta_{G}+1
\end{aligned}
$$

Therefore, Lemma 3.1 yields $\operatorname{dim}\left(\Delta_{G}\right)=n-1$ and hence $\beta(G)=n$. Suppose that $F \in \Delta_{G}$ is a maximum independent set of $G$. Then

$$
\begin{aligned}
W(G) & =\sum_{u, v \in G \backslash F} d(u, v)+\sum_{u, v \in F} d(u, v)+\sum_{u \in G \backslash F, v \in F} d(u, v) \\
& \geq\binom{ N-\beta(G)}{2}+2\binom{\beta(G)}{2}+\beta(G)(N-\beta(G)) \\
& =\frac{1}{2}(2 n-\beta(G))(2 n-\beta(G)-1)+\beta(G)(2 n-1) \\
& =\frac{1}{2}(2 n-n)(2 n-n-1)+n(2 n-1) \\
& =\frac{1}{2} n(5 n-3) .
\end{aligned}
$$

If the equality holds, then for every $u, v \in V(G) \backslash F$ the edge $\{u, v\} \in E(G)$ and every vertex of $F$ is adjacent to all vertices of $V(G) \backslash F$. This implies $G \cong K_{n} * E_{n}$. Conversely, suppose that $G \cong K_{n} * E_{n}$. Then

$$
\begin{aligned}
W\left(K_{n} * E_{n}\right) & =\sum_{u, v \in K_{n}} d(u, v)+\sum_{u, v \in E_{n}} d(u, v)+\sum_{u \in K_{n}, v \in E_{n}} d(u, v) \\
& =\frac{1}{2} n(n-1)+n(n-1)+n^{2} \\
& =\frac{1}{2} n(5 n-3),
\end{aligned}
$$

and the assertion follows.
Example 3.3. In [4], the authors introduced $B$-grafted graphs, which are a generalization of grafted graphs introduced by Faridi [8]. Let $H_{0}$ be a graph with the labeled vertices $1,2, \ldots, q$. For every $i=1, \ldots, q$, let $B_{i}$ be a bipartite graph with labeled partition $X_{i}$ and $Y_{i}$ such that $\left|X_{i}\right|=\left|Y_{i}\right|=n_{i}$. (We do not give a label to
each vertex of $B_{i}$, but we distinguish the partition $X_{i}$ and $Y_{i}$ ). We assume that $B_{i}$ has no isolated vertex for every $i=1, \ldots, q$. Let

$$
G=G\left(H_{0} ; B_{1}, \ldots, B_{q}\right)
$$

be a B-grafted graph with the vertex set $V(G):=X \cup Y$, where $X=X_{1} \cup \cdots \cup X_{q}$ and $Y=Y_{1} \cup \cdots \cup Y_{q}$.

The edge set $E(G)$ of $G$ is $x y \in E(G)$ if and only if either there exist $i, j$ such that $x \in X_{i}, y \in X_{j}$, and $i j \in E\left(H_{0}\right)$ or there exists $i$ such that $x \in X_{i}, y \in Y_{i}$, and $x y \in E\left(B_{i}\right)$.

Note that $X$ is a minimal vertex cover of $G$ and that $Y$ is a maximal independent set of $G$. If $G$ is an unmixed B-grafted graph, then by Theorem 3.2 we have

$$
\begin{aligned}
W(G) & \geq \frac{1}{2}\left(2\left(\sum_{i=1}^{q} n_{i}\right)-\sum_{i=1}^{q} n_{i}\right)\left(2\left(\sum_{i=1}^{q} n_{i}\right)-\sum_{i=1}^{q} n_{i}-1\right)+\left(\sum_{i=1}^{q} n_{i}\right)\left(2\left(\sum_{i=1}^{q} n_{i}\right)-1\right) \\
& =\frac{1}{2}\left(\sum_{i=1}^{q} n_{i}\right)\left(5\left(\sum_{i=1}^{q} n_{i}\right)-3\right)
\end{aligned}
$$

and the equality holds if and only if $G \cong K_{\sum_{i=1}^{q} n_{i}} * E_{\sum_{i=1}^{q} n_{i}}$.

## 4 Wiener index of Boolean graphs

In this section we obtain a lower bound for the Wiener index of Boolean graphs. Let $[n]=\{1, \ldots, n\}$ and let $2^{[n]}$ denote the power set of $[n]$. Recall from [15] that a finite Boolean graph, denoted by $B_{n}$, is a graph defined on the vertex set $2^{n} \backslash\{[n], \emptyset\}$, in which two vertices $u$ and $v$ are adjacent if $u \cap v=\emptyset$. Clearly, $B_{n}$ is also the zero-devisor graph of the finite Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_{2}$. Note that a finite or infinite Boolean graph has a unique corresponding zero-divisor commutative semigroup.

Theorem 4.1. For any $n \geq 1$, let $G=B_{n}$ be the Boolean graph. Then

$$
W(G) \geq \frac{5}{8} 2^{2 n}-\frac{13}{4} 2^{n}+4
$$

Proof. Suppose that $G=B_{n}$ is a Boolean graph for all $n \geq 1$. A subset $\Upsilon=$ $\left\{b_{1}, \ldots, b_{t}\right\}$ of $V(G)$ is an independent vertex set if and only if $b_{i} \cap b_{j} \neq \emptyset$ holds for any distinct $b_{i}, b_{j}$ in $\Upsilon$. By [14, Theorem 2.1] all maximal independent vertex sets $\Upsilon$ of $V(G)$ have the same cardinality $2^{n-1}-1$ and for any $b_{i} \in V(G)$, only one of $\left\{b_{i}, b_{i}^{c}\right\}$ is in $\Upsilon$, where $b_{i}^{c}=[n] \backslash b_{i}$. Thus the edge ideal of the graph $B_{n}$ has height $2^{n-1}-1$. Hence, by applying (1) and [14, Theorem 2.4] we obtain

$$
\operatorname{pd}\left(B_{n}\right)=2^{n-1}-1 .
$$

Therefore, Theorem 2.4 yields

$$
\begin{aligned}
W(G) & \geq \frac{1}{2}\left(2^{n-1}-1\right)\left(2^{n-1}-2\right)+\left(2^{n}-2^{n-1}-1\right)\left(2^{n}-3\right) \\
& =\left(2^{n}-2\right)^{2}+\left(1+2^{n-1}-1\right)\left(\frac{1}{2}\left(2^{n-1}-1\right)-\left(2^{n}-2\right)\right) \\
& =\frac{5}{8} 2^{2 n}-\frac{13}{4} 2^{n}+4
\end{aligned}
$$

Then the desired conclusion follows.
Example 4.2. Let $G=B_{4}$ be a Boolean graph. The edge ideal of $B_{4}$ is

$$
\begin{aligned}
I\left(B_{4}\right)= & \left(x_{1} x_{2}, x_{1} x_{3}, x_{1} x_{4}, x_{1} x_{8}, x_{1} x_{9}, x_{1} x_{10}, x_{1} x_{14},\right. \\
& x_{2} x_{3}, x_{2} x_{4}, x_{2} x_{6}, x_{2} x_{7}, x_{2} x_{10}, x_{2} x_{13}, x_{3} x_{4}, \\
& x_{3} x_{5}, x_{3} x_{7}, x_{3} x_{9}, x_{3} x_{12}, x_{4} x_{5}, x_{4} x_{6}, x_{4} x_{8}, \\
& \left.x_{4} x_{11}, x_{5} x_{10}, x_{6} x_{9}, x_{7} x_{8}\right) .
\end{aligned}
$$

We calculate the primary decomposition of $I\left(B_{4}\right)$ by CoCoA [3] as follows:

$$
\begin{aligned}
I\left(B_{4}\right)= & \left(x_{1}, x_{2}, x_{3}, x_{4}, x_{7}, x_{9}, x_{10}\right) \cap\left(x_{1}, x_{2}, x_{4}, x_{5}, x_{7}, x_{9}, x_{12}\right) \\
& \cap\left(x_{1}, x_{2}, x_{3}, x_{5}, x_{6}, x_{8}, x_{11}\right) \cap\left(x_{1}, x_{2}, x_{4}, x_{5}, x_{6}, x_{8}, x_{9}\right) \\
& \cap\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}\right) \cap\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{6}, x_{7}, x_{10}\right) \\
& \cap\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{6}, x_{8}, x_{10}\right) \cap\left(x_{1}, x_{3}, x_{4}, x_{6}, x_{7}, x_{10}, x_{13}\right) \\
& \cap\left(x_{1}, x_{2}, x_{3}, x_{4}, x_{8}, x_{9}, x_{10}\right) \cap\left(x_{2}, x_{3}, x_{4}, x_{8}, x_{9}, x_{10}, x_{14}\right) .
\end{aligned}
$$

Hence by (1) we have height $\left(I\left(B_{4}\right)\right)=\operatorname{pd}\left(I\left(B_{4}\right)\right)=7$. Therefore, Theorem 4.1 yields

$$
W(G) \geq \frac{5}{8} 2^{8}-\frac{13}{4} 2^{4}+4=112 .
$$

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(Received 15 Feb 2020; revised 7 June 2021)

