On the Wiener index of Cohen-Macaulay and very well-covered graphs

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Abstract

In this paper, we obtain some lower bounds for the Wiener index of Cohen-Macaulay graphs. We also give a lower bound for the Wiener index of very well-covered graphs.

1 Introduction

In a molecule, if we represent atoms by vertices and bonds by edges, we obtain a molecular graph [16, 18]. Graph theoretic invariants of molecular graphs, which predict properties of the corresponding molecule, are known as topological indices. The oldest topological index is the Wiener index [26], which was introduced in 1947 as the path number.

At first the Wiener index was used for predicting the boiling points of paraffins [26], but later a strong correlation between the Wiener index and the chemical properties of a compound was found. Nowadays this index is a tool used for preliminary screening of drug molecules [1]. The Wiener index also predicts binding energy of protein-ligand complex at a preliminary stage. A great deal of knowledge on the Wiener index is accumulated in several survey papers [5, 6, 11].

In this paper, a graph is assumed to be finite and simple. Denote by G = (V(G), E(G)) the graph with vertex set V(G) and edge set E(G). The distance between the vertices u and v of G is denoted by $d_G(u, v)$ or d(u, v) which is defined as the length of a shortest path between u and v in G. The Wiener index of a graph G, denoted by W(G), is the sum of the distances between all (unordered) pairs of vertices of G,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v).$$

Let \mathbb{K} be a field and let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over \mathbb{K} with each x_i of degree 1. Let $I \subset S$ be a monomial ideal and G(I) its unique minimal monomial generators.

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We consider the polynomial ring $\mathbb{K}[V(G)]$ whose variables are $x_v, v \in V(G)$. The ideal of $\mathbb{K}[V(G)]$ generated by quadratic squarefree monomial ideals $x_u x_v, \{u, v\} \in E(G)$ is called the edge ideal of G and is denoted by I(G). A graph G is called Cohen-Macaulay over the field \mathbb{K} if $\mathbb{K}[V(G)]/I(G)$ is a Cohen-Macaulay ring (see [2, 24]). A subset F of V(G) is a stable set or independent set if $e \notin F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. Here Gis called well-covered if every maximal stable set has the same cardinality. On the other hand, a subset D of V(G) is a vertex cover of G if $D \cap e \neq \emptyset$ for every $e \in E(G)$. The number of vertices in a minimum vertex cover of G is called the covering number of G and is denoted by $\alpha(G)$. This number coincides with height(I(G)), the height of I(G). If the minimal vertex covers have the same cardinality, then G is called an unmixed graph. The Stanley-Reisner complex of I(G), denoted by Δ_G , is the simplicial complex whose faces are the stable sets of G. Recall that a simplicial complex Δ is called pure if every facet has the same number of elements. Thus, Δ_G is pure if and only if G is well-covered.

Some properties of G, Δ_G and I(G) allow an interaction between commutative algebra and combinatorial theory. Examples of these properties are: Cohen-Macaulayness, shellability, vertex decomposability and well-coveredness. These properties have been studied in [2, 7, 12, 20, 22, 23, 24].

The present paper is organized as follows. In Section 1 we connect the Wiener index with some homological and algebraic invariants, such as projective dimension and regularity. In Theorem 2.4, we give a lower bound for the Wiener index of general Cohen-Macaulay graphs in terms of projective dimension of the graphs. We use this result to obtain a lower bound for the Wiener index of a Cohen-Macaulay graph G by using the regularity of $J(G) = I(G)^{\vee}$, see Corollary 2.6.

In Section 2 we consider a class of graphs G such that the height of the edge ideal I(G) is half of the number |V(G)| of the vertices. Such a class of graphs is rich, because it includes all the unmixed bipartite graphs and all the grafted graphs. In Theorem 3.2, we obtain a lower bound for the Wiener index of very well-covered graphs. In the last section we give a lower bound for the Wiener index of Boolean graphs, see Theorem 4.1.

2 Cohen-Macaulay graphs

In this section we give some lower bounds for the Wiener index of Cohen-Macaulay graphs. Since calculating the Wiener index of a graph can be computationally expensive, it is of some interest to know the extreme values of the Wiener index.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field \mathbb{K} in the variables x_1, \ldots, x_n , and let $I \subset S$ be a monomial ideal. Let G be a *finite simple graph* on the vertex set V(G) with edge set E(G). In other words, $|V(G)| < \infty$ and $E(G) \subset V(G) \times V(G) \setminus \{\{v, v\} : v \in V(G)\}$. All graphs considered in this paper are finite simple graphs, which henceforth will simply be called graphs. Let $\mathbb{K}[V(G)]$ be the polynomial ring over a field \mathbb{K} whose variables are vertices of G. The empty

graph on n vertices is denoted by E_n .

A vertex cover of a graph G on V(G) is a subset $D \subset V(G)$ such that $\{u, v\} \cap D \neq \emptyset$ for all $\{u, v\} \in E(G)$. A vertex cover D is called *minimal* if D is a vertex of G, and no proper subset of D is a vertex cover of G. We denote by $\alpha(G)$ the set of minimal vertex covers of G.

An independent set of G is a set $F \subset V(G)$ such that $\{u, v\} \notin E(G)$ for all $u, v \in F$. Obviously, F is an independent set of G if and only if $V(G) \setminus F$ is a vertex cover of G. Thus the maximal independent sets of G correspond to the minimal vertex covers of G. The vertex independence number, denoted by $\beta(G)$, is the number of vertices in any largest independent set of vertices.

Consider the minimal free graded resolution of $M = \mathbb{K}[V(G)]/I(G)$ as a $\mathbb{K}[V(G)]$ -module.

$$0 \to \bigoplus_{j \in \mathbf{Z}} \mathbb{K}[V(G)](-j)^{\beta_{pj}}(M) \to \dots \to \bigoplus_{j \in \mathbf{Z}} \mathbb{K}[V(G)](-j)^{\beta_{0j}}(M) \to M \to 0$$

The Castelnuovo-Mumford regularity (or simply the regularity) of $M = \mathbb{K}[V(G)]/I(G)$ is defined as

 $\operatorname{reg}(\mathbb{K}[V(G)]/I(G)) := \max\{j - i : \beta_{i,j} \neq 0\}.$

Also, the *projective dimension* of M is defined as

$$pd(M) := max\{i : \beta_{i,j} \neq 0 \text{ for some } j\}.$$

We define $pd(G) := pd(\mathbb{K}[V(G)]/I(G)).$

G is said to be a Cohen-Macaulay graph over \mathbb{K} if

$$depth(\mathbb{K}[V(G)]/I(G)) = \dim(\mathbb{K}[V(G)]/I(G)).$$

Remark 2.1. The edge ideal I(G) is Cohen-Macaulay if and only if $\mathbb{K}[V(G)]/I(G)$ is Cohen-Macaulay, that is,

$$depth(\mathbb{K}[V(G)]/I(G)) = \dim(\mathbb{K}[V(G)]/I(G)).$$

A finite graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality.

Let G_1 and G_2 be graphs on the vertex sets $V(G_1) = \{x_1, \ldots, x_n\}$ and $V(G_2) = \{y_1, \ldots, y_m\}$, respectively. Then the *join* of G_1 and G_2 , denoted by $G_1 * G_2$, is a graph on the vertex set $V(G_1) \cup V(G_2)$ and edge set

$$E(G_1) \cup E(G_2) \cup \{\{x_i, y_j\} : 1 \le i \le n, \ 1 \le j \le m\}.$$

Example 2.2. Let E_m be the empty graph on the vertex set $\{y_1, \ldots, y_m\}$ and let $K_n * E_m$ be the join of complete graph K_n and E_m . It is easy to see that $K_1 * E_n \cong S_n$ and $K_n * E_1 \cong K_n$, where S_n is the star graph on n + 1 vertices.

In the following, we compute the minimal prime ideals of joins of graphs.

Proposition 2.3. Let G_1 and G_2 be graphs on the vertex sets $V(G_1) = \{x_1, \ldots, x_n\}$ and $V(G_2) = \{y_1, \ldots, y_m\}$, respectively. Suppose that $R = \mathbb{K}[V(G_1 * G_2)]$. *i)* If n + height $(I(G_2)) = m$ + height $(I(G_1))$ then

$$|\operatorname{Min}(I(G_1 * G_2))| = |\operatorname{Min}(I(G_1))| + |\operatorname{Min}(I(G_2))|.$$

ii) If $n + \text{height}(I(G_2)) > m + \text{height}(I(G_1))$ then

$$|\operatorname{Min}(I(G_1 * G_2))| = |\operatorname{Min}(I(G_1))|.$$

iii) If $n + \text{height}(I(G_2)) < m + \text{height}(I(G_1))$ then

$$|\operatorname{Min}(I(G_1 * G_2))| = |\operatorname{Min}(I(G_2))|.$$

Proof. For a graph G, it is well-known that the minimal prime ideals of I(G) correspond to the minimal vertex covers of G. Note that if $\mathbf{p} \in \operatorname{Min}(I(G_1 * G_2))$, then either $\mathbf{p} = p_1 + (y_1, \ldots, y_m)$ or $\mathbf{p} = p_2 + (x_1, \ldots, x_n)$, where $p_1 \in \operatorname{Min}(I(G_1))$ and $p_2 \in \operatorname{Min}(I(G_2))$. Hence the assertion follows. \Box

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V such that (i) $x_i \in \Delta$ for all $x_i \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ . For $F \subset V$, we define the dimension of F by dim F = |F| - 1. Let $d = \max\{|F| : F \in \Delta\}$ and define the *dimension* of Δ to be dim $\Delta = d - 1$. A maximal face of Δ with respect to inclusion is called a *facet* of Δ . If all facets of Δ have the same dimension, then Δ is called *pure*.

If I is an ideal of S generated by squarefree monomials, the Stanley-Reisner simplicial complex Δ_I associated to I has vertex set $V = \{x_i \mid x_i \notin I\}$ and its faces are defined by

$$\Delta_I = \{\{x_{i_1}, \dots, x_{i_k}\} \mid i_1 < \dots < i_k, \ x_{i_1} \cdots x_{i_k} \notin I\}.$$

Conversely if Δ is a simplicial complex with vertex set V contained in $\{x_1, \ldots, x_n\}$, the *Stanley-Reisner ideal* I_{Δ} is defined as

$$I_{\Delta} = (\{x_{i_1} \cdots x_{i_r} \mid i_1 < \cdots < i_r, \{x_{i_1}, \dots, x_{i_r}\} \notin \Delta\}),$$

and its Stanley-Reisner ring $\mathbb{K}[\Delta]$ is defined as the quotient ring S/I_{Δ} .

A simplicial complex Δ is said to be *Cohen-Macaulay* over \mathbb{K} if the Stanley-Reisner ring $\mathbb{K}[\Delta]$ is a Cohen-Macaulay ring.

Note that the Stanley-Reisner complex of I(G) is given by $\Delta_{I(G)} = \Delta_G$, where Δ_G is the simplicial complex whose faces are the independent vertex sets of G. Thus

$$\mathbb{K}[\Delta_G] = \mathbb{K}[V(G)]/I(G),$$

where $\mathbb{K}[\Delta_G]$ is the Stanley-Reisner ring of Δ_G . The simplicial complex Δ_G whose faces are the independent vertex sets of G is called the *independence complex* of G.

Let $I \subset S$ be a monomial ideal and G(I) its unique minimal monomial generators. The Auslander and Buchsbaum Theorem says that depth $S/I = \dim S - \operatorname{pd} S/I$, see for example [2].

Since height(I) = dim S – dim S/I, and since S/I is Cohen-Macaulay if and only if dim S/I = depth S/I, it follows that

$$S/I$$
 is Cohen–Macaulay \iff height $(I) = pd S/I.$ (1)

As the main result of this section we have the following.

Theorem 2.4. Let G be a Cohen-Macaulay graph on N vertices. Then

$$W(G) \ge N^2 + (1 + \mathrm{pd}(G))(\frac{1}{2}\mathrm{pd}(G) - N)$$

and the equality holds if and only if pd(G) = N - 1.

Proof. Let G be a Cohen-Macaulay graph on the vertex set V(G). We set |V(G)| = N. Then [12, Lemma 9.1.10] implies that G is unmixed, i.e., all of its associated primes have the same height. According to [12, Lemma 9.1.4] we have that the associated primes of an edge ideal correspond to the minimal vertex covers of G. Then [25, Corollary 7.2.5] implies that

height
$$I(G) = \dim(\mathbb{K}[V(G)]) - \dim(\mathbb{K}[V(G)]/I(G)) = N - \dim(\mathbb{K}[V(G)]/I(G)).$$

The complement of a vertex cover is an independent set, that is, a face of Δ_G . It follows from [19, Theorem 1.3] that

$$\dim(\mathbb{K}[V(G)]/I(G)) = \dim(\mathbb{K}[V(G)]/I_{\Delta_G}) = \dim \Delta_G + 1 = \beta(G) - 1 + 1 = \beta(G).$$

Since all the minimal vertex covers have the same cardinality, so do the facets of Δ_G , that is, Δ_G is pure and dim $(\Delta_G) = N - \text{height}(I(G)) - 1$. Hence, by applying [2, Theorem 1.3.3] we obtain:

$$pd(G) = \dim(\mathbb{K}[V(G)]) - \operatorname{depth}(\mathbb{K}[V(G)]/I(G)) = N - \dim(\mathbb{K}[V(G)]/I(G)).$$

Suppose that $F \in \Delta_G$ is a maximum independent set of G. Thus [25, Corollary 6.3.5], [25, Corollary 7.2.5] together with (1) yield

$$\begin{split} W(G) &= \sum_{u,v \in G \setminus F} d(u,v) + \sum_{u,v \in F} d(u,v) + \sum_{u \in G \setminus F, v \in F} d(u,v) \\ &\geq \binom{N - \beta(G)}{2} + 2\binom{\beta(G)}{2} + \beta(G)(N - \beta(G)) \\ &= \frac{1}{2}(N - \beta(G))(N - \beta(G) - 1) + \beta(G)(N - 1) \\ &= \frac{1}{2}(N - \dim \Delta_G - 1)(N - \dim \Delta_G - 2) + (\dim \Delta_G + 1)(N - 1) \\ &= \frac{1}{2}\operatorname{pd}(G)(\operatorname{pd}(G) - 1) + (N - \operatorname{pd}(G))(N - 1) \\ &= N^2 + (1 + \operatorname{pd}(G))(\frac{1}{2}\operatorname{pd}(G) - N). \end{split}$$

If the equality holds, then for every $u, v \in V(G) \setminus F$ the edge $\{u, v\} \in E(G)$ and every vertex of F is adjacent to all vertices of $V(G) \setminus F$. This implies

$$G \cong K_{\operatorname{height}(I(G))} * E_{N-\operatorname{height}(I(G))}.$$

One can see that if C is a minimal vertex cover of $K_{\text{height}(I(G))} * E_{N-\text{height}(I(G))}$, then either $C = A \cup V(E_{N-\text{height}(I(G))})$ or $C = V(K_{\text{height}(I(G))}) \cup B$, where A and B are minimal vertex covers of $K_{\text{height}(I(G))}$ and $E_{N-\text{height}(I(G))}$, respectively. Then by [2, Proposition 1.2.9] we have

$$\operatorname{depth}(\mathbb{K}[V(K_{\operatorname{height}(I(G))} * E_{N-\operatorname{height}(I(G))})]/I(K_{\operatorname{height}(I(G))} * E_{N-\operatorname{height}(I(G))})) = 1.$$

Since G is a Cohen-Macaulay graph, by Lemma 2.3 together with (1) we have

$$N - \mathrm{pd}(K_{\mathrm{pd}(G)} * E_{N-\mathrm{pd}(G)}) = 1.$$

Thus pd(G) = N - 1.

Conversely, suppose that pd(G) = N - 1; then it is obvious that

$$W(G) = N^{2} + (1 + \mathrm{pd}(G))(\frac{1}{2}\mathrm{pd}(G) - N),$$

and the proof is complete.

For a monomial ideal $I = (x_{11} \dots x_{1n_1}, \dots, x_{t1} \dots x_{tn_t})$ of the polynomial ring S, the *Alexander dual* ideal of I, denoted by I^{\vee} , is defined as

$$I^{\vee} := (x_{11}, \ldots, x_{1n_1}) \cap \cdots \cap (x_{t1}, \ldots, x_{tn_t}).$$

The *cover ideal* associated to a graph G is the monomial ideal

$$J(G) := I(G)^{\vee} = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j)$$

The following theorem, which was proved in [21], is one of our main tools in the study of the regularity of the ring $\mathbb{K}[V(G)]/I(G)$.

Theorem 2.5. [21, Theorem 2.1] Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then $pd(I^{\vee}) = reg(S/I)$.

By using Theorems 2.4 and 2.5, we have the following corollary.

Corollary 2.6. Let G be a Cohen-Macaulay graph on N vertices. Then

$$W(G) \ge N^2 + (1 + \operatorname{reg}(J(G)))(\frac{1}{2}\operatorname{reg}(J(G)) - N)$$

and the equality holds if and only if reg(J(G)) = N - 1.

Example 2.7. Consider the class SQC of well-covered graphs from [17]. A vertex v of a graph G is said to be simplicial if the induced subgraph of G on the set N[v] is a complete graph and we say this complete graph is a simplex of G.

A 5-cycle C_5 of a graph G is called basic if C_5 does not contain two adjacent vertices of degree 3 or more in G; a 4-cycle is called basic if it contains two adjacent vertices of degree 2, and the remaining two vertices belong to a complete subgraph or a basic 5-cycle of G.

A graph is in the class SQC if there are simplicial vertices x_1, \ldots, x_m ; basic 5-cycles C^1, \ldots, C^s ; and basic 4-cycles Q^1, \ldots, Q^t such that

$$V(G) = \bigcup_{j=1}^{m} N[x_j] \cup \bigcup_{j=1}^{s} V(C_j) \cup \bigcup_{j=1}^{t} B(Q^j)$$

and this forms a partition of V(G), where $B(Q^j)$ is the set of two vertices of degree 2 of the basic 4-cycle Q^j . Such a graph is Cohen-Macaulay [13, Theorem 2.3]. Therefore by Theorem 2.4 and [17, Theorem 3.1], we have

$$\begin{split} W(G) &\geq |V(G)|^2 + (|V(G)| - m - 2s - t + 1)(\frac{1}{2}(|V(G)| - m - 2s - t) - |V(G)|) \\ &= |V(G)|^2 + (|V(G)| - m - 2s - t + 1)(-\frac{1}{2}|V(G)| - m - 2s - t) \\ &= \frac{1}{2}(|V(G)|)(|V(G)| - 1) + (-m - 2s - t)(\frac{1}{2}|V(G)| - m - 2s - t + 1) \end{split}$$

and the equality holds if and only if m + 2s + t = 1.

3 Wiener index of very well-covered graphs

In [9], Gitler and Valencia proved that if G is a well-covered graph without isolated vertices, then $h(I(G)) \ge \frac{|V(G)|}{2}$.

A graph G is called *very well-covered* if it is unmixed without isolated vertices and with $h(I(G)) = \frac{|V(G)|}{2}$. Since the class of very well-covered graphs contains unmixed bipartite graphs, whiskered graphs and grafted graphs (see [4, 8]), it is interesting in the algebraic sense as well.

The main goal of this section is to study the Wiener index of very well-covered graphs. The following is a useful result on very well-covered graphs that allows us to assume a certain order on their vertices and edges.

Lemma 3.1. [10, Corollary 3.2] Let G be a very well-covered graph with 2n vertices. Then there is a relabeling of vertices $V(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that the following two conditions hold:

(1) $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of G;

(2) for all $1 \le i \le n$, $\{x_i, y_i\} \in E(G)$.

Theorem 3.2. Let G be a very well-covered graph with 2n vertices. Then

$$W(G) \ge \frac{1}{2}n(5n-3)$$

and the equality holds if and only if $G \cong K_n * E_n$.

Proof. Let G be a very well-covered graph with 2n vertices, and let Δ_G be its independence complex. By [25, Corollary 6.3.5] together with [25, Corollary 7.2.5] it follows that

$$2n = \operatorname{height}(I(G)) + \beta(G) = \operatorname{height}(I(G)) + \dim(\mathbb{K}[V(G)]/I(G))$$

=
$$\operatorname{height}(I(G)) + \dim(\mathbb{K}[V(G)]/I_{\Delta_G})$$

=
$$\operatorname{height}(I(G)) + \dim \Delta_G + 1.$$

Therefore, Lemma 3.1 yields $\dim(\Delta_G) = n - 1$ and hence $\beta(G) = n$. Suppose that $F \in \Delta_G$ is a maximum independent set of G. Then

$$\begin{split} W(G) &= \sum_{u,v \in G \setminus F} d(u,v) + \sum_{u,v \in F} d(u,v) + \sum_{u \in G \setminus F,v \in F} d(u,v) \\ &\geq \binom{N - \beta(G)}{2} + 2\binom{\beta(G)}{2} + \beta(G)(N - \beta(G)) \\ &= \frac{1}{2}(2n - \beta(G))(2n - \beta(G) - 1) + \beta(G)(2n - 1) \\ &= \frac{1}{2}(2n - n)(2n - n - 1) + n(2n - 1) \\ &= \frac{1}{2}n(5n - 3). \end{split}$$

If the equality holds, then for every $u, v \in V(G) \setminus F$ the edge $\{u, v\} \in E(G)$ and every vertex of F is adjacent to all vertices of $V(G) \setminus F$. This implies $G \cong K_n * E_n$. Conversely, suppose that $G \cong K_n * E_n$. Then

$$W(K_n * E_n) = \sum_{u,v \in K_n} d(u,v) + \sum_{u,v \in E_n} d(u,v) + \sum_{u \in K_n, v \in E_n} d(u,v)$$

= $\frac{1}{2}n(n-1) + n(n-1) + n^2$
= $\frac{1}{2}n(5n-3),$

and the assertion follows.

Example 3.3. In [4], the authors introduced *B*-grafted graphs, which are a generalization of grafted graphs introduced by Faridi [8]. Let H_0 be a graph with the labeled vertices $1, 2, \ldots, q$. For every $i = 1, \ldots, q$, let B_i be a bipartite graph with labeled partition X_i and Y_i such that $|X_i| = |Y_i| = n_i$. (We do not give a label to

each vertex of B_i , but we distinguish the partition X_i and Y_i). We assume that B_i has no isolated vertex for every i = 1, ..., q. Let

$$G = G(H_0; B_1, \ldots, B_q)$$

be a B-grafted graph with the vertex set $V(G) := X \cup Y$, where $X = X_1 \cup \cdots \cup X_q$ and $Y = Y_1 \cup \cdots \cup Y_q$.

The edge set E(G) of G is $xy \in E(G)$ if and only if either there exist i, j such that $x \in X_i, y \in X_j$, and $ij \in E(H_0)$ or there exists i such that $x \in X_i, y \in Y_i$, and $xy \in E(B_i)$.

Note that X is a minimal vertex cover of G and that Y is a maximal independent set of G. If G is an unmixed B-grafted graph, then by Theorem 3.2 we have

$$W(G) \geq \frac{1}{2} (2(\sum_{i=1}^{q} n_i) - \sum_{i=1}^{q} n_i)(2(\sum_{i=1}^{q} n_i) - \sum_{i=1}^{q} n_i - 1) + (\sum_{i=1}^{q} n_i)(2(\sum_{i=1}^{q} n_i) - 1)$$
$$= \frac{1}{2} (\sum_{i=1}^{q} n_i)(5(\sum_{i=1}^{q} n_i) - 3)$$

and the equality holds if and only if $G \cong K_{\sum_{i=1}^{q} n_i} * E_{\sum_{i=1}^{q} n_i}$.

4 Wiener index of Boolean graphs

In this section we obtain a lower bound for the Wiener index of Boolean graphs. Let $[n] = \{1, \ldots, n\}$ and let $2^{[n]}$ denote the power set of [n]. Recall from [15] that a *finite Boolean graph*, denoted by B_n , is a graph defined on the vertex set $2^n \setminus \{[n], \emptyset\}$, in which two vertices u and v are adjacent if $u \cap v = \emptyset$. Clearly, B_n is also the zero-devisor graph of the finite Boolean ring $\prod_{i=1}^{n} \mathbb{Z}_2$. Note that a finite or infinite Boolean graph has a unique corresponding zero-divisor commutative semigroup.

Theorem 4.1. For any $n \ge 1$, let $G = B_n$ be the Boolean graph. Then

$$W(G) \ge \frac{5}{8}2^{2n} - \frac{13}{4}2^n + 4.$$

Proof. Suppose that $G = B_n$ is a Boolean graph for all $n \ge 1$. A subset $\Upsilon = \{b_1, \ldots, b_t\}$ of V(G) is an independent vertex set if and only if $b_i \cap b_j \ne \emptyset$ holds for any distinct b_i, b_j in Υ . By [14, Theorem 2.1] all maximal independent vertex sets Υ of V(G) have the same cardinality $2^{n-1} - 1$ and for any $b_i \in V(G)$, only one of $\{b_i, b_i^c\}$ is in Υ , where $b_i^c = [n] \setminus b_i$. Thus the edge ideal of the graph B_n has height $2^{n-1} - 1$. Hence, by applying (1) and [14, Theorem 2.4] we obtain

$$pd(B_n) = 2^{n-1} - 1.$$

Therefore, Theorem 2.4 yields

$$W(G) \geq \frac{1}{2}(2^{n-1}-1)(2^{n-1}-2) + (2^n - 2^{n-1} - 1)(2^n - 3)$$

= $(2^n - 2)^2 + (1 + 2^{n-1} - 1)(\frac{1}{2}(2^{n-1} - 1) - (2^n - 2))$
= $\frac{5}{8}2^{2n} - \frac{13}{4}2^n + 4.$

Then the desired conclusion follows.

Example 4.2. Let $G = B_4$ be a Boolean graph. The edge ideal of B_4 is

$$I(B_4) = (x_1x_2, x_1x_3, x_1x_4, x_1x_8, x_1x_9, x_1x_{10}, x_1x_{14}, x_2x_3, x_2x_4, x_2x_6, x_2x_7, x_2x_{10}, x_2x_{13}, x_3x_4, x_3x_5, x_3x_7, x_3x_9, x_3x_{12}, x_4x_5, x_4x_6, x_4x_8, x_4x_{11}, x_5x_{10}, x_6x_9, x_7x_8).$$

We calculate the primary decomposition of $I(B_4)$ by CoCoA [3] as follows:

$$I(B_4) = (x_1, x_2, x_3, x_4, x_7, x_9, x_{10}) \cap (x_1, x_2, x_4, x_5, x_7, x_9, x_{12})$$

$$\cap (x_1, x_2, x_3, x_5, x_6, x_8, x_{11}) \cap (x_1, x_2, x_4, x_5, x_6, x_8, x_9)$$

$$\cap (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \cap (x_1, x_2, x_3, x_4, x_6, x_7, x_{10})$$

$$\cap (x_1, x_2, x_3, x_4, x_6, x_8, x_{10}) \cap (x_1, x_3, x_4, x_6, x_7, x_{10}, x_{13})$$

$$\cap (x_1, x_2, x_3, x_4, x_8, x_9, x_{10}) \cap (x_2, x_3, x_4, x_8, x_9, x_{10}, x_{14}).$$

Hence by (1) we have height $(I(B_4)) = pd(I(B_4)) = 7$. Therefore, Theorem 4.1 yields

$$W(G) \ge \frac{5}{8}2^8 - \frac{13}{4}2^4 + 4 = 112.$$

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