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Edge precoloring extension of trees

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Abstract

We consider the problem of extending partial edge colorings of trees. We obtain analogues of classical results on extending partial Latin squares by characterizing exactly which partial edge colorings with at most $\Delta(T)$ precolored edges of a tree T with maximum degree $\Delta(T)$ are extendable to proper $\Delta(T)$-edge colorings of T. Furthermore, we prove sharp conditions on when it is possible to extend a partial edge coloring where the precolored edges form a matching or a collection of connected subgraphs. Finally, we consider the problem of avoiding a given (not necessarily proper) partial edge coloring.

1 Introduction

Given a graph G, the classical edge coloring problem is to minimize the number of colors required to properly color the edges of G; this number is denoted by $\chi'(G)$ and is known as the *chromatic index*. The two classical theorems on edge coloring are Vizing's theorem, which states that the chromatic index of a graph G is either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G, and König's theorem which asserts that $\Delta(G)$ colors suffices for a proper edge coloring if G is bipartite. In this paper we are interested in extending partial edge colorings of graphs. A *partial edge coloring* (or *precoloring*) of a graph G is a coloring of some subset $E' \subseteq E(G)$. Unless otherwise stated, we shall assume that all (partial) edge colorings are proper. A partial t-edge coloring φ of G is called *extendable* if there is a proper t-edge coloring f of G such that $f(e) = \varphi(e)$ for every edge e that is colored under φ; f is called an *extension* of φ. Usually, we shall be interested in partial edge colorings, and extensions thereof, using $\chi'(G)$ colors.

Questions on extending partial edge colorings have immediate applications in scheduling problems where some activities have been prescheduled and we ask for a schedule of minimum length which respects the prescheduled activities.

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Edge precoloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares and the well-known Evans' conjecture (see e.g. [1, 6, 10]); by a well-known correspondence, the problem of completing a partial Latin square is equivalent to asking if a partial edge coloring with $\Delta(G)$ colors of a balanced complete bipartite graph G is extendable to a $\Delta(G)$-edge coloring. Another early reference on edge precoloring extension is [9], where the authors study the problem from the viewpoint of polyhedral combinatorics.

More recently, the problem of extending a precoloring of a matching has been considered in [5]. In particular, it is conjectured that for every graph G, if φ is an edge precoloring of a matching M in G using $\Delta(G) + 1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G) + 1)$-edge coloring of G; here, by the *distance* between two edges e and e' we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e'; a *distance-t matching* is a matching where any two edges are at distance at least t from each other. In [5], it is proved that this conjecture holds for e.g. bipartite multigraphs and subcubic multigraphs, and in [8] it is proved that a version of the conjecture with the distance increased to 9 holds for general graphs.

Quite recently, with motivation from results on completing partial Latin squares, questions on extending partial edge colorings of d-dimensional hypercubes Q_d was studied in [3]. In particular, a characterization of which partial edge colorings with at most d precolored edges are extendable to d-edge colorings of Q_d is obtained, thereby establishing an analogue for hypercubes of the characterization by Andersen and Hilton [1] of which $n \times n$ partial Latin squares with at most n non-empty cells are completable to Latin squares.

In this paper, we study edge precoloring extension problems of trees. First, we obtain an analogue for trees of the aforementioned result of Andersen and Hilton by characterizing exactly which precolorings of at most $\Delta(T)$ edges in a tree T are extendable to $\Delta(T)$-edge colorings of T. Furthermore, we prove sharp conditions on when it is possible to extend a precolored matching or a collection of connected precolored subgraphs of a tree T to a $\Delta(T)$-edge coloring of T.

Finally, we consider the problem of *avoiding* a (not necessarily proper) partial edge coloring; that is, given a partial edge coloring f of T, we prove sharp conditions on when it is possible to find a proper $\Delta(T)$-edge coloring g of T such that $g(e) \neq f(e)$ for every edge e that is colored under f.

All our proofs are self-contained, and yield explicit polynomial time algorithms for finding the required colorings.[1]

[1] A referee made us aware of the related paper [4], where the authors prove necessary and sufficient conditions for the existence of a edge list multicoloring of a tree. Since the problem of extending, as well as avoiding, an edge coloring has a natural interpretation as a list coloring problem, the results in this paper could, at least in principle, be deduced from results therein. Nevertheless, our focus, and approach, here is different, so we believe it is of independent interest, and we have not made any attempt to apply their results.

2 Extending precolorings with a bounded number of precolored edges

We first consider partial edge colorings with a bounded number of precolored edges; our first result gives an analogue for trees of Andersen and Hilton's resolution of a stronger form of the well-known Evans' conjecture on completing partial Latin squares.

We shall prove that a proper precoloring of at most $\Delta(T)$ edges in a tree T is always extendable unless the precoloring ϕ satisfies any of the following conditions:

(C1) there is an uncolored edge uv in T such that u is incident with edges of $k < \Delta(T)$ distinct colors and v is incident to $\Delta(T)-k$ edges colored with $\Delta(T)-k$ other distinct colors (so uv is adjacent to edges of $\Delta(T)$ distinct colors);

(C2) there is a vertex u that is incident with edges of $\Delta(T) - k$ distinct colors $c_1, \dots, c_{\Delta(T)-k}$, and k vertices $v_1, \dots, v_k$, where $1 \leq k < \Delta(T)$, such that for $i = 1, \dots, k$, uv_i is uncolored but v_i is incident with an edge colored by a fixed color $c \notin \{c_1, \dots, c_{\Delta(T)-k}\}$;

(C3) there is a vertex u of degree $\Delta(T)$ and a fixed color c satisfying that every edge incident with u is uncolored and adjacent to an edge colored c,

(C4) $\Delta(T) = 2$ and there are two precolored edges using the same color if they are at even distance, and using different colors if they are at odd distance.

For $i = 1, 2, 3, 4$, we denote by C_i the set of all partial edge colorings of a tree T, $\Delta(T) \geq 2$, satisfying the corresponding condition above, and we set $C = \cup C_i$. Note that if such a partial edge coloring is in C, then at least $\Delta(T)$ edges in T are precolored.

Let us briefly verify that if ϕ is a precoloring of T with at most $\Delta(T)$ precolored edges and $\phi \in C$, then it is not extendable. Suppose first that the precoloring ϕ satisfies condition (C1). Since the edge uv is adjacent to edges of $\Delta(T)$ distinct colors, there is no proper $\Delta(T)$-edge coloring of T that agrees with ϕ. If, on the other hand, ϕ satisfies condition (C2), then since u has degree $\Delta(T)$, any extension of ϕ satisfies that the color c must appear on one of the edges in $\{uv_1, \dots, uv_k\}$; however, such a $\Delta(T)$-edge coloring cannot be proper.

Suppose that ϕ satisfies condition (C3). If f is an extension of ϕ, then since u has degree $\Delta(T)$, exactly one edge incident with u must be colored c; since such a $\Delta(T)$-edge coloring cannot be proper, ϕ is not extendable. Finally, if ϕ satisfies condition (C4), then since every vertex of T has degree at most two, the two precolored edges must lie on a $(1, 2)$-colored path in an extension of ϕ, where an *(i, j)-colored path* is a path whose edges are colored i and j alternately. Thus, ϕ is not extendable if it satisfies (C4).

More generally, we shall say that a precoloring ϕ of a forest T is in C_i (or C), if there is a component of T such that the restriction of ϕ to this component is in C_i (or C).

Theorem 2.1. *Let T be a forest and ϕ be a $\Delta(T)$-edge precoloring of T with at most $\Delta(T)$ precolored edges. If $\phi \notin C$, then ϕ is extendable to a proper $\Delta(T)$-edge coloring of T.*

Proof. The proof is by induction on $|E(T)|$. The statement is trivial for any forest with at most two edges; thus we assume that T is a forest with $|E(T)| \geq 3$ and that for any forest T', with $|E(T')| < |E(T)|$, any proper partial coloring of at most $\Delta(T')$ edges, which is not in C, is extendable to a proper $\Delta(T')$-edge coloring.

So assume that $\phi \notin C$ is a partial proper coloring of at most $\Delta(T)$ edges in T. If $\Delta(T) \leq 2$, then the proposition trivially holds, so assume that $\Delta(T) \geq 3$. Moreover, if T has an uncolored pendant edge e such that $\Delta(T-e) = \Delta(T)$, then the restriction of ϕ to $T-e$ cannot be in C; thus it is extendable by the induction hypothesis, which implies that ϕ is extendable to a $\Delta(T)$-edge coloring of T. Therefore, we assume that T has no such pendant edge e. Note that this implies that T contains only one unique vertex v of degree $\Delta(T)$.

Suppose first that there is some color c that appears on at least two edges in T and let E_c be the set of all edges colored c. If some edge of E_c is incident with v, then the restriction of ϕ to $T - E_c$ is not in C, so by the induction hypothesis, it is extendable to a proper edge coloring with colors from $\{1, \dots, \Delta(T)\} \setminus \{c\}$; hence ϕ is extendable. If, on the other hand, no edge from E_c is incident with v, then we pick an uncolored edge e incident to v that is not adjacent to any edge from E_c; since $\phi \notin C$, there is such an edge e. By coloring e by the color c, and applying the induction hypothesis to the restriction of ϕ to $T - (E_c \cup \{e\})$, we obtain an extension of ϕ.

Suppose now that every color appears on at most one edge in T. We first consider the case when $\Delta(T) = 3$; if T contains at most two precolored edges, then ϕ is trivially extendable (e.g. by first properly coloring all edges in a path between the two precolored edges if they are not adjacent and are in the same component), so we assume that T contains three precolored edges e_1, e_2, e_3, where $\phi(e_i) = i$. Suppose first that the three precolored edges lie on a path; without loss of generality we assume that e_1 is contained in a path from e_2 to e_3. If e_1 is incident with v, then the restriction of ϕ to $T - e_1$ is not in C_4; thus we are done by applying the induction hypothesis to $T - e_1$; otherwise we select an uncolored edge e incident with v that is not adjacent to e_1 (this is possible unless $\phi \in C$), color e by color 1, and apply the induction hypothesis to $T - \{e, e_1\}$. If all three edges e_1, e_2, e_3 are not contained in the same component, then we proceed similarly.

It remains to consider the case when e_1, e_2, e_3 are in the same component but not contained in a single path. Without loss of generality assume that the length of the path between e_2 and e_3 is even; since we have 3 precolored edges with 3 mutual distances between them (and all three precolored edges are not contained in a common path), at least one of them is even. Thus we may proceed as in the preceding paragraph.

Suppose now that $\Delta(T) \geq 4$. Let e be an uncolored edge that is adjacent to a maximum number of precolored edges; then, since $\phi \notin C_1$, there is some color

$c \in \{1, \dots, \Delta(G)\}$ that does not appear on an edge adjacent to e. Without loss of generality, we assume that there is an edge e_c precolored c in T; otherwise we just add such an edge e_c to T by attaching it to a vertex of degree 1 in T that is not incident with e. Any extension of this precoloring of $T + e_c$ also yields an extension of ϕ of T.

If e or e_c is incident with v, then we color e with color c, and apply the induction hypothesis to $T - \{e, e_c\}$; otherwise, we pick an uncolored edge e' that is incident with v and not adjacent to e or e_c; since $\phi \notin C_1$, and by our assumption on e, there is such an edge e'. Now we may color the edges e and e' by the color c, and apply the induction hypothesis to $T - \{e, e', e_c\}$ to obtain an extension of ϕ. □

We note the following corollary.

Corollary 2.2. *Every partial edge coloring of a forest T with at most $\Delta(T) - 1$ precolored edges is extendable to a proper $\Delta(T)$-edge coloring of T.*

3 Extending precolored matchings

In this section we consider the problem of extending an edge coloring of a matching in a tree to a coloring of the full tree. We shall prove sharp sufficient conditions for when this is possible.

For a tree T with maximum degree two, it is trivial to determine when a precolored matching is extendable to a proper edge coloring of T; for the case $\Delta(T) \geq 3$, we have the following.

Theorem 3.1. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$, and M a precolored distance-2 matching of T. If no vertex v satisfies that $k - 1$ uncolored edges incident with v are all adjacent to edges precolored by a fixed color c, then the precoloring can be extended to a proper k-edge coloring of T.*

Proof. Let ϕ be a precoloring satisfying the conditions in the theorem. We shall describe an algorithm based on Breadth First Search for coloring the full tree using k colors.

First, we designate one of the vertices not incident to a precolored edge as root and call it u. Since no two edges of M are at distance 1 from each other, there is such a vertex u. Given the root vertex u, we begin by coloring all the edges incident to u (level 1 edges); since every edge incident with u is adjacent to at most one precolored edge, and at most $k - 2$ edges incident with u are adjacent to precolored edges colored by the same color, this is possible. Next, we examine the edges incident to the children of u (level 2 edges) in the same way across the different subtrees of T. More precisely, we proceed as follows: suppose that v is a vertex considered at some stage and that one edge incident with v was colored at a previous step (or is precolored); without loss of generality we assume that v has degree k and consider the following two different cases. By the edges in the *present level* we mean the edges joining v to its children; by the edges in the *next level* we mean the edges that are not incident to v and adjacent to some edge in the present level; by the *previous level* edges we mean the edge incident with v that was considered at a previous step of the algorithm.

Case 1. *v is incident to a precolored edge e_1 in the present level:*

Suppose that the precolored edge e_1 incident with v is colored c_1 and that the edge incident with v from the previous level is colored c_2. Since $d(v) = k$, there are $k-2$ uncolored edges incident with v, and none of these edges are adjacent to another precolored edge distinct from e_1, since M is a distance-2 matching. We assign $k-2$ colors from $\{1, \dots, k\} \setminus \{c_1, c_2\}$ to the uncolored edges incident with v in an arbitrary way.

Case 2. *v is not incident to a precolored edge in the present level:*

Suppose that the edge in the previous level incident to v is colored c_2. Since M is a distance-2 matching, all the $k-1$ uncolored edges incident with v may be adjacent to precolored edges. However, by assumption, all these $k-1$ edges incident to v are not adjacent to edges precolored by a fixed color c. Hence, without loss of generality we assume that there are two edges e and e' in the present level that are adjacent to edges, from the next level, precolored with two different colors c and c', respectively, where $c \neq c_2$. We color the uncolored edges incident with v by first assigning c to e', and then coloring the $k-3$ uncolored edges incident with v, distinct from e and e', greedily; finally we color e with a color c'' that does not appear on any edge incident with v. Since e' was colored by c, $c'' \neq c$. This yields a proper k-coloring of the edges incident with v.

This coloring process is implemented level by level in every subtree of T until we finish coloring all the edges. When this process terminates we have a proper k-edge coloring which agrees with the precoloring of T. $\square$

Remark 3.2. The graph in Figure 1 shows that the condition in the preceding theorem that at most $k-2$ edges incident to a vertex of maximum degree k can be adjacent to edges precolored with a fixed color c is necessary.

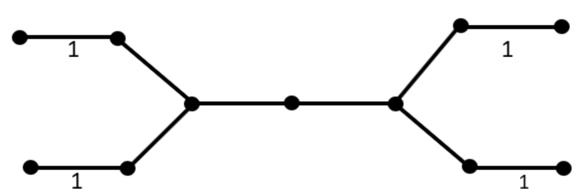

Figure 1: A representative T of a class of trees with a non-extendable precolored distance-2 matching.

Corollary 3.3. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$. Any precoloring of a distance-3 matching in T can be extended to a proper k-edge coloring of T.*

Edwards et al. [5] proved that if G is a bipartite graph, then any precolored matching of G can be extended to a proper edge coloring of G using $\Delta(G) + 1$ colors; their proof relied on list coloring techniques [2] based on Galvin's well-known resolution of the list coloring conjecture for bipartite graphs [7]. We note that by proceeding as in the proof of the preceding theorem, we obtain a simple proof of this result for trees. Thus, we get a short self-contained proof of the following theorem, the details of which we omit.

Theorem 3.4. *If T is a tree with maximum degree $\Delta(T) = k$ and ϕ is a $(k+1)$-precoloring of a matching of T, then ϕ is extendable to a proper $(k+1)$-coloring of the edges of T.*

4 Extending precolorings of connected subgraphs

In this section we consider the problem of extending a precoloring of a tree where the precolored edges form several connected subgraphs. We shall prove sharp sufficient conditions for when it is possible to extend such a partial edge coloring. By the *distance* between two subgraphs of a graph we mean the shortest distance between any two vertices of the subgraphs.

Theorem 4.1. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$. If at most $k-1$ connected subgraphs of T are properly k-edge-colored, and the distance between any two such precolored subgraphs is at least 3, then this partial edge coloring is extendable to a proper k-edge coloring of T.*

Proof. Let ϕ be a precoloring satisfying the conditions in the theorem and H be the precolored subgraph of T; that is, the subgraph induced by all precolored edges of T. We set $J = T - E(H)$.

Since the distance between two different precolored connected subgraphs is at least three, the partial coloring ϕ may be extended greedily to a proper k-edge coloring of the spanning subgraph of T whose edge set consists of $E(H)$ and all edges of J that are adjacent to an edge of H. Consider the restriction of this coloring to the graph J. Since in J, all edges incident with a vertex of H are colored, the colored edges of J induce a subgraph of J consisting of stars.

Let J' be the graph obtained from J by removing all isolated vertices. By splitting all vertices of H that are in J' into vertices of degree 1, we obtain a graph J'' from J' where every component contains at most $k-1$ precolored edges, because T is acyclic and contains at most $k-1$ connected precolored subgraphs. Hence by applying Corollary 2.2 to each component of J'', we can properly color the remaining uncolored edges in J'' with k colors. By merging the split vertices, this yields a proper k-edge coloring of T. □

Remark 4.2. Note that Theorem 4.1 does not hold if we have k connected precolored subgraphs rather than $k-1$; for instance, consider the precoloring of a tree shown in Figure 2 with maximum degree 3. Note also that the theorem is sharp with respect to the distance between connected precolored subgraphs; the graph in Figure 3 shows that the theorem becomes false if we replace distance 3 with distance 2.

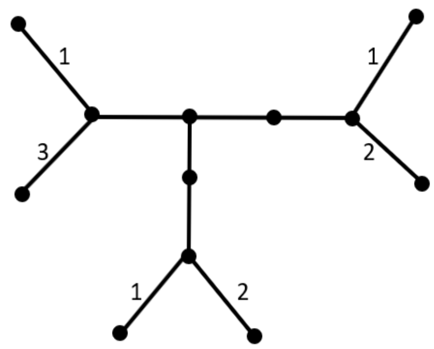

Figure 2: A representative T of a class of trees with a non-extendable precoloring of $\Delta(T)$ connected subgraphs.

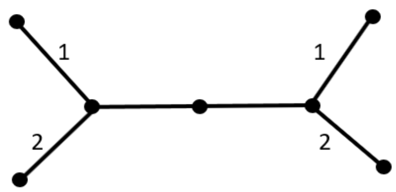

Figure 3: A representative of a class of trees with a non-extendable precoloring of $\Delta(T) - 1$ connected subgraphs at distance 2.

For trees with any number of precolored connected subgraphs, we have the following.

Theorem 4.3. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$, where the edges of some connected subgraphs of T have been precolored. If the distance between any two precolored such subgraphs is at least 5, then the precoloring can be extended to a proper k-edge coloring of T.*

Proof. Our proof of this theorem is similar to the proof of the preceding result.

Let ϕ be a precoloring of T satisfying the conditions of the theorem, and let H be the subgraph induced by all precolored edges of T. As above, we properly color every edge that is adjacent to a precolored edge; this yields a proper k-edge coloring of a subgraph of T. Consider the restriction of this coloring to the graph $J = T - E(H)$, and define the subgraphs J' and J'' as in the proof of the preceding theorem. Now, notice that the distance between any two colored edges of J'' is at least 3, since by assumption the distance between any two connected precolored subgraphs of T

is at least 5. Thus the colored edges of every connected component of J'' form a distance-3 matching. Hence, by applying Corollary 3.3 to each component of J'', we can properly color the uncolored edges in J'' with k colors. Finally, by merging the split vertices we obtain a proper k-edge coloring of T. □

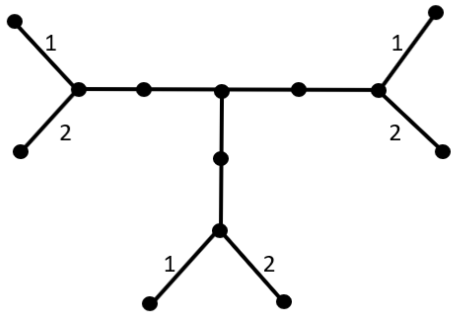

Figure 4: A representative T of a class of trees with a non-extendable precoloring where the distance between any two precolored subgraphs is 4.

Remark 4.4. Theorem 4.3 becomes false if we have precolored subgraphs at distance 4 rather than at distance 5; see Figure 4.

If we are allowed to use $\Delta(T) + 1$ colors, rather than $\Delta(T)$, then the distance condition between the precolored subgraphs can be weakened.

Theorem 4.5. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$, where the edges of some connected subgraphs of T have been precolored by using $k + 1$ colors. If the distance between any two such subgraphs with precolored edges is at least 3, then the precoloring can be extended to a $(k + 1)$-edge coloring.*

Proof. Once again, we use the technique from the above proofs.

Let H be the subgraph of T induced by the precolored edges of T. We define a proper edge coloring of $J = T - E(H)$ by properly coloring (by $k+1$ colors) all edges of J that are adjacent to a precolored edge. Next, define J' and J'' as in the proofs of the two preceding theorems. Now, since the distance between any two colored edges of J'' is at least 1, the colored edges of every component of J'' form a matching. Hence, by applying Theorem 3.4 to each component of J'', we can properly color the uncolored edges in J'' with $k + 1$ colors. Finally, by merging the split vertices we obtain a proper $(k + 1)$-edge coloring of T. □

Remark 4.6. The number of colors used in Theorem 4.5 is best possible; see e.g. the example in Figure 2. Note also that by the example in Figure 5, the distance condition cannot be weakened.

Moreover, the class of graphs described in Figure 5 shows that we may need up to $2\Delta(T) - 1$ colors for an extension of a partial edge coloring with $\Delta(T) - 1$ colors if the precolored connected subgraphs are at distance 2.

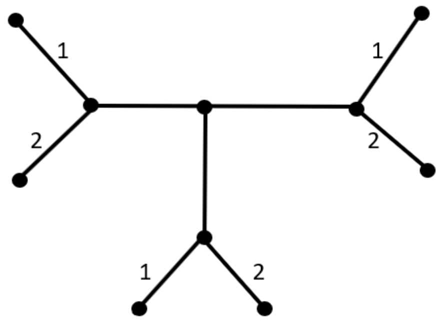

Figure 5: A representative T of a class of trees with a non-extendable precoloring where the distance between any two precolored subgraphs is 2.

5 Avoiding partial edge colorings of trees

Next, we consider the problem of avoiding partial edge colorings; note that in this section partial edge colorings are not necessarily proper.

Theorem 5.1. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$. If ϕ is a partial k-edge coloring of T where every vertex is incident with at most $k-2$ edges colored with a fixed color c, then ϕ is avoidable.*

Proof. Let ϕ be a partial coloring of T that satisfies the conditions in the theorem. We shall show that we can avoid the coloring ϕ by proceeding as in the proof of Theorem 3.1 using the idea of Breadth First Search for defining a proper k-edge coloring f of T that avoids ϕ.

We first designate one of the vertices as a root and denote it by u; then we properly color all the edges incident to u (level 1 edges) in the following way: without loss of generality, we assume that u has degree k and that there are colors c_1 and c_2, $c_1 \neq c_2$, and edges e_1 and e_2 incident with u so that $\phi(e_i) = c_i$, $i = 1, 2$. We properly color all edges incident to u, except e_1 and e_2, greedily, so that the resulting partial coloring avoids ϕ. Finally, we color e_1 and e_2 by the two colors from $\{1, \dots, k\}$ not used on the other edges incident with v so that the resulting coloring avoids ϕ on the edges incident with u.

Suppose now that v is a vertex considered at some stage and that an edge e_1 incident with v was colored by, say, color c_1 at a previous step. Without loss of generality we assume that v has degree k. Now, since at most $k-2$ edges incident with v are colored by a fixed color c, we may assume that there are two edges e and e' incident with v that are uncolored under f, and satisfying that $\phi(e) = c$ and $\phi(e') = c'$, where $c \neq c'$. Again, we can properly color all edges incident to v, except e and e', greedily so as to avoid ϕ. Finally, we color e and e' by the two colors from $\{1, \dots, k\}$ not used on the other edges incident with v so that the resulting coloring avoids ϕ on the edges incident with v.

We continue this process until all the edges of T have been colored. The obtained coloring f avoids the partial coloring ϕ of T. □

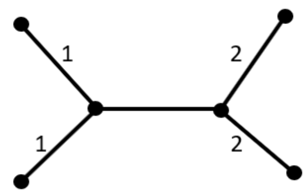

Figure 6: A representative T of a class of trees with a non-avoidable partial k-edge coloring.

Remark 5.2. The coloring of a tree given in Figure 6 shows that $k-2$ cannot be replaced by $k-1$ in Theorem 5.1.

6 Concluding remarks

In this short note, we have proved sharp conditions on when a partial edge coloring of a tree T can be extended to a proper $\Delta(T)$-edge coloring. The relatively strong edge precoloring extension results obtained are of course due to the fact that trees, being 1-degenerate graphs, have a very simple structure; for general graphs, the type of problems considered here quickly becomes untractable; see e.g. [3, 5]. Nevertheless, edge precoloring extension problems constitute a rather unexplored area of research, and we believe that the results of this paper may serve as a good starting point for further investigation; it would be interesting to study similar questions for e.g. k-degenerate graphs, or, for instance, graphs with large girth.

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