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# An upper bound on the order of graphs of diameter two arising as abelian lifts of multigraphs

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**Abstract**

McKay, Miller and Širáň (1998) constructed, for an infinite set of values $d$, a vertex-transitive graph of diameter 2, degree $d$ and order $\nu_d = \frac{8}{9}(d+\frac{1}{2})^2$, as an abelian lift of a regular complete bipartite graph with loops at each vertex. We examine the order of abelian lifts of complete multigraphs and also prove that $\nu_d$ is an upper bound on the order of any (not necessarily vertex-transitive) graph of diameter 2 and any degree $d \geq 11$, obtained as an abelian lift of a regular complete bipartite graph (with loops and semi-edges) of order at least 4.

## 1 Introduction and preliminaries

The well known degree-diameter problem is to find, for given $d$ and $k$, the largest order (that is, the number of vertices) of a graph of maximum degree $d$ and diameter $k$. We will be interested in the special case of diameter $k = 2$ and in a somewhat restricted class of regular, finite and undirected graphs of degree $d$ which we will describe in what follows. For literature and advances on the general problem we refer to the survey [7].

The upper bound on the order of a graph of diameter 2 and a maximum degree $d$ is $d^2 + 1$, which is a special case of the so-called Moore bound for general degree and diameter. The value of $d^2 + 1$ is attained only for $d = 2, 3, 7$, and, possibly, 57 [5]; for the remaining $d \geq 4$ the upper bound is $d^2 - 1$, see [7]. The best currently known graphs of diameter 2 and order 'close' to the Moore bound are the Brown graphs [2] (also known as polarity graphs [1], originally introduced by Erdős and Rényi in [3]) of order $d^2 - d + 1$ for all values of $d$ such that $d - 1$ is an odd prime power (with an improvement on the order by +1 if $d - 1$ is a power of 2). Brown graphs, however,

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are not regular; by results of [12] they can be extended to regular graphs of degree $d$ but never to vertex-transitive graphs of such degree. As shown in [1] this is also not possible when polarity graphs are extended to regular graphs of degree $d + 2$.

The largest currently known vertex-transitive graphs — in fact, Cayley graphs — of diameter 2 and a given degree $d$ for an infinite (but rather sparse) set of degrees $d$ were constructed in [11] and have order at least $d^2 - 6\sqrt{2}d^{\frac{3}{2}}$. The largest available vertex-transitive non-Cayley graphs of diameter 2 and degree $d = (3q - 1)/2$ where $q$ is a prime power congruent to 1 mod 4, known as the McKay-Miller-Širáň graphs [6, 9], have order $\nu_d = \frac{8}{9}(d + \frac{1}{2})^2$. The graphs were originally constructed in [6] as abelian lifts of graphs arising from $K_{q,q}$ by adding an appropriate number of loops to every vertex. It is quite remarkable that the McKay-Miller-Širáň graphs also turn out to be (ordinary) lifts of dipoles (multigraphs of order 2 with the same number of loops at each vertex), with voltages in abelian groups, as was shown in [9]. Such lifts, however, cannot be used to approach the Moore bound, since it was proved in [8] that their order does not exceed $0.932d^2$ in the asymptotic sense.

A natural generalisation of dipoles are the complete multigraphs $K_n(m, l, s)$ of order $n \geq 2$ and degree $d = (n - 1)m + 2l + s$, with edge multiplicity $m$, $l$ loops, and $s$ semi-edges at every vertex, which we will call *$n$-poles*. Another generalisation of dipoles retaining the bipartiteness are the complete bipartite multigraphs $K_{n,n}(m, l, s)$ for $n \geq 2$ and degree $d = nm + 2l + s$, also with edge multiplicity $m$, $l$ loops, and $s$ semi-edges at every vertex, and such graphs will be called *$(n, n)$-bipoles*. We recall here that a semi-edge is incident to just one vertex and contributes just 1 to the degree of the vertex. Semi-edges arise naturally in graph coverings; see e.g. [7].

In this note we will focus on deriving upper bounds on the orders of lifts of $n$-poles and $(n, n)$-bipoles with voltages in abelian groups. As a consequence we will obtain a generalisation of the upper bound for lifts of dipoles (without semi-edges) proved in [8].

We will assume throughout that the reader is familiar with the theory of lifts of graphs by voltage assignments (see eg. [4]); for an elementary introduction to the degree-diameter problem we refer to [7]. We recall here just a few basic facts. If $G$ is a graph (possibly with loops and parallel edges), then every edge $h$ of $G$ can be viewed as consisting of two oppositely directed darts $x, x^{-1}$, and we write $h = \{x, x^{-1}\}$. Let $V(G)$ and $D(G)$ be the vertex set and the dart set of $G$. Given a group $\Gamma$, a *voltage assignment* on $G$ in $\Gamma$ is a mapping $\alpha : D(G) \to \Gamma$ such that $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for every $x \in D(G)$. The *lift* $G^\alpha$ of $G$ by $\alpha$ has vertex set $V(G^\alpha) = V(G) \times \Gamma$ and dart set $D(G^\alpha) = D(G) \times \Gamma$, and for any dart $x$ of $D(G)$ from a vertex $u$ to a vertex $v$ and for any $g \in \Gamma$ there is a dart $(x, g)$ in $G^\alpha$ from the vertex $(u, g)$ to the vertex $(v, g\alpha(x))$; the darts $(x, g)$ and $(x^{-1}, g\alpha(x))$ form an edge of $G^\alpha$. Note that if $\Gamma$ is an abelian group, then the lift will be called *abelian*. The original graph $G$ is called the *base graph* of the lift, and for each vertex $u$ of $G$ the set $\{(u, g), g \in \Gamma\}$ of vertices of the lift is the *fibre* above $u$. Let us note here that the degree of every vertex in the fibre above $u$ is equal to the degree of $u$.

There is an easy way to control the diameter of a lift in terms of the base graph and the voltage assignment, proved in [6]. Given a base graph $G$ and a voltage assignment $\alpha$ on $G$ in a group $\Gamma$ and a walk $W = x_0 x_1 \dots x_t$ in $G$ (that is, a sequence of darts such that the terminal vertex of $x_{i-1}$ is the initial vertex of $x_i$ for $i \in \{1, \dots, t\}$), the voltage $\alpha(W)$ is simply the product $\alpha(x_0)\alpha(x_1)\dots\alpha(x_t)$.

**Lemma 1.1** *Let $\alpha$ be a voltage assignment on a connected graph $G$ in a group $\Gamma$, let $k$ be a positive integer. Then, the lift $G^\alpha$ has diameter at most $k$ if and only if for any pair of vertices $u$, $v$ of $G$, and for every element $g$ of the group $\Gamma$ there is a $u \to v$ walk $W$ in $G$ of length at most $k$ such that $\alpha(W) = g$.*

## 2 Lifting given base graphs to graphs of diameter two

In this section we will consider lifts of graphs $K_n(m,l,s)$ and $K_{n,n}(m,l,s)$ introduced in the previous section. We will apply Lemma 1.1 to prove upper bounds on the orders of lifts of $n$-poles and $(n,n)$-bipoles with voltages in abelian groups.

**Proposition 2.1** *Let $G = K_n(m,l,s)$ and let $\alpha$ be a voltage assignment on $G$ in an abelian group $A$ such that the lift $G^\alpha$ has diameter two. Then the order of $G^\alpha$ is bounded above by*

$$\begin{aligned}\omega_1(m,l,s) = n \cdot \min\{&(n-1)m(m-1) + 2l(l+1) + 2ls + \frac{s(s+1)}{2} + 1, \\ &(n-2)m^2 + (4l+2s+1)m\}.\end{aligned} \tag{1}$$

**Proof.** Let $\alpha$ be a voltage assignment on the base graph $G = K_n(m,l,s)$ in an abelian group $A$. At this point there are no restrictions on $\alpha$, except that voltages on semi-edges must be involutions of $A$.

The number of vertices in a fibre above any particular vertex $u \in V(G)$ is, by Lemma 1.1, bounded above by the number of distinct voltages on the closed $u \to u$ walks of length at most 2 in the base graph. By vertex-transitivity of $G$, our considerations do not depend on the choice of $u$. We obviously have the trivial walk of length 0 that carries the zero voltage. Next, there are $2l + s$ walks of length 1 (each loop traversed in either direction and each semi-edge traversed), which can carry at most $2l + s$ distinct voltages. The $u \to u$ walks of length 2 in the base graph not containing a loop or a semi-edge can only be obtained by traversing from $u$ to any vertex distinct from $u$ and back, which gives at most $(n-1)m(m-1)$ distinct non-zero voltages.

Finally, let us estimate the number of distinct voltages of the $u \to u$ walks of length 2 that use loops and/or semi-edges. Every loop can be traversed twice in the same direction, giving at most $2l$ distinct voltages. There are $\binom{l}{2}$ *unordered* pairs of distinct loops at $u$ and, since $A$ is abelian, walks of length 2 using two loops in such a pair give at most 4 distinct voltages. Similarly, each unordered pair of distinct semi-edges, and also each unordered pair of a semi-edge and a loop with a chosen direction, induce two different walks with equal voltages. It follows that the number of voltages on such paths of length 2 is bounded above by $2l+\binom{l}{2}\cdot 4+2ls+s(s-1)/2$. Summing up, we can have at most

$$p_1(m,l,s) = (n-1)m(m-1) + 2l(l+1) + 2ls + \frac{s(s+1)}{2} + 1$$

vertices of the lift in a fibre above any vertex of the base graph.

The second limitation comes, by Lemma 1.1, from examining the walks between the vertices in two different fibres. Since the base graph is arc-transitive, it is sufficient to take an arbitrary ordered pair $u$, $v$ of distinct vertices and derive an upper bound on the number of distinct voltages of the $u \to v$ walks of length at most 2 in $G$. The $m$ edges between $u$ and $v$ give $m$ paths of length 1, and hence at most $m$ distinct voltages on such walks. There are $2(2l+s)m$ distinct $u \to v$ walks of length 2 using a loop or a semi-edge exactly once; these carry at most $2(2l+s)m$ distinct voltages. It remains to take into account the $u \to v$ paths of length 2 through a third vertex $w$ distinct from $u$ and $v$, carrying at most $(n-2)m^2$ distinct voltages. In summary, this gives the upper bound of the form

$$p_2(m,l,s) = (n-2)m^2 + (4l+2s+1)m$$

vertices in a fibre over a vertex in the lift.

Then, the minimum of polynomials $p_1(m,l,s)$ and $p_2(m,l,s)$, appearing in the statement of our proposition, is an upper bound on the number of vertices in a fibre, and this quantity multiplied by $n$ is therefore an upper bound on the order of $G$. □

Now we turn our attention to lifts of $(n,n)$-bipoles.

**Proposition 2.2** *Let $G = K_{n,n}(m,l,s)$ for $n \geq 2$ and let $\alpha$ be a voltage assignment on $G$ in an abelian group $A$ such that the lift $G^\alpha$ has diameter two. Then the order of $G^\alpha$ is bounded above by*

$$\omega_2(m,l,s) = 2n\cdot\min\{nm(m-1)+2l(l+1)+2ls+\frac{s(s+1)}{2}+1, (4l+2s+1)m, nm^2\}. \tag{2}$$

**Proof.** We will use an analogous approach as in the proof of Proposition 2.1, with only few differences. Let $G = K_{n,n}(m,l,s)$, $n \geq 2$ be the base graph that is a complete bipartite multigraph with two distinct sets of vertices $U = \{u_k, k \in \{1,\ldots,n\}\}$ and $V = \{v_k, k \in \{1,\ldots,n\}\}$. Let $\alpha$ be a voltage assignment on this base graph in an abelian group $A$, without any restrictions on $\alpha$, except that voltages on semi-edges must be involutions of $A$.

We recall here that the number of vertices in a fibre above any particular vertex is bounded above, as in the previous proof, by the number of distinct voltages assigned on the closed walks of length at most 2 at a chosen vertex in the base graph. By vertex-transitivity of the base graph this number is equal for any vertex of $G$.

For any $u \in U$ examination of all possible $u \to u$ walks of length 2 on the base graph $G$ shows that there is only one difference from the previous case of $n$-poles.

Namely, the number of distinct voltages of the walks traversing from $u$ to some vertex distinct from $u$ and back is at most $nm(m-1)$ in this case. Next, let $u \in U$ and $v \in V$ and consider all possible $u \to v$ walks of length 2 on $G$. Again, this is similar to the case of $n$-poles, but since $u$ and $v$ are in different parts of bipartite graph $G$, there will be no walks of length 2 through a third vertex distinct from vertices $u$ and $v$. The sum of the possibilities for $u \to u$ and $u \to v$ walks gives an upper bound on the number of vertices in a fibre above any vertex of the base graph $G$ in the form of two polynomials

$$q_1(m,l,s) = nm(m-1) + 2l(l+1) + 2ls + \frac{s(s+1)}{2} + 1, \quad \text{and}$$

$$q_2(m,l,s) = (4l+2s+1)m.$$

Finally, we look on the walks between two distinct vertices in the same partition. Let $u$, $u'$ be a pair of distinct vertices in $U$ and consider all $u \to u'$ walks of length 2. Every such walk will go from $u$ to one of the $n$ vertices in $V$ and back to $u'$, both by $m$ paths. This gives an upper bound of the form

$$q_3(m,l,s) = nm^2$$

vertices in a fibre over a vertex in the lift of $G$. Therefore, the minimum of polynomials $q_1(m,l,s)$, $q_2(m,l,s)$ and $q_3(m,l,s)$ is an upper bound on the number of vertices in a fibre, and therefore an upper bound on the order of $G$ is this quantity multiplied by $2n$. □

Observe that the 'min' terms in (1) and (2) also give an upper bounds on the orders of the corresponding abelian voltage groups.

## 3 Main results

We now prove an estimate on the order of an abelian lift of an $n$-pole.

**Theorem 3.1** *Let $n \geq 2$ be an integer and let $\alpha$ be a voltage assignment on an $n$-pole $G$ of degree $d$ in an abelian group such that the lift $G^\alpha$ has diameter 2. Then the order of $G^\alpha$ is bounded above by*

$$\frac{n^4 + 4n^3 + (2\sqrt{2}-1)n^2 - (2\sqrt{2}+2)n}{(n^2+2n-1)^2} d^2 + O(d^{3/2})$$

*as $d \to \infty$.*

**Proof.** Let $G = K_n(m,l,s)$, with $m \geq 1$ and $l, s \geq 0$. By Proposition 2.1 we have $|G^\alpha| \leq \omega_1(m,l,s)$. Since we would like to have a bound on the order of the lift in terms of $d = (n-1)m + 2l + s$, we evaluate $l = (d-(n-1)m-s)/2$ and substitute this into $\omega_1(m,l,s)$. If we consider the two arguments of the 'min' term as polynomials in $m$, the substitution gives

$$p_1(m) = \left(\frac{n^2}{2} - \frac{1}{2}\right) m^2 + (-dn - 2n + d + 2)m + \frac{d^2}{2} + d + 1 - \frac{s}{2}\ , \quad \text{and}$$

$$p_2(m) = -nm^2 + (2d+1)m$$

where in both cases $1 \le m \le d/(n-1)$ and $0 \le s \le d - (n-1)$.

Since we want to establish an upper bound on $|G^\alpha|$ we determine

$$M = \max_m \ \min\ \{p_1(m), p_2(m)\}.$$

For an upper bound on $M$ it is sufficient to replace $m$ by a continuous real argument $x$ and determine the quantity

$$M^* = \max_x \ \min\ \{p_1(x), p_2(x)\}$$

where $x$ ranges over all real numbers; we clearly have $M \le M^*$.

Observe that the parabola $p_1(x)$ is concave up while the parabola $p_2(x)$ is concave down. It follows that to determine $M^*$ it is necessary and sufficient to identify the points of intersection of $p_1(x)$ and $p_2(x)$. The value $M^*$ will be attained at one of the intersection points of the parabolas. The roots $x_1$ and $x_2$ of the equation $p_1(x) = p_2(x)$ are

$$x_1 = \frac{dn + 2n + d - 1 - \sqrt{2d^2 + (2n^2 - 2n)d + 2n^2 - 8n + 3 + (n^2 + 2n - 1)s}}{n^2 + 2n - 1}$$

$$x_2 = \frac{dn + 2n + d - 1 + \sqrt{2d^2 + (2n^2 - 2n)d + 2n^2 - 8n + 3 + (n^2 + 2n - 1)s}}{n^2 + 2n - 1}$$

and the $x$-coordinates of the vertices of the parabolas $p_1(x)$ and $p_2(x)$, respectively, are $V_1 = \frac{d+2}{n+1}$ and $V_2 = \frac{2d+1}{2n}$. An inspection shows that if $V_1 \le V_2$, that is, if $3n \le 2d + 1$, then $M^* = \max_x\ \min\ \{p_1(x), p_2(x)\} = p_2(x_2)$, and if $V_1 \ge V_2$, that is, if $3n \ge 2d + 1$, then $M^* = p_2(x_1)$. Since we are interested in the case when $d \to \infty$ for any fixed $n \ge 2$, we only need to consider the first case and so $M^* = p_2(x_2)$. The resulting upper bound on the number of vertices of the lift is in this case $n \cdot M^* = n \cdot p_2(x_2)$. An evaluation of this quantity finally yields

$$n \cdot p_2(x_2) = \frac{n^4 + 4n^3 + (2\sqrt{2} - 1)n^2 - (2\sqrt{2} + 2)n}{(n^2 + 2n - 1)^2} d^2 + O(d^{3/2})$$

as in the statement of the theorem. □

In the following Theorem 3.2 we derive an upper bound on the order of an abelian lift of an $(n,n)$-bipole.

**Theorem 3.2** *Let $n \geq 2$ be an integer and let $\alpha$ be a voltage assignment in an abelian group on an $(n,n)$-bipole $G$ of degree $d$ such that the lift $G^\alpha$ has diameter 2. If $d \geq 11$, then the order of $G^\alpha$ is bounded above by*

$$\frac{8}{9}\left(d+\frac{1}{2}\right)^2 .$$

**Proof.** Let $G = K_{n,n}(m,l,s)$ of degree $d = nm + 2l + s$, where $n \geq 2$, $m \geq 1$ and $l, s \geq 0$. Let $\alpha$ be a voltage assignment on $G$ in an abelian group $A$ of order at most $\omega_2(m,l,s)$, cf. Proposition 2.2. We substitute $s = d - nm - 2l$ into $\omega_2(m,l,s)$ to derive an upper bound $\omega(d)$ on the order of the lift in terms of $d$. As a result of this substitution we have a modified form of the three polynomials of $\omega_2(m,l,s)$ in variable $m$ as an argument:

$$q_1(m) = \left(\frac{n^2}{2}+n\right)m^2 + \left(-\frac{3n}{2} - dn\right)m + \frac{d(d+1)}{2} + l + 1,$$

$$q_2(m) = (-2n)m^2 + (2d+1)m,$$

$$q_3(m) = nm^2,$$

where $1 \leq m \leq \frac{d}{n}$ and $0 \leq l \leq \lfloor\frac{d-n}{2}\rfloor$.

To derive an upper bound on $|G^\alpha|$ we replace $m$ by a continuous real argument $x$ and determine the quantity

$$M = \max_x \ \min\{q_1(x), q_2(x), q_3(x)\}.$$

For determining the value of $M$ it is necessary to examine the position of parabolas $q_1(x)$, $q_2(x)$ and $q_3(x)$, and the points of their intersection, since the value of $M$ will be attained at one of these points.

Observe that the parabolas $q_1(x)$ and $q_3(x)$ are concave up while the parabola $q_2(x)$ is concave down, and the $x$-coordinates of their vertices are $V_1 = \frac{2d+3}{2n+4}$, $V_2 = \frac{2d+1}{4n}$ and $V_3 = 0$. It is straightforward to check that the inequalities $V_3 < V_2 < V_1$ hold for all $d \geq 11$ and $n \geq 2$. The roots $x_1^\sigma$ and $x_2^\sigma$ of the equation $\sigma : q_1(x) = q_3(x)$ are given by

$$\frac{2d+3 \pm \sqrt{8d-8l+1}}{2n},$$

where the minus belongs to $x_1^\sigma$ and the plus belongs to $x_2^\sigma$. It is easy to verify that since $d \geq l$ it holds that both $x_1^\sigma$ and $x_2^\sigma$ are real. Then the roots $x_1^\tau$ and $x_2^\tau$ of the equation $\tau : q_2(x) = q_3(x)$ are $x_1^\tau = 0$ and $x_2^\tau = \frac{2d+1}{3n}$.

Based on the fact that $q_1(x)$ and $q_3(x)$ are both concave up and the $x$-coordinates of their points of intersection are $x_1^\sigma, x_2^\sigma$ we have $q_1(x) \leq q_3(x)$ for $x \in (x_1^\sigma, x_2^\sigma)$. Considering the position of the parabolas $q_1(x)$ and $q_2(x)$ in the sense of determining the value of $M$ we can show that the case where

$$x_1^\sigma > x_2^\tau \tag{3}$$

holds for $d \geq 11$. This is equivalent to showing that

$$\frac{2d+3-\sqrt{8d-8l+1}}{2n} > \frac{2d+1}{3n} \ . \tag{4}$$

If we consider any value of $l$ from the interval $0 \leq l \leq \lfloor \frac{d-n}{2} \rfloor$, the character of the position of parabolas $q_1(x)$, $q_2(x)$ and $q_3(x)$ will not be changed and therefore we only need to consider the case where $l = 0$. Then, the inequality (4) can, for $n \geq 2$, be simplified to

$$2d + 7 - 3\sqrt{8d+1} > 0 \ . \tag{5}$$

Squaring both sides of (5) and further manipulating the resulting inequality yields $(d-10)(d-1) > 0$, and reversing the process gives (3) for $d > 10$.

The inequality (3) implies that for all $d \geq 11$ the $M = \max_x \ \min \{q_1(x), q_2(x), q_3(x)\}$ is attained at $x_2^\tau$ with value $M = q_3(x_2^\tau)$, and then the resulting upper bound on the number of vertices of the lift will be determined as $2n \cdot M = 2n \cdot q_3(x_2^\tau)$. This finally gives

$$2n \cdot q_3(x_2^\tau) = \frac{8}{9}\left(d + \frac{1}{2}\right)^2$$

for any fixed $n \geq 2$, regardless of the number of loops and semi-edges at each vertex, as in the statement of the Theorem 3.2. Note here that we omit roots of the equation $q_1(x) = q_2(x)$ since in the case (3) for $d \geq 11$ the $M = \max_x \ \min \{q_1(x), q_2(x), q_3(x)\}$ is not achieved in any of these roots. □

For $d \leq 10$, that is, for the opposite case than (3), the maximum value of $\omega(d)$ is attained for $n = 2$ and $l = \lfloor \frac{d-2}{2} \rfloor$, and the corresponding values of the upper bound (obtained by computer search) are $\omega(8) = 64$, $\omega(9) = 80$ and $\omega(10) = 98$. We do not include the calculations for the remaining values of $d \leq 7$ because in this range of $d$ the orders of the largest known $(d, 2)$-graphs is larger that $\omega(d)$.

The last part of this section is dedicated to the conclusions of theorems stated above. Theorem 3.1 may be regarded as an improvement over the Moore bound for graphs of degree $d$ and diameter two obtained as lifts of $n$-poles of degree $d$ in an abelian group. In the case where $n = 2$ we obtain, up to the 'big O' term, the same upper bound as in [8]. For $n = 3$ the upper bound from the Theorem 3.1 gives approximately $\frac{87+6\sqrt{2}}{98}d^2 + O(d^{3/2}) \doteq 0.974d^2 + O(d^{3/2})$ as $d \to \infty$. For $n$ up to 7 we obtain the following approximate upper bounds on the order of a diameter-two lift of an $n$-pole with voltages in an abelian group:

| $n$ | **Order of lift of $n$-pole** |
|---|---|
| 2 | $0.932d^2 + O(d^{3/2})$ |
| 3 | $0.974d^2 + O(d^{3/2})$ |
| 4 | $0.987d^2 + O(d^{3/2})$ |
| 5 | $0.992d^2 + O(d^{3/2})$ |
| 6 | $0.995d^2 + O(d^{3/2})$ |
| 7 | $0.996d^2 + O(d^{3/2})$ |

This behavior is, of course, expected, since the limit of the leading term in the upper bound in Theorem 3.1 tends to 1 as $n \to \infty$.

Theorem 3.2 shows that for every $d \geq 11$ and $n \geq 2$ the upper bound on the order of abelian lifts of complete bipartite multigraphs based on $K_{n,n}$ of diameter 2 is $\frac{8}{9}\left(d+\frac{1}{2}\right)^2$. Since this is equal to the order of McKay-Miller-Širáň graphs [6, 9], the upper bound is sharp.

We would like to point out that our proof of the upper bound from Theorem 3.2 works for $n \geq 2$. For $n = 1$ the $(1,1)$-bipole in our terminology is just a dipole, that is, in the notation of Theorem 3.1, a 2-pole. Thus, for abelian lifts of 2-poles we currently have no better bound than the one of Theorem 3.1; elaborating on it the bound turns out to be

$$\frac{4(10+\sqrt{2})}{49}(d+0,35)^2 \approx 0,932(d+0,35)^2,$$

for odd degrees $d \geq 7$, which is mildly better than the one presented in [8] where semiedges have not been considered.

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