AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **82(1)** (2022), Pages 59–73

On the metric dimension of incidence graphs of Möbius planes

Ákos Beke

Department of Mathematics
ELTE Eötvös Loránd University, Savaria University Centre
Károlyi Gáspár tér 4., 9700, Szombathely
Hungary
beke.akos@sek.elte.hu

Abstract

We study the metric dimension and optimal split-resolving sets of the point-circle incidence graph of a Möbius plane. We prove that the metric dimension of a Möbius plane of order q is $2q + \mathcal{O}(\log q)$, and that an optimal split-resolving set has cardinality between $5q-10$ and $2.5q \log q + \mathcal{O}(q)$. We also prove that a smallest blocking set of a Möbius plane of order q has at most $2q(1 + \log(q + 1))$ points.

1 Introduction

The concept of metric dimension can be discussed in any metric space, and it already appeared in 1953 [7]. In graph theory, resolving sets and metric dimension were first introduced independently by Slater [21], and Harary and Melter [15]. The topic has been studied in several articles, and many results have been gathered in [2] and [9]. Since then, the metric dimension of various graph classes has been studied, including numerous graphs arising from finite geometries [1, 3, 4, 5, 16, 17].

In this paper we study the point-circle incidence graphs of Möbius planes and give lower and upper bounds for the metric dimension and for the size of smallest split-resolving sets of the incidence graphs of Möbius planes.

Definition 1.1. Let $G = (V, E)$ be a graph. We say that a set $W \subseteq V$ is *resolved by the set* $S \subseteq V$ if for any two different vertices $v, u \in W$ there is a vertex $s \in S$ such that $d(v, s) \neq d(u, s)$.

S is called a *resolving set* of G if it resolves the set V. The cardinality of a smallest resolving set is called the *metric dimension of the graph* and it is denoted by $\mu(G)$.

Let $G = (V, E)$ be a bipartite graph with parts A and B. We say that S is a *split-resolving set* if $S \cap A$ resolves B and $S \cap B$ resolves A.

ISSN: 2202-3518

Note that if G is a bipartite graph with parts A and B and the set S resolves the classes, then S is also a resolving set, because if $a \in A$ and $b \in B$, then for any element s of S, $d(a, s)$ is odd if and only if $d(b, s)$ is even.

In some cases we will reformulate the problem into a blocking problem of a hypergraph. A blocking set of a hypergraph is a subset of vertices such that every edge has at least one common vertex with the subset. The goal is to determine the size of a smallest blocking set.

It is standard to reformulate such a problem to an Integer Linear Programming (IP) task: for each vertex v of the hypergraph we introduce a binary variable x_v, which indicates whether the vertex v is included in a subset of vertices or not. Then the subset $\{v \in V : x_v = 1\}$ is a blocking set if and only if for each hyperedge e, the constraint

$$\sum_{v \in e} x_v \geq 1$$

holds. The objective function of this IP problem is

$$\sum_{v \in V} x_v,$$

which we want to minimize.

If we change the constraints $x_v \in \{0, 1\}$ to $x_v \geq 0$, $x_v \in \mathbb{R}$, then the solution of this relaxed Linear Programming (LP) task is called the fractional solution of the blocking problem.

We will consider points and circles of a Möbius plane as vertices and hyperedges of a hypergraph. The blocking set of this hypergraph is called a blocking set of the Möbius plane. These kinds of blocking sets have been studied in several articles (see [8, 11, 13, 18, 22]). We will give an upper bound to an optimal blocking set in Theorem 4.4.

We will use the following theorem to give upper bounds for the considered combinatorial problems.

Theorem 1.2 (Lovász [20])**.** *Let τ denote the optimum of the blocking set problem of a hypergraph $H = (V, E)$. Then*

$$\tau < \tau^*(1 + \log(d)),$$

where d is the greatest degree of the hypergraph, that is

$$d = \max\{|\{e \in E \ : \ v \in e\}| \ : \ v \in V\}$$

and τ^* is the fractional optimal solution.

We will use this theorem by constructing a hypergraph such that a subset of its vertices is a blocking set of the hypergraph if and only if it resolves a particular subset of the graph.

We mention some results about the metric dimension of the incidence graphs of some finite incidence geometries:

Theorem 1.3 ([17]). *The metric dimension of the incidence graph of a projective plane of order q is $4q + \mathcal{O}(1)$.*

Theorem 1.4 ([3]). *The metric dimension of the incidence graph of an affine plane of order q is $3q + \mathcal{O}(1)$.*

Theorem 1.5 ([3]). *The metric dimension of the incidence graph of a generalized quadrangle of order (q, q) is at least $\max\{6q - 27, 4q - 7\}$, and it is at least $8q$ for the classical generalized quadrangle $W(q)$ and $Q(4, q)$.*

We note that the metric dimension of a graph is related to other graph parameters which are also studied in graphs arising from finite geometries, see [6, 12], for example.

2 Möbius planes and their incidence graphs

Definition 2.1. Let $\mathcal{M} = (\mathcal{P}, \mathcal{Z})$ a hypergraph. We call this hypergraph a *Möbius plane*, the elements of $\mathcal{P}$ points and the elements of $\mathcal{Z}$ circles if the following axioms hold:

1. For every three pairwise distinct points there is exactly one circle through them.
2. If $z \in \mathcal{Z}$, $P \in z$ and $Q \in \mathcal{P} \setminus z$, there is exacly one circle z' through P and Q such that $z \cap z' = \{P\}$.
3. There is at least one circle, and every circle has at least three points.
4. For every circle z there is at least one point P such that $P \notin z$.

If $|\mathcal{P}| < \infty$ then $\mathcal{M}$ is a *finite Möbius plane*.

In a finite Möbius plane, every circle has the same number of points. If a circle has $q + 1$ points, then q is called the *order* of the Möbius plane. In this case there are $q^2 + 1$ points and $q(q^2 + 1)$ circles in the plane, and there are $q(q + 1)$ circles through every point. For a point $P \in \mathcal{P}$ let us define the sets

$$\mathcal{P}' = \mathcal{P} \setminus \{P\}, \qquad \mathcal{L} = \{z \setminus \{P\} \ : \ P \in z \in \mathcal{Z}\}.$$

Then the hypergraph $(\mathcal{P}', \mathcal{L})$ is an affine plane, called the *affine residue at point P*. More details and constructions of Möbius planes can be found in [10] and [19].

We give a simple example for the smallest Möbius plane:

Example 2.2. [23, p. 755] By axioms 3 and 4, there is at least one circle z, three points on z and a fourth point not on z. By axioms 2 and 3 there are at least two points not on z. So there are at least 5 points, and if there are only three points on z then, by axiom 1, there are $\binom{5}{3}$ circles.

Let $\mathcal{P} = \{1, 2, 3, 4, 5\}$ and $\mathcal{Z} = \{z \subseteq \mathcal{P} \ : \ |z| = 3\}$.

It is easy to see that $(\mathcal{P}, \mathcal{Z})$ is a Möbius plane of order 2, it has five points and ten circles.

From now on let $\mathcal{M}(q) = (\mathcal{P}, \mathcal{Z})$ be a Möbius plane of order q.
We introduce some notation:

- The set of circles which go through a point P is denoted by $[P]$.
- For any three points A, B and C we denote the circle through them by ABC.
- We say the circle a is *skew to the circle* b if they have no common points.
- We say the circle a is *tangent to the circle* b if they have one common point.

In the next lemma we summarize combinatorial statements that are important for us.

Lemma 2.3. *[10, p. 264] Let $z \in \mathcal{Z}$ be a circle.*

1. *There are $q + 1$ circles through two distinct points.*
2. *There are $\frac{(q+1)q^2}{2}$ circles with two common points with z.*
3. *There are $q - 1$ circles tangent to z through any one point of z.*
4. *There are $q^2 - 1$ circles tangent to z.*
5. *There are $\frac{q^3-3q^2+2q}{2}$ circles skew to z.*
6. *There are $\frac{q^3+3q^2-2}{2}$ circles which have one or two common points with z.*

Proof.

1. Let P and Q be two different points and $H := \mathcal{P} \setminus \{P, Q\}$. Consider the circles through P and Q and let k denote the number of such circles. All of them covers $q - 1$ points of H. By the first axiom, every point $X \in H$ is covered by exactly one of them. Hence
$$k(q - 1) = |H| = q^2 - 1.$$
So there are $k = q + 1$ circles through P and Q.
2. For every two points on z there are q circles through them different from z. Counting them we get $\binom{|z|}{2} q = \frac{(q+1)q^2}{2}$ such circles.
3. Let P be a point on z. There are $q(q + 1)$ circles through the point P. One of them is z. We can choose another point Q on z in q different ways, and there are q circles through P and Q different from z. Hence the number of circles tangent to z on P is
$$q(q + 1) - 1 - qq = q - 1.$$

4. The circle z has $q+1$ points and, by the previous statement, there are $q-1$ circles tangent to z on each of them. Thus there are $(q+1)(q-1) = q^2-1$ circles tangent to z.

5. There are $q(q^2+1)$ circles, one of them is z itself. Subtracting the number of circles which have at least one common point with z, we get

$$q(q^2+1) - 1 - (q^2-1) - \frac{(q+1)q^2}{2} = \frac{q^3-3q^2+2q}{2}.$$

6. Adding the number of circles with one or two common points with z we get

$$\frac{(q+1)q^2}{2} + q^2 - 1 = \frac{q^3+3q^2-2}{2}.$$

□

Definition 2.4. The *point-circle incidence graph of a Möbius plane* $\mathcal{M}(q)$ is $G = (V,E)$, where $V := \mathcal{P} \cup \mathcal{Z}$ and $E := \{\{P,z\} \ : \ P \in z\}$.

This is obviously a bipartite graph with vertex classes $\mathcal{P}$ and $\mathcal{Z}$. The metric dimension of G will be considered as the metric dimension of the geometry and we use the notation $\mu(\mathcal{M}(q))$ instead of $\mu(G)$. For every $P, Q \in \mathcal{P}$ and $a, b \in \mathcal{Z}$ we have

$$d(P,Q) = \begin{cases} 0 & \text{if } P = Q \\ 2 & \text{if } P \neq Q \end{cases}; \qquad d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 2 & \text{if } a \cap b \neq \emptyset \\ 4 & \text{if } a \cap b = \emptyset \end{cases};$$

$$d(a,P) = \begin{cases} 1 & \text{if } P \in a \\ 3 & \text{if } P \notin a \end{cases}.$$

Definition 2.5. For a subset $S \subseteq V$ we call the circles *outer circles* and the points *outer points* if they are not elements of S.

3 Metric dimension of Möbius planes

We give a construction that resolves the set of points, and then we will use Theorem 1.2 to find an upper bound on the minimum cardinality of sets that resolve the set of circles.

Let us construct a hypergraph $H = (V, E')$ with the same vertex set as G. For every two different circles a and b we construct a hyperedge $e_{a,b}$ which contains all vertices $v \in V$ for which $d(v,a) \neq d(v,b)$. That means

- $a, b \in e_{a,b}$,
- a circle $c \in \mathcal{Z} \setminus \{a,b\}$ is an element of $e_{a,b}$ if and only if c has a common point with a or b but not with both,

- a point $P \in \mathcal{P}$ is in $e_{a,b}$ if and only if P is incident with a or b but not with both.

By definition, $e_{a,b}$ denotes the same hyperedge as $e_{b,a}$. Any subset of the graph G resolves the set of circles if and only if it is a blocking set of the hypergraph H.

Lemma 3.1. *There are at least* $\frac{q^3}{2} - 3q^2 + \frac{11q}{2} - 1$ *vertices in any hyperedge of* H.

Proof. Let $e_{a,b}$ be a hyperedge. There are three cases depending on how many common points a and b have.

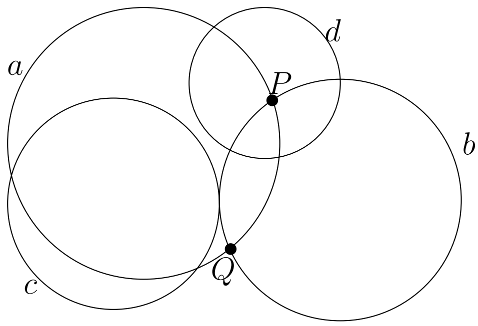

Figure 1: Two types of circles not in $e_{a,b}$.

First let a and b be two circles which have two common points P and Q, and let Z denote the set of circles with two common points with a containing neither P nor Q. Then $|Z| = \binom{q-1}{2}q$. Among them, there are at most $(q-1)^2$ circles tangent to b (like circle c in Figure 1). Let

$$X = \big\{(\{A,B\},C) \ : \ A,B \in b,\ A \neq B,\ C \in a,\ ABC \in Z\big\}.$$

There are $\binom{q-1}{2}$ unordered pairs $\{A,B\}$ and there are $q-1$ choices of C such that $A,B \in b$, $A \neq B$, $C \in a$ and $A,B,C \notin \{P,Q\}$. Since the circle ABC has to be an element of Z,

$$|X| \leq \binom{q-1}{2}(q-1).$$

If $(\{A,B\},C_1) \in X$ then $ABC_1 \in Z$, therefore it has two common points with a, thus there is a pair $(\{A,B\},C_2) \in X$ such that $C_1 \neq C_2$ and the circles ABC_1 and ABC_2 are the same. So we have at most $\frac{1}{2}\binom{q-1}{2}(q-1)$ circles in Z with two common points with b (like circle d in Figure 1). So there are at least $\binom{q-1}{2}q-(q-1)^2-\frac{1}{2}\binom{q-1}{2}(q-1)$ circles with two common points with a skew to b. There are at least the same number of circles that have two common points with b skew to a. All of them are elements of the hyperedge $e_{a,b}$. The points of a and b except P and Q and the circles a and b are in $e_{a,b}$, too. Therefore

$$\begin{aligned} |e_{a,b}| &\geq 2\left(\binom{q-1}{2}q - \frac{1}{2}\binom{q-1}{2}(q-1) - (q-1)^2\right) + 2(q-1) + 2 \\ &= \frac{q^3}{2} - 3q^2 + \frac{11q}{2} - 1. \end{aligned}$$

If a and b have one common point, then in the same way the lower bound for the edge is

$$|e_{a,b}| \geq 2\left(\binom{q}{2}q - \frac{1}{2}\binom{q}{2}q - q(q-1)\right) + 2q + 2 = \frac{q^3}{2} - \frac{5q^2}{2} + 4q + 2.$$

Finally, if a and b have no common points, then the lower bound is

$$\begin{aligned} |e_{a,b}| &\geq \left(\binom{q+1}{2}q - \frac{1}{2}\binom{q+1}{2}(q+1) - (q+1)(q-1)\right) + 2(q+1) + 2 \\ &= \frac{q^3}{2} - 2q^2 + \frac{3q}{2} + 6. \end{aligned}$$

It is easy to see that the first case is the smallest of the three lower bounds. □

Theorem 3.2. *If $q \geq 4$, then*

$$\mu(\mathcal{M}(q)) \leq 2q - 2 + \left(2 + \frac{14q^2 - 20q + 6}{q^3 - 6q^2 + 11q - 2}\right)\left(1 + \log\left(\frac{q^6}{4}\right)\right).$$

In particular, if $q \geq 156$, then

$$\mu(\mathcal{M}(q)) \leq 2q + 12\log(q).$$

Proof. We give a construction that resolves the set of points if q is at least 3. Let P be a point and let us consider the affine residue at point P. Let P_1 and P_2 be two different parallel classes in this affine plane (See Figure 2).

These are circle classes in the Möbius plane such that any two circles in a class are tangent to each other in the point P. Let $a \in P_1$ and $b \in P_2$. We show that the set $S_1 = P_1 \cup P_2 \setminus \{a, b\}$ resolves the set of points.

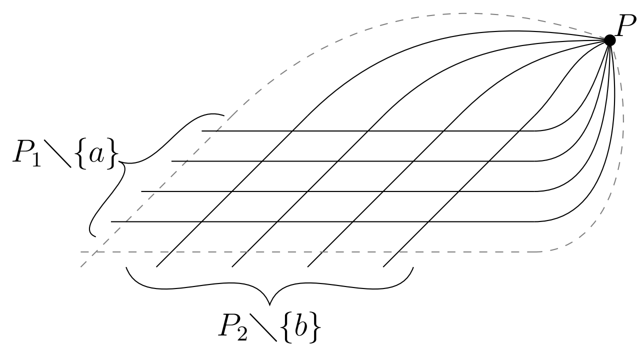

Figure 2: The set S_1 resolves the set of points.

Let $A, B \in \mathcal{P}\setminus\{P\}$ be two different points. If $ABP \notin P_1$ then A and B lie on two different circles of P_1, and if $ABP \in P_1$ then A and B lie on two different circles of P_2. Since A is on at most two lines of S_1 and the point P lies on every circle of S_1 and $|S_1| = 2q-2 > 2$ there is a circle incidence with P but not with A. Therefore the set of points is resolved by the set S_1, and this set has $2q-2$ elements.

If $z_1, z_2 \in \mathcal{Z} \diagdown (P_1 \cup P_2)$ and they are circles of the affine residue at point P, then for any $s \in S_1$, $d(z_1, s) = d(z_2, s) = 2$. So the set S_1 does not resolve the set $\mathcal{Z}$. Using Theorem 1.2 we prove that there is a set S_2 with at most roughly $12\log(q)$ vertices that resolves the set of circles.

Let a and b be two different circles. Setting all the variables $\frac{2}{q^3-6q^2+11q-2}$, by Lemma 3.1 all the constraints hold:

$$\sum_{v \in e_{a,b}} x_v \geq \left(\frac{q^3}{2} - 3q^2 + \frac{11q}{2} - 1\right) \frac{2}{q^3 - 6q^2 + 11q - 2} = 1.$$

The objective value is

$$\tau^* \leq \frac{2}{q^3 - 6q^2 + 11q - 2}(q^3 + q + q^2 + 1) = 2 + \frac{14q^2 - 20q + 6}{q^3 - 6q^2 + 11q - 2}.$$

By Lemma 2.3, for every circle x there are $\frac{q^3-3q^2+2q}{2}$ circles skew to x and there are $\frac{q^3+3q^2-2}{2}$ circles which have one or two common points with x. Also, x is an element of the edge $e_{a,x}$ for every $a \in \mathcal{Z} \diagdown \{x\}$. Therefore, in the hypergraph H, the degree of a circle is

$$\begin{aligned} \frac{q^3 - 3q^2 + 2q}{2} \cdot \frac{q^3 + 3q^2 - 2}{2} + q^3 + q - 1 &= \frac{q^6}{4} - \frac{7q^4}{4} + 2q^3 + \frac{3q^2}{2} - 1 \\ &< \frac{q^6}{4}. \end{aligned}$$

It is easy to see that the degree of a point is $(q^2 + q)(q^3 - q^2)$. If $q \geq 4$ then the degree of a circle is greater then the degree of a point. By Theorem 1.2, there is a set S_2 with cardinality less than

$$\left(2 + \frac{14q^2 - 20q + 6}{q^3 - 6q^2 + 11q - 2}\right)\left(1 + \log\left(\frac{q^6}{4}\right)\right)$$

that resolves the set of circles, so the set $S = S_1 \cup S_2$ is a resolving set. □

The given upper bound of the metric dimension is $2q + \mathcal{O}(\log(q))$. We have a similar lower bound too:

Theorem 3.3.

$$\mu(\mathcal{M}(q)) \geq \left\lceil 2q - 4 + \frac{8}{q+2} \right\rceil \geq 2q - 3.$$

Moreover, if $q \geq 156$, then every optimal resolving set for $\mathcal{M}(q)$ contains at least $2q - 4$ circles.

Proof. Let S be an optimal resolving set. Let $\mathcal{Z}_S$ denote the set of circles, and $\mathcal{P}_S$ the set of points of S. Let t denote the number of outer points that are covered by one circle:

$$t = |\{P \in \mathcal{P} \setminus S : |[P] \cap S| = 1\}|.$$

Then $t \leq |\mathcal{Z}_S|$, because if there were two outer points P and Q which are covered by the same circle, and only by that circle, then the constraint of $\{P, Q\}$ would not be resolved. Also, there could be only one outer point not covered by $\mathcal{Z}_S$. Let us double count the set

$$\{(P, a) \in \mathcal{P} \times \mathcal{Z} : a \in \mathcal{Z}_S,\ |[P] \cap \mathcal{Z}_S| \geq 2,\ P \in a\}$$

to obtain

$$|\mathcal{Z}_S|(q+1) - t \geq 2(q^2 + 1 - t - 1 - |\mathcal{P}_S|).$$

By rearranging the inequality we get

$$\begin{aligned} |\mathcal{Z}_S| q &\geq 2(q^2 - t - |\mathcal{P}_S|) + t - |\mathcal{Z}_S| \\ &= 2q^2 - t - |\mathcal{Z}_S| - 2|\mathcal{P}_S| \\ &\geq 2q^2 - 2|S|, \end{aligned}$$

and thus

$$|\mathcal{Z}_S| \geq 2q - 2\frac{|S|}{q}. \tag{1}$$

As $|S| \geq |\mathcal{Z}_S|$, (1) yields

$$|S| \;\geq\; \frac{2q^2}{q+2} \;=\; 2q - 4 + \frac{8}{q+2},$$

which proves the assertion on $|S|$. If $q \geq 156$, we can combine (1) with the upper bound in Theorem 3.2 to obtain

$$|\mathcal{Z}_S| \;\geq\; 2q - 2\frac{2q + 12\log(q)}{q} \;=\; 2q - 4 - \frac{24\log(q)}{q}.$$

If $q \geq 114$, then $24\log(q) < q$, so for $q \geq 156$ we have $|\mathcal{Z}_S| \geq 2q - 4$. □

4 Split-resolving sets of Möbius planes

In this section we give a lower and an upper bound for the cardinality of an optimal split-resolving set of a Möbius plane.

Let $\mathcal{P}_S$ and $\mathcal{Z}_S$ denote the set of points and the set of circles of a split-resolving set S.

Proposition 4.1. *Let S be an optimal split-resolving set. If $q > 5$, then*

$$2q - 3 \leq |\mathcal{Z}_S| \leq 2q - 2.$$

If $3 \leq q \leq 5$ then

$$|\mathcal{Z}_S| = 2q - 2.$$

Proof. We can use the same construction as in the proof of Theorem 3.2, where we gave a circle set which resolves $\mathcal{P}$ with $2q-2$ circles.

To obtain a lower bound, we can do almost the same as in the proof of Theorem 3.3. Let t denote the number of points that are covered by one circle. Then $|\mathcal{Z}_S| \geq t$, and there could be only one point not covered by $\mathcal{Z}_S$. Let us double count the set

$$\{(P,a) \in \mathcal{P} \times \mathcal{Z} : a \in \mathcal{Z}_S,\ |[P] \cap \mathcal{Z}_S| \geq 2,\ P \in a\}.$$

Now

$$|\mathcal{Z}_S|(q+1) - t \geq 2(q^2+1-t-1),$$

and by using the upper bound for t,

$$|\mathcal{Z}_S|(q+1) \ \geq\ 2q^2 - t \ \geq\ 2q^2 - |\mathcal{Z}_S|;$$

thus $|\mathcal{Z}_S|(q+2) \geq 2q^2$, and the obtained lower bound is

$$|\mathcal{Z}_S| \geq 2q - 4 + \frac{8}{q+2}.$$

If $q < 6$ then $\frac{8}{q+2} > 1$, and therefore, if $q \in \{3,4,5\}$ then $|\mathcal{Z}_S| \geq 2q-2$, and if $q > 5$ then $|\mathcal{Z}_S| \geq 2q-3$. □

Proposition 4.2. *If S is an optimal split-resolving set, then*

$$3q - 7 \ \leq\ |\mathcal{P}_S| \ \leq\ \frac{q+2}{2}\left(1 + \log\left(q^5\right)\right).$$

Proof. Since the bounds are trivial for $q = 2$, we can assume that $q \geq 3$. First we prove the upper bound. Let us construct a hypergraph $H = (\mathcal{P}, E')$ such that for every two different circles a and b we construct a hyperedge $e_{a,b}$ which contains all points P for which $d(P,a) \neq d(P,b)$. That means a point $P \in \mathcal{P}$ is in $e_{a,b}$ if and only if P is incident with a or b, but not with both.

Let a and b be two different circles. Setting all the variables to $\frac{1}{2q-2}$, all the constraints hold because there are at least $2q-2$ vertices in any hyperedge of H. In this case the objective value is

$$\tau^* \ \leq\ \frac{1}{2q-2}(q^2+1) \ =\ \frac{q}{2} + \frac{1}{2} + \frac{1}{q-1} \ \leq\ \frac{q}{2} + 1.$$

In the hypergraph H the degree of a point is $(q^2+q)(q^3-q^2) = q^5 - q^3 < q^5$. By Theorem 1.2, $\dfrac{q+2}{2}\,(1 + \log(q^5))$ vertices resolve all the constraints.

To have a lower bound, let

$$t_k = |\{z \in \mathcal{Z} : |z \cap \mathcal{P}_S| = k\}|, \quad k \in \{0,1,2\}.$$

There can be only one unblocked circle, and for every $P \in \mathcal{P}_S$ there is at most one circle that is blocked by only P. Thus

$$t_0 \leq 1, \quad t_1 \leq |\mathcal{P}_S|.$$

For any two points $P_1, P_2 \in \mathcal{P}_S$ there is at most one double blocked circle z that is blocked by P_1 and P_2. Thus

$$t_2 \leq \binom{|\mathcal{P}_S|}{2}.$$

Let us double count the set $\{(P, z) \in \mathcal{P} \times \mathcal{Z} : P \in \mathcal{P}_S,\ |z \cap \mathcal{P}_S| \geq 3,\ P \in z\}$ to get

$$|\mathcal{P}_S|(q^2 + q) - t_1 - 2t_2 \geq 3(q^3 + q - t_0 - t_1 - t_2).$$

By using the upper bound for t_0, t_1 and t_2,

$$|\mathcal{P}_S|(q^2 + q) \geq 3(q^3 + q - 1) - 2|\mathcal{P}_S| - \binom{|\mathcal{P}_S|}{2}.$$

This yields the quadratic inequality

$$|\mathcal{P}_S|^2 + |\mathcal{P}_S|(2q^2 + 2q + 3) + 6(1 - q^3 - q) \geq 0. \tag{2}$$

If we substitute $3q - 8$ into $|\mathcal{P}_S|$, we get the inequality

$$q^2 + 61q - 46 \leq 0.$$

Since both roots are less than 2, this inequality does not hold. Also, (2) clearly fails for $|\mathcal{P}_S| = 0$, and thus

$$|\mathcal{P}_S| \geq 3q - 7.$$

□

A corollary of the above propositions is the following.

Theorem 4.3. *If S is an optimal split-resolving set of $\mathcal{M}(q)$, then*

$$5q - 10 \ \leq\ |S| \ \leq\ \frac{q+2}{2}\left(1 + \log\left(q^5\right)\right) + 2q - 2.$$

Note that the bound $t_0 \leq 1$, in the proof of Proposition 4.2, implies that the set $\mathcal{P}_S$ blocks all circles with one possible exception. This means that there is a point $P \in \mathcal{P}$ such that $B = \mathcal{P}_S \cup \{P\}$ is a blocking set of the Möbius plane. In [8] Bruen and Rothschild proved that if B is a blocking set of the Möbius plane of order $q \geq 9$, then $|B| \geq 2q$; thus

$$\text{if } q \geq 9, \text{ then } |\mathcal{P}_S| \geq 2q - 1.$$

As far as we know, the best upper bound for the size of a blocking set in a Mobius plane of order q was given by Greferath and Rössing in [13]. They proved that there exists a blocking set that has approximately $3q\log(q)$ points. We prove that there exists a blocking set of size approximately $2q\log(q)$.

Theorem 4.4. *If B is an optimal blocking set of $\mathcal{M}(q)$, then*

$$|B| < \frac{q^2+1}{q+1}\left(1 + \log(q(q+1))\right).$$

Proof. Let $\mathcal{M}(q) = (\mathcal{P}, \mathcal{Z})$ be a Möbius plane. We consider the points as variables and circles as constraints of an LP; that is, the constraints are the inequalities

$$\sum_{P \in z} x_P \geq 1$$

for every circle z. First we give a fractional solution. Since every circle has $q+1$ points, if we set all variables to $\frac{1}{q+1}$, then all constraints hold with equality. There are q^2+1 variables, so the objective value is

$$\tau^* \leq \frac{q^2+1}{q+1}.$$

All points are incident with the same number of circles, and thus the degree d of the hypergraph is the number of circles incident with a point:

$$d = q(q+1).$$

Using Theorem 1.2, we get the upper bound

$$|B| \ < \ \tau^*(1 + \log(d)) \ \leq \ \frac{q^2+1}{q+1}\left(1 + \log(q(q+1))\right).$$

□

5 Results for small orders

In this section we deal with optimal resolving sets and split-resolving sets for Möbius planes of small order. Let us consider first $\mathcal{M}(2)$ in detail. We use the construction that we gave in Section 2.

Lemma 5.1. *For any three different points A, B and C there is no circle which resolves all the constraints $\{A, B\}$, $\{A, C\}$ and $\{B, C\}$.*

Proof. Let us check whether a circle z can resolve all the considered constraints. Without loss of generality we may assume that $A \in z$ and $B \notin z$, so z resolves $\{A, B\}$. If $C \in z$ then $\{A, C\}$ is not resolved and if $C \notin z$ then $\{B, C\}$ is not resolved by z. □

We use again the notation $\mathcal{P}_S$ and $\mathcal{Z}_S$ to denote the set of points and the set of circles of a resolving or split-resolving set S.

Theorem 5.2. $\mu(\mathcal{M}(2)) = 4$.

Proof. We prove that any four-element subset of $\mathcal{P}$ is a resolving set. For any two different points P and Q one can assume $P \in S$, so $d(P, P) = 0 \neq 2 = d(P, Q)$. For any two different circles a and b there are points $A \in a \setminus b$ and $B \in b \setminus a$. We can assume that $A \in S$. Then $d(a, A) = 1 \neq 3 = d(b, A)$.

Next we prove the lower bound. In $\mathcal{M}(2)$, any two circles intersect, so two distinct circles cannot be resolved by a third circle. Hence there can be at most one unblocked outer circle. If $|\mathcal{P}_S| = 0$ then $|\mathcal{Z}_S| \geq 9$. Since one point blocks six circles, if $|\mathcal{P}_S| = 1$ then there are four unblocked circles, so $|\mathcal{Z}_S| \geq 3$. If $|\mathcal{P}_S| = 2$ then by Lemma 5.1, $|\mathcal{Z}_S| \geq 2$. Finally, if $|\mathcal{P}_S| = 3$, then the constraint of the two outer points is not resolved by $\mathcal{P}_S$, so $|\mathcal{Z}_S| \geq 1$. ☐

Theorem 5.3. *If S is an optimal split-resolving set of $\mathcal{M}(2)$, then*

$$|\mathcal{P}_S| = 4 \quad \textit{and} \quad |\mathcal{Z}_S| = 3.$$

Proof. We have already proved that any four-element point set resolves the set of circles. Suppose that there are at most three points in the set $\mathcal{P}_S$. Without loss of generality we may assume that 1 and 2 are outer points. Then the circles $\{1, 3, 4\}$ and $\{2, 3, 4\}$ are at the same distance from any points of $\mathcal{P}_S$.

Now let us consider the set $\mathcal{Z}_S$. We may assume that $\{1, 2, 3\} \in \mathcal{Z}_S$. This circle does not resolve the constraints $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$. So by Lemma 5.1, we need at least two more circles.

We prove that the set $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$ resolves the set of points. The circle $\{1, 2, 3\}$ resolves every constraint $\{A, B\}$, where $A \in \{1, 2, 3\}$ and $B \in \{4, 5\}$. So $\{4, 5\}$, $\{2, 3\}$, $\{1, 3\}$ and $\{1, 2\}$ are the constraints not resolved by $\{1, 2, 3\}$. The first three are resolved by $\{1, 2, 4\}$ and the last one by $\{1, 3, 4\}$. ☐

We have investigated Miquelian planes and obtained results for small orders. We used Gurobi [14] to solve the problems. The optimals of resolving sets and split resolving sets are summarized in the following table:

Order of the plane	Resolving set	Split-resolving set
3	8	11
4	11	15
5	MIN:9 MAX:13	21

Acknowledgements

The author would like to sincerely thank Tamás Héger and János Ruff for the valuable discussions and remarks about the results in this paper. The author is grateful to the anonymous referees for their thorough proofreading of the manuscript and their valuable comments.

References

[1] R. F. Bailey, On the metric dimension of incidence graphs, *Discrete Math.* 341 (2018), 1613–1619.

[2] R. F. Bailey and P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, *Bull. Lond. Math. Soc.* 43(2) (2011), 209–242.

[3] D. Bartoli, T. Héger, G. Kiss and M. Takáts, On the metric dimension of affine planes, biaffine planes and generalized quadrangles, *Australas. J. Combin.* 72 (2018), 226–248.

[4] D. Bartoli, G. Kiss, S. Marcugini and F. Pambianco, Resolving sets for higher dimensional projective spaces, *Finite Fields Appl.* 67 (2020), 101723, 14 pp.

[5] D. Bartoli, G. Kiss and F. Pambianco, On resolving sets in the point-line incidence graph of PG(n,q), *Ars Math. Contemp.* 19(2), 231–247. doi: https://doi.org/10.26493/1855-3974.2125.7b0.

[6] A. Bonato, M. A. Huggan and T. G. Marbach, The localization number of designs, *J. Combin. Des.* 29(3) (2021), 175–192.

[7] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1953.

[8] A. A. Bruen and B. L. Rothschild, Lower bounds on blocking sets, *Pacific J. Math.* 118(2) (1985), 303–311.

[9] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.* 105 (2000), 99–113.

[10] P. Dembowski, *Finite Geometries*, Springer-Verlag, 1968.

[11] A. Gács and T. Szőnyi, Random constructions and density results, *Des. Codes Cryptogr.* 47(1) (2008), 267–287.

[12] S. Gravier, A. Parreau, S. Rottey, L. Storme and É. Vandomme, Identifying Codes in Vertex-Transitive Graphs and Strongly Regular Graphs, *Electron. J. Combin.* 22(4) (2015), #P4.6.

[13] M. Greferath and C. Rösing, On the Cardinality of Intersection Sets in Inversive Planes, *J. Combin. Theory Ser. A* 100 (2002), 181–188.

[14] Gurobi Optimization, Gurobi Optimizer 8.1, https://www.gurobi.com.

[15] F. Harary and R. Melter, On the metric dimension of a graph, *Ars Combin.* 2 (1976), 191–195.

[16] T. Héger, P. Szilárd and M. Takáts, The metric dimension of the incidence graphs of projective and affine planes of small order, *Australas. J. Combin.* 78(3) (2020), 352–375.

[17] T. Héger and M. Takáts, Resolving Sets and Semi-Resolving Sets in Finite Projective Planes, *Electron. J. Combin.* 19(4) (2012), #P30.

[18] G. Kiss, S. Marcugini and F. Pambianco, On blocking sets of inversive planes, *J. Combin. Des.* 13(4) (2004), 268–275.

[19] G. Kiss and T. Szőnyi, *Finite Geometries*, Chapman and Hall/CRC, 2019.

[20] L. Lovász, On the ratio of optimal integral and fractional covers, *Discrete Math.* 13 (1975), 383–390.

[21] P. J. Slater, Leaves of Trees, *Congr. Numer.* 14 (1975), 549–559.

[22] T. Szőnyi, Blocking sets in finite planes and spaces, *Ratio Math.* 5 (1992), 93–106.

[23] B. L. van der Waerden and L. J. Smid, Eine Axiomatik der Kreisgeometrie und der Laguerregeometrie, *Math. Ann.* 110 (1935), 753–776.

(Received 9 Dec 2020; revised 19 Sep 2021, 29 Nov 2021)