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Matching extendability in Cartesian products of cycles

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Abstract

In a bipartite graph G, a set $S \subseteq V(G)$ is deficient if $|N(S)| < |S|$. A matching M with vertex set U is k-suitable if $G - U$ has no deficient set of size less than k. Define the extremal function $f_k(G)$ to be the largest integer r such that every k-suitable matching in G with at most r edges extends to a perfect matching. Let $G(2m)_d$ be the d-fold Cartesian product of the cycle C_{2m}, where $m \geq 2$. We extend results of Vandenbussche and West by showing that for any integers k and d such that $1 \leq k \leq d$, $f_k(G(2m)_d) = k(2d - k) + \binom{k-1}{2}$, except when $m = 2$ and $d = 1$.

1 Introduction

A graph G is *m-extendable* if $m < |V(G)|/2$ and every matching of size m extends to a perfect matching. The largest value for which a graph G with a perfect matching is m-extendable, called the *extendability* of G, has been studied at length. Plummer has written three extensive surveys on this topic [18, 19, 20], the most recent of which appeared in 2008. Since then, additional results have been obtained on graphs on surfaces [2, 3, 11], distance-regular graphs [10], Cayley graphs [9], bi-Cayley graphs [16], eigenvalue conditions [23], the interrelation between extendability and criticality [7], and many others.

An early result of particular relevance to the current paper is by Györi and Plummer [12], who proved that the Cartesian product of an a-extendable and a b-extendable graph is an $(a + b + 1)$-extendable graph. Also relevant is the study of extendability under additional structural conditions. Aldred and Plummer [5] proved that induced matchings of size three extend to perfect matchings in planar triangulations, even though these graphs are not 3-extendable [17]. Vandenbussche and West [21] proved that all induced matchings extend to perfect matchings in the d-dimensional hypercube. Since induced matchings are those in which the matched

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edges are at distance at least two from each other, a natural generalization is the extendability of matchings in which more general distance constraints are placed on the edges. Tseng and Anstee (unpublished) proved that in the Cartesian product of d copies of the path P_n, every matching whose edges are separated by distance at least 3 extends to a perfect matching. Other results on distance-restricted matchings can be found in [1] and [6]. A common variation of this problem also considers matching extensions that must avoid some edges [4, 8].

An easy way to prevent a matching from extending to a perfect matching is to saturate the neighborhood of an uncovered vertex; prohibiting such a configuration may allow larger matchings to extend. This idea led Vandenbussche and West to introduce another structural requirement on matchings in [21] based on Hall's Condition. For a bipartite graph G, a set $S \subseteq V(G)$ is *deficient* if $|N(S)| < |S|$, where $N(S) = \bigcup_{v \in S} N(v)$. In 1935, Philip Hall [13] proved the famous characterization of bipartite graphs that admit perfect matchings: If G is a bipartite graph with partite sets X and Y, then G has a matching saturating X if and only if no subset of X is deficient. We say such a graph satisfies Hall's condition on X if X has no deficient subset, i.e., for all $S \subseteq X$, $|N(S)| \geq |S|$. The condition introduced in [21] prohibits small deficient sets in order to increase the extendability of the graph. More precisely, a matching M (with vertex set U) is *k-suitable* if $G - U$ has no deficient set of size less than k. Every matching is trivially 1-suitable, and a matching is 2-suitable if no set of endpoints of the matching edges cover the neighborhood of a vertex not in the matching. Let the extremal function $f_k(G)$ be the largest integer r such that every k-suitable matching in G with at most r edges extends to a perfect matching. Let Q_d be the d-dimensional hypercube. The main result of [21] was the following:

Theorem 1.1 (Vandenbussche and West [21]). *For $k \leq d - 3$,*

$$f_k(Q_d) = k(d-k) + \binom{k-1}{2}.$$

This result was a generalization of the result by Limaye and Sarvate [14] that proved (in different language) that $f_1(Q_d) = d - 1$ and $f_2(Q_d) \geq d$.

Note that Q_d is the d-fold Cartesian product graph of K_2, that is, $Q_d \cong K_2 \square \cdots \square K_2 = K_2^d$. Stretching the methodology in [21], the main result of this paper shows that the same extremal function applies to a much larger class of graphs, namely the d-fold Cartesian product graph of the cycle graph C_{2m}, where $m \geq 2$ and $d \geq 1$. We will denote this graph as $G(2m)_d$.

Theorem 1.2 (Main Theorem). *For $k \leq d$ and $m \geq 2$, every k-suitable matching in $G(2m)_d$ having at most $k(2d-k) + \binom{k-1}{2}$ edges extends to a perfect matching.*

Moreover, we will also show the result is sharp: If $1 \leq k \leq d$ and $d \neq 1$, then there exists a k-suitable matching having $k(2d-k) + \binom{k-1}{2} + 1$ edges that does not extend to a perfect matching in $G(2m)_d$. Together, this will establish that $f_k(G(2m)_d) = k(2d-k) + \binom{k-1}{2}$.

Note that for any $d \geq 0$, $Q_{2d} \cong G(4)_d$. When $m = 2$, the two results align, except Theorem 1.1 is true for a larger range of values of k (e.g., $k \leq 2d-3$ versus $k \leq d$). Therefore, throughout this paper, we will generally assume $m \geq 3$, as the case $m = 2$ is resolved by Theorem 1.1 (though many of the statements do in fact hold for $m = 2$ as well). We will discuss the motivation for requiring $k \leq d$ in our Main Theorem at the end of the paper.

Although it may not be surprising that the extremal function for $G(2m)_d$ and Q_{2d} are the same, we should note that the two families of graphs do behave differently with respect to matchings in another important way. As mentioned above, in [21], it was also shown that every induced matching in Q_d extends to a perfect matching. At the end of this paper, we will give an example of an induced matching that shows that this is not true in $G(2m)_d$ when $m > 2$. We also include a variety of open questions in the area of k-suitability.

2 Definitions and preliminaries

Throughout, d will be a positive integer and $[d] = \{1, \dots, d\}$. In this paper, $\mathbb{Z}_n = \{0, \dots, n-1\}$ is the usual cyclic group of integers modulo n and $\mathbb{Z}_n^d = \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ (d copies of $\mathbb{Z}_n$) is the d-fold external direct product group of $\mathbb{Z}_n$. For each $i \in [d]$, let $e_i \in \mathbb{Z}_n^d$ be the d-dimensional vector consisting of 0's in all coordinates except coordinate i, in which the entry is 1. For example, in $\mathbb{Z}_n^3$, we have $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$. For any $n \geq 2$ and $d \geq 1$, the set $\{e_1, \dots, e_d\}$ is a generating set (often called the standard generating set) for the group $\mathbb{Z}_n^d$.

For any integer $m \geq 2$, let $G(2m)_d$ denote the graph with vertex set $\mathbb{Z}_{2m}^d$, and two vertices $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d)$ are adjacent if and only if their coordinate-wise difference (modulo $2m$), $v - w$, is equal to $\pm e_i$ for some $i \in [d]$. Note $G(2m)_d$ can be thought of in two equivalent ways: $G(2m)_d \cong \mathrm{Cay}(\mathbb{Z}_{2m}^d; \{e_1, \dots, e_d\})$ (the Cayley graph on $\mathbb{Z}_{2m}^d$ whose connection set is the standard generating set of $\mathbb{Z}_{2m}^d$) and $G(2m)_d \cong C_{2m} \square \cdots \square C_{2m}$ (the Cartesian product graph of d copies of the cycle C_{2m}). $G(2m)_d$ is a $2d$-regular, connected, bipartite graph of order $(2m)^d$, and its partite sets, X and Y, satisfy $|X| = |Y| = (2m)^{d-1}$.

For each $i \in \mathbb{Z}_{2m}$, let $G(2m)_d^i$ denote the subgraph of $G(2m)_d$ induced by the vertices of the form $v = (v_1, \dots, v_{d-1}, i)$ (i.e., those with an i in coordinate d). We will refer to each such subgraph as a *layer* in $G(2m)_d$, so $G(2m)_d$ has $2m$ layers. Note that $G(2m)_d^i \cong G(2m)_{d-1}$, which will be crucial to the many inductive arguments throughout. The two layers $G(2m)_d^{i\pm1}$ are said to be the *adjacent layers* of $G(2m)_d^i$. Throughout this paper, we will usually abbreviate $G(2m)_d$ and $G(2m)_d^i$ as G_d and G_d^i, respectively.

The following remark introduces some fundamental facts and terminology used throughout the paper.

Remark 2.1. If S is a nonempty subset of one of the partite sets of G_d, then we can represent S as the union of subsets

$$S = S_0 \cup S_1 \cup \cdots \cup S_{2m-1},$$

where $S_i = S \cap V(G_d^i)$ for each $i \in \mathbb{Z}_{2m}$. For ease of notation, where appropriate, we will use s_i to denote $|S_i|$ and s to denote $|S|$. Provided $s \geq 2$, since at least two elements of S must differ in some coordinate position, we may assume without loss of generality that they differ in the last coordinate. Hence when $s \geq 2$, we will assume that there exist distinct i and j such that S_i and S_j are nonempty.

Any $v \in S_i$ has three types of neighbors (all sums that follow are taken modulo $2m$): The vertex $v - e_d$, contained in G_d^{i-1}; the vertex $v + e_d$, contained in G_d^{i+1}; and the neighbors of v contained in its own layer. Accordingly, the neighborhood of S_i in G_d can be partitioned into three sets, which we call the *left shadow* $N_{i-1}(S_i)$, the *right shadow* $N_{i+1}(S_i)$, and the *local neighborhood* $N_i(S_i)$, where

$$\begin{aligned} N_{i-1}(S_i) &= \{v - e_d : v \in S_i\} \subset V(G_d^{i-1}), \\ N_{i+1}(S_i) &= \{v + e_d : v \in S_i\} \subset V(G_d^{i+1}), \text{ and} \\ N_i(S_i) &= \{v \pm e_j : v \in S_i, j \in \{1, \ldots, d-1\}\} \subset V(G_d^i). \end{aligned}$$

Note $|N_{i-1}(S_i)| = |N_{i+1}(S_i)| = |S_i|$. Collectively, the left shadow and right shadow of S_i are the *shadows* of S_i.

In Section 1, we introduced the function $f_k(G)$. The goal of this paper is to prove that $f_k(G(2m)_d) = f_k(d)$, for all $1 \leq k \leq d$, where $f_k(d)$ is defined below. We also define an additional function, $g_k(d)$, that will be used frequently throughout.

Definition 2.2. Let k and d be integers such that $1 \leq k \leq d$, and let m be a fixed positive integer where $m \geq 2$. Define $f_k(d)$ and $g_k(d)$ to be the following functions:

$$f_k(d) = k(2d - k) + \binom{k-1}{2},$$

$$g_k(d) = \frac{1}{2}(2m)^d - f_k(d) - (k - 1).$$

While $g_k(d)$ does also depend on m, the value of m throughout the paper will always be clear from the context, so for convenience of notation it is omitted. Note that any subset of a single partite set of size at least $g_k(d)$ consists of almost all vertices in that set.

3 Neighborhood sizes in G_d

The general idea behind the proof of the Main Theorem is very similar to that in [21]. The theorem follows easily after we establish a result showing that most sets in G_d have a neighborhood size that is sufficiently large to guarantee that Hall's condition will be satisfied in G_d even after the endpoints of a k-suitable matching M are removed:

Lemma 3.1 (Main Lemma)**.** *Let X and Y be the partite sets of $G(2m)_d$, where $m \geq 3$, and let S be an arbitrary subset of X. If $1 \leq k \leq d$ and $k \leq |S| \leq g_k(d)$, then*

$$|N(S)| - |S| \geq f_k(d).$$

In this section, we will prove this lemma. First, we provide a short summary in order to help the reader navigate it. Our goal is to find "extra" neighbors for most subsets of $V(G_d)$. The result will essentially follow by induction: When we restrict a subset S of X to its subsets S_i (defined in Remark 2.1) within each $(d-1)$-dimensional layer G_d^i, if the inductive hypothesis is satisfied, it guarantees us many "extra" local neighbors within each $(d-1)$-dimensional subgraph. Since none of these neighborhoods are overlapping, each can contribute to the extra neighbors of S. However, there are anomalous cases that need to be carefully considered, such as when a set S_i is too small or too large to satisfy the inductive hypothesis. The lemmas that precede the proof of the Main Lemma will help us prove some of these cases.

We introduce some machinery that will help with many of the proofs. As discussed in Remark 2.1, we shall let $S_0, \ldots, S_{2m-1}$ be the pairwise disjoint subsets of S in each $(d-1)$-dimensional layer G_d^i. Subdivide $\mathbb{Z}_{2m}$ into two sets I_0 and I_1 such that $S_i = \varnothing$ for all $i \in I_0$, and $|S_i| \geq 1$ for all $i \in I_1$. Let $|I_1| = t$; recall we may assume $t \geq 2$. For each $i \in \mathbb{Z}_{2m}$, the existence of a perfect matching in G_d^i ensures that we have $|N_i(S_i)| - |S_i| \geq 0$, where $N_i(S_i)$ is the local neighborhood of S_i. Hence if we can find a subset $\mathcal{S}$ of $\{S_j \ : j \in I_1\}$ such that

$$\sum_{S_i \in \mathcal{S}} (|N_i(S_i) - |S_i|) \geq f_k(d),$$

then it follows that $|N(S)| - |S| \geq f_k(d)$. In some cases, we may also need to consider shadows of sets S_i where $i \in I_1$ but $i+1 \in I_0$ or $i-1 \in I_0$.

To begin, the following lemma provides a lower bound on the neighborhood size of subsets of $V(G_d)$ that will be helpful when a set S_i is small. There is an analogous result in [21].

Lemma 3.2. *If $\varnothing \neq S \subseteq V(G_d)$ and S is contained in a single partite set in G_d, then*

$$|N(S)| \geq 2ds - \frac{1}{2}s^2 - \frac{1}{2}s + 1.$$

Proof. When $s = 1$, $|N(S)| = 2d$, so the inequality holds. Consider when $s > 1$. We use induction on d. When $d = 1$, $G_d \cong C_{2m}$, so $|N(S)| \geq s$. The right side of the inequality simplifies to $\frac{3s}{2} - \frac{s^2}{2} + 1$, which is at most s for all $s \geq 2$, so the inequality holds.

Suppose that $d > 1$. The induction hypothesis applied to the t subgraphs G_d^i for $i \in I_1$ guarantees at least $2(d-1)s_i - \frac{1}{2}s_i^2 - \frac{1}{2}s_i + 1$ vertices in the local neighborhood of S_i. All of these local neighborhoods are disjoint, so

$$\begin{aligned} |N(S)| &\geq 2(d-1)\left(\sum_{i\in I_1} s_i\right) - \frac{1}{2}\left(\sum_{i\in I_1} s_i^2\right) - \frac{1}{2}\left(\sum_{i\in I_1} s_i\right) + t \\ &= 2(d-1)s - \frac{1}{2}\left(\sum_{i\in I_1} s_i^2\right) - \frac{s}{2} + t. \end{aligned}$$

For a fixed t, the sum of squares is maximized when exactly one s_i is equal to $s-(t-1)$ and all others are equal to 1. Hence

$$\begin{aligned} |N(S)| &\geq 2(d-1)s - \frac{1}{2}\left((s-t+1)^2 + t - 1\right) - \frac{s}{2} + t \\ &= 2ds - \frac{s^2}{2} + \left(t - \frac{7}{2}\right)s - \frac{t^2}{2} + \frac{3t}{2}. \end{aligned}$$

We consider three cases based on t:

Case 1: $t = 2$. Suppose $I_1 = \{i, j\}$. Since $m \geq 3$, at least one shadow of S_i is disjoint from G_d^j and at least one shadow of S_j is disjoint from G_d^i. Hence these two shadows contribute an additional $s_i + s_j = s$ neighbors of S not counted in the local neighborhoods, so

$$|N(S)| \geq 2ds - \frac{s^2}{2} - \frac{3s}{2} + 1 + s = 2ds - \frac{s^2}{2} - \frac{s}{2} + 1.$$

Case 2: $t = 3$. Since $m \geq 3$, there exists $i \in I_0$ with either $i+1 \in I_1$ or $i-1 \in I_1$. Hence a vertex in S has a shadow neighbor in G_d^i, which contributes at least one more to $|N(S)|$:

$$|N(S)| \geq 2ds - \frac{s^2}{2} - \frac{s}{2} + 1.$$

Case 3: $t \geq 4$. It suffices to show that $\left(t - \frac{7}{2}\right)s - \frac{t^2}{2} + \frac{3t}{2} \geq -\frac{s}{2} + 1$. Note that $s \geq t$ so

$$\left(t-\frac{7}{2}\right)s - \frac{t^2}{2} + \frac{3t}{2} + \frac{s}{2} - 1 \geq \left(t-\frac{7}{2}\right)t - \frac{t^2}{2} + \frac{3t}{2} + \frac{t}{2} - 1 = \frac{1}{2}(t^2 - 3t - 2).$$

Since $\frac{1}{2}(t^2 - 3t - 2) \geq 1$ for all $t \geq 4$, the inequality holds.

□

Applying Definition 2.2 to the result of Lemma 3.2, we see that for appropriate subsets S of $V(G_d)$, $|N(S)| - |S| \geq f_s(d)$. In fact, this proves the Main Lemma for small sets S:

Corollary 3.3. *If $1 \leq k \leq |S| \leq d$ and S is contained in a single partite set in G_d, then*

$$|N(S)| - |S| \geq f_k(d).$$

Proof. For fixed d, the function $f_s(d)$ is increasing on the interval $1 \le s \le d$. Hence if $k \le s \le d$, then $f_s(d) \ge f_k(d)$ and the result follows. □

Lemma 3.2 is sharp when S is small with respect to d:

Proposition 3.4. *For any k such that $1 \le k \le d$, there exists a subset S in a single partite set in G_d such that $|S| = k$ and*

$$|N(S)| = f_k(d) + k.$$

In particular, $S = \{e_1, \dots, e_k\}$ is such a set.

Proof. Clearly all vertices of S are in a single partite set, and $|S| = k$. Note $N(S)$ consists of the zero vector 0^d, exactly $\binom{k}{2}$ vertices of the form $e_i + e_j$ where $1 \le i < j \le k$, and for each vertex v in S, there are $2d-(k-1+1) = 2d-k$ vertices adjacent to v and no other vertex in S. Hence,

$$|N(S)| = 1 + \binom{k}{2} + k(2d-k) = f_k(d) + k.$$

□

The following lemma guarantees that a large set S_i within G_d^i will have a large neighborhood. Recall that sets larger than $g_k(d)$ contain nearly all vertices of a single partite set.

Lemma 3.5. *Let X and Y be the partite sets of G_d and X_i and Y_i be the partite sets of G_d^i where $i \in \mathbb{Z}_{2m}$. Suppose a subset S_i of X_i satisfies $|S_i| > g_{k-1}(d-1)$ and $S_\star \in \{S_{i-1}, S_{i+1}\}$. If $S_i' \subset S_i$ such that $|S_i'| = g_{k-1}(d-1)$, and $|N_i(S_i')| - |S_i'| \ge f_{k-1}(d-1)$, then*

$$|N_i(S_i)| \ge |Y_i| - (k-2)$$

and

$$|N(S_i \cup S_\star) \cap \left(V(G_d^i) \cup V(G_d^\star)\right)| - |S_i \cup S_\star| \ge |Y_i| - (k-2) - |S_\star|.$$

Proof. Recall that for each subgraph G_d^i, $|X_i| = |Y_i| = \frac{1}{2}(2m)^{d-1}$. By the hypothesis and applying Definition 2.2,

$$|N_i(S_i')| \ge g_{k-1}(d-1) + f_{k-1}(d-1) = \frac{1}{2}(2m)^{d-1} - (k-2) = |Y_i| - (k-2).$$

Since $S_i' \subset S_i$, $|N_i(S_i)| \ge |N_i(S_i')|$, so this establishes the first claim. Further,

$$|N_i(S_i)| - |S_i| \ge |Y_i| - (k-2) - |S_i|.$$

By definition, $|N_\star(S_i)| = |S_i|$. Therefore,

$$\begin{aligned} |N(S_i \cup S_\star) \cap \left(V(G_d^i) \cup V(G_d^\star)\right)| - |S_i \cup S_\star| &\ge |N_i(S_i)| - |S_i| + |N_\star(S_i)| - |S_\star| \\ &= |Y_i| - (k-2) - |S_\star|. \end{aligned}$$

□

We will require the following elementary result (a proof can be found in [21]).

Proposition 3.6. *Let G be a bipartite graph with partite sets X and Y, where $|X| = |Y|$. If $S \subseteq X$ and $T = Y - N(S)$, then*

$$|N(S)| - |S| \geq |N(T)| - |T|.$$

We apply the proposition in the following way.

Remark 3.7. Suppose $S \subseteq X$ and $|S| \geq (2m)^d/4$, and let $T = Y - N(S)$. The graph G_d satisfies Hall's condition, so $|N(S)| \geq |S|$, hence $|T| \leq (2m)^d/4$. Then by Proposition 3.6, it would suffice to verify the Main Lemma holds for T instead of S. Hence, in the proofs of Lemmas 3.8, 3.9, and the Main Lemma, we may further assume that $|S| \leq (2m)^d/4$. Recall that we also assume $m \geq 3$.

The next two lemmas will serve as the basis step in the inductive proof of the Main Lemma.

Lemma 3.8. *Let X and Y be the partite sets of G_d and let $S \subseteq X$. If $1 \leq k \leq d \leq 2$ and $k \leq |S| \leq g_k(d)$, then*

$$|N(S)| - |S| \geq f_k(d).$$

Proof. When $d = 1$, $G_d \cong C_{2m}$, and it is easy to verify that $|N(S)| - |S| \geq 1 = f_1(1)$ for every nonempty subset S of X satisfying $|S| \leq m - 1 = g_1(1)$.

When $d = 2$, $G_d \cong C_{2m} \,\square\, C_{2m}$. Consider the typical representation of such a graph with $V(G_d) = \{(i,j) : 0 \leq i \leq 2m-1, 0 \leq j \leq 2m-1\}$. (All sums involving pairs (i,j) indicated in this proof are taken mod $2m$ and $m \geq 3$.) If $|S| = 1$, then $k = 1$, and $f_1(2) = 3$; a single vertex has four neighbors, and the result holds. Suppose $|S| \geq 2$. Furthermore, by Remark 3.7, we may assume $|S| \leq m^2$. Mathematica [22] directly verified that the result is true for $m = 3$: For all S such that $2 \leq |S| \leq 9$, $|N(S)| - |S| \geq 4 = f_2(2)$ (This claim can also be verified by hand.) Assume now that $m \geq 4$. Note $4 \geq f_k(2)$, so it suffices to show $|N(S)| - |S| \geq 4$. Define a *column* $\mathcal{C}_j = \{(i,j) : 0 \leq i \leq 2m-1\}$ and define $\mathcal{C}_j(X) = \mathcal{C}_j \cap X$. We say a column $\mathcal{C}_j$ is either *full* (if $\mathcal{C}_j(X) \cap S = \mathcal{C}_j(X)$), *empty* (if $\mathcal{C}_j(X) \cap S = \varnothing$), or *partial* (otherwise). Let $N^+(S) = \{(i+1,j) : (i,j) \in S\}$; we call such vertices *down neighbors* of S. Clearly $|N^+(S)| = |S|$ and $N^+(S) \subseteq N(S)$. We now show $|N(S) - N^+(S)| \geq 4$, which will complete the proof.

If some $\mathcal{C}_j$ is a partial column, then there exists an i such that $(i,j) \in S$ and $(i-2,j) \notin S$. Hence $\mathcal{C}_j$ contributes an additional neighbor, the vertex $(i-1,j)$ (we call such vertices *up neighbors*) to $N(S) - N^+(S)$. Hence we may assume there are at most three partial columns, otherwise the result holds.

If some $\mathcal{C}_j$ is full and $\mathcal{C}_{j+1}$ (or symmetrically $\mathcal{C}_{j-1}$) is empty, then we are also done; matching (i,j) to $(i,j+1)$ (or symmetrically to $(i,j-1)$) produces m extra neighbors (we call such vertices *side neighbors*), which is enough as $m \geq 4$. Therefore we may assume there are no full columns adjacent to any empty columns. Moreover, we may assume there are at most $m-1$ full columns (for otherwise either $|S| = m^2$, which forces a full column to be adjacent to an empty column, or $|S| > m^2$, which is contrary to our assumption on $|S|$), and since there are at most three partial columns, there must be at least $m-2 \geq 2$ empty columns. In particular, there must be at least two partial columns, each adjacent to a different empty column. Each partial column provides at least one up neighbor and at least one side neighbor. This contributes at least 4 extra neighbors, so $|N(S) - N^+(S)| \geq 4$. $\square$

Lemma 3.9. *Let $d \geq 1$ and let X and Y be the partite sets of G_d. If $S \subseteq X$ and $1 \leq |S| \leq g_1(d)$, then*

$$|N(S)| - |S| \geq f_1(d).$$

Proof. We will induct on d. Lemma 3.8 establishes the basis for $d = 1$ and $d = 2$. Therefore, suppose that $d \geq 3$ and S is an arbitrary subset of X such that $1 \leq |S| \leq g_1(d)$. Recall from Remark 3.7, we may also assume $|S| \leq (2m)^d/4$. We need to show $|N(S)| - |S| \geq f_1(d) = 2d - 1$. Note a subset S_i satisfies the induction hypothesis if $1 \leq |S_i| \leq g_1(d-1)$.

Case 1: *No subset satisfies the induction hypothesis.* Here, for each $i \in \mathbb{Z}_{2m}$, S_i is either empty or contains more than $g_1(d-1)$ vertices. If every S_i is nonempty, then

$$|S| > 2m \cdot g_1(d-1) \geq (2m)^d/4,$$

contradicting our upper bound on S.

Hence there must exist an i such that $|S_i| > g_1(d-1)$ and either $i-1 \in I_0$ or $i+1 \in I_0$. This implies some shadow of S_i contains more than $g_1(d-1) \geq 2d-1$ vertices, and since these vertices are not in the local neighborhood of any S_j, $|N(S)| - |S| \geq 2d - 1$.

Case 2: *At least one subset satisfies the induction hypothesis.* If there are two such sets, then the inductive hypothesis guarantees

$$\sum_{i \in I_1} (|N_i(S_i)| - |S_i|) \geq 4d - 6,$$

and this is at least $2d - 1$ when $d \geq 3$. If not, the set S_i satisfying the inductive hypothesis provides $|N_i(S_i)| - |S_i| \geq 2d-3$, and we seek two additional "extra" neighbors. If some S_j is empty, then again we obtain at least two extra neighbors: Choose an empty S_j such that either S_{j+1} or S_{j-1} is nonempty. Since only S_i satisfies the inductive hypothesis, either $|S_{j-1}| \geq 2$ or $|S_{j+1}| \geq 2$. Hence either the right shadow of S_{j-1} or the left shadow of S_{j+1} contributes two additional vertices.

Otherwise, all sets S_j, where $i \neq j$, have more than $g_1(d-1)$ elements. Each contains a subset S'_j of size $g_1(d-1)$, so by the induction hypothesis, $|N_j(S'_j)| - |S'_j| \geq f_1(d-1)$. Hence applying Lemma 3.5, for each S_j with $j \neq i$, $|N_j(S_j)| = |Y_j| = |X_i|$. If there exists a $j \neq i$ such that $|S_j| \leq |X_i| - 2$, then S_j contributes an additional two local neighbors. If there exists two sets S_{j_1} and S_{j_2}, where

$j_1 \neq j_2 \neq i$, such that each has size at most $|X_i| - 1$, then each will contribute at least one additional local neighbor. Otherwise, for each $j \neq i$, $S_j = X_j$, except for possibly one set, S_ℓ, which contains all but one vertex from X_ℓ. But in this case, since $|S_i| \geq 1$,

$$|S| \geq 1 + (2m-1)\frac{1}{2}(2m)^{d-1} - 1 > \frac{1}{4}(2m)^d,$$

which contradicts our upper bound on S.

□

We now prove the Main Lemma which will serve as the foundation for the proof of the Main Theorem.

Proof of Lemma 3.1. Let S be an arbitrary subset of X such that $k \leq |S| \leq g_k(d)$. If $|S| \leq d$, then the result is immediate from Corollary 3.3. Therefore, together with Remark 3.7, we can assume $d < |S| \leq (2m)^d/4$.

Divide S into sets $S_0, \dots, S_{2m-1}$ as discussed in Remark 2.1, and recall we subdivide $\mathbb{Z}_{2m}$ into two sets I_0 and I_1 such that $S_i = \varnothing$ for all $i \in I_0$, and $|S_i| \geq 1$ for all $i \in I_1$. We proceed by induction on d. Lemmas 3.8 and 3.9 establish the basis: $d \geq 1$ and $k = 1$ and $d = 2$ and $k = 1, 2$. Therefore suppose that d is an arbitrary integer such that $d \geq 3$ and $2 \leq k \leq d$.

Note that when we apply the inductive hypothesis to the sets S_i within each G_d^i, we will also need to reduce k so that it does not exceed the dimension. A subset $S_i \subset S$ satisfies the induction hypothesis if $k-1 \leq |S_i| \leq g_{k-1}(d-1)$. We consider four cases, based on *how many* nonempty subsets S_i satisfy the induction hypothesis in the corresponding subgraphs $G_d^i \cong G_{d-1}$. In Cases 2 through 4, we say a subset S_i is *too small* if $|S_i| < k-1$ or *too big* if $|S_i| > g_{k-1}(d-1)$.

Case 1: *There exist at least two subsets that satisfy the induction hypothesis.*

Considering local neighbors only in these two layers, we obtain

$$|N(S)| - |S| \geq \sum_{i \in I_1} (|N_i(S_i)| - |S_i|) \geq 2f_{k-1}(d-1).$$

Note that

$$2f_{k-1}(d-1) - f_k(d) = 2d(k-2) - \frac{1}{2}k(k+7) + 7,$$

and since $d \geq k$, the expression above is at least $\frac{1}{2}(3k^2 - 15k + 14)$ which is positive when $k \geq 4$. Furthermore, if $d > k$, then $2d(k-2) - \frac{1}{2}k(k+7) + 7 \geq \frac{1}{2}(3k-2)(k-3)$ which is non-negative for all $k \geq 3$. Hence $2f_{k-1}(d-1) \geq f_k(d)$ when $k \geq 4$ or $d > k$.

When $d = k = 3$ or $d = 3$ and $k = 2$, we follow the argument from Case 2 of Lemma 3.9. From the above calculation, the local neighbors of the two sets S_i and S_j satisfying the inductive hypothesis collectively provide at least $2f_{k-1}(d-1)$ extra neighbors. In both cases, $2f_{k-1}(d-1) = f_k(d)-2$, and we seek two more neighbors. If there is a third set satisfying the inductive hypothesis, we obtain two more extra local neighbors. If $k = 3$ and there is a set with $|S_a| = 1$, we also obtain two extra local neighbors by applying Corollary 3.3. If S_a is empty for some a, we obtain two extra neighbors from a shadow. Hence we may assume there are two sets satisfying the inductive hypothesis and for all others, $|S_a| > g_k(d)$. As long as there is some set satisfying $|S_a| \leq |X_a| - 2$ or two sets S_{a_1}, S_{a_2} with $|S_{a_i}| \leq |X_{a_i}| - 1$, applying the reasoning in Case 2 of Lemma 3.9 yields two extra local neighbors. Otherwise, there are $2m-3$ sets of cardinality $\frac{1}{2}(2m)^{d-1}$ and one of cardinality at least $\frac{1}{2}(2m)^{d-1}-1$. Therefore, since $m \geq 3$ and $k \geq 2$,

$$\begin{aligned}|S| &\geq |S_i| + |S_j| + (2m-2)\frac{1}{2}(2m)^{d-1} - 1\\ &\geq 2(k-1) + \frac{1}{2}(2m)^d - (2m)^{d-1} - 1 > \frac{1}{4}(2m)^d,\end{aligned}$$

contradicting our assumption that $|S| \leq (2m)^d/4$.

Case 2: *No subset satisfies the induction hypothesis because all are too small.*

Here, $1 \leq |S_i| \leq k-2$ for all $i \in I_1$, so $k \geq 3$. Recall $|S_i| = s_i$ and $|S| = s$. By applying Lemma 3.2 to count local neighbors for each S_i we obtain

$$\begin{aligned}|N(S)| - |S| &\geq \sum_{i\in I_1} (|N_i(S_i)| - |S_i|) \geq \sum_{i\in I_1}\left(2(d-1)s_i - \frac{1}{2}s_i^2 - \frac{3}{2}s_i + 1\right)\\ &= 2(d-1)s - \frac{3}{2}s + t - \frac{1}{2}\sum_{i\in I_1} s_i^2.\end{aligned}$$

This quantity is minimized when $t - \frac{1}{2}\sum_{i\in I_1} s_i^2$ is minimized. This occurs when $t = \lceil s/(k-2)\rceil$, and there are $\lfloor s/(k-2)\rfloor$ sets of size $k-2$ and one of size ℓ, where $\ell = s \mod (k-2)$. (Otherwise, we can assume $1 \leq s_1 \leq s_2 < k-2$, and replacing s_1 with s_1-1 and s_2 with s_2+1 yields $(s_1-1)^2+(s_2+1)^2 > s_1^2+s_2^2$ without increasing t.)

Thus, setting $\epsilon = 0$ when $\ell = 0$ and $\epsilon = 1$ otherwise, $t = \frac{s-\ell}{k-2} + \epsilon$, and

$$\begin{aligned}|N(S)| - |S| &\geq 2(d-1)s - \frac{3}{2}s + \frac{s-\ell}{k-2} + \epsilon - \frac{1}{2}\left(\frac{s-\ell}{k-2}(k-2)^2 + \ell^2\right)\\ &= \left(2d - \frac{5}{2} - \frac{k}{2} + \frac{1}{k-2}\right)s + \left(\frac{k-2}{2} - \frac{1}{k-2}\right)\ell - \frac{1}{2}\ell^2 + \epsilon.\end{aligned}$$

This expression is quadratic in ℓ, and it is maximized when $\ell = \left(\frac{k-2}{2} - \frac{1}{k-2}\right)$ and minimized at the boundary: $\ell = 0$ and $\ell = k-3$. Hence is suffices to establish the result when $\ell \in \{0, k-3\}$.

When $\ell = \epsilon = 0$, we have

$$|N(S)| - |S| \geq \theta(k, d, s),$$

where $\theta(k, d, s) = \left(2d - \frac{5}{2} - \frac{k}{2} + \frac{1}{k-2}\right) s$. Since $d \geq k \geq 3$, the coefficient on s in θ is positive. Hence for fixed k and d, $\theta(k, d, s)$ is an increasing function of s. Since $s > d$, we have $s \geq k+1$, and thus $\theta(k, d, s) - f_k(d) \geq 0$ for all $3 \leq k \leq d$ and $d \geq 3$.

When $\ell = k - 3$ (and thus $\epsilon = 1$),

$$|N(S)| - |S| \geq \theta(k, d, s) + \left(\frac{k-2}{2} - \frac{1}{k-2}\right)(k-3) - \frac{1}{2}(k-3)^2 + 1.$$

In an almost identical calculation to the one used in the case $\ell = 0$, by the hypotheses on k, d, and s, the right side of the inequality above is at least $f_k(d)$ for all $3 \leq k \leq d$ and $d \geq 3$.

Case 3: *Exactly one subset satisfies the induction hypothesis and all other subsets are too small.*

Suppose for some S_i, $k - 1 \leq |S_i| \leq g_{k-1}(d-1)$, and $|S_j| < k - 1$ for all $j \in I_1 \setminus \{i\}$. Applying the induction hypothesis to S_i yields

$$|N_i(S_i)| - |S_i| \geq f_{k-1}(d-1).$$

Note $f_{k-1}(d-1) - f_k(d) = -2d - k + 3$, hence it suffices to show that

$$\sum_{j \in I_1 \setminus \{i\}} (|N_j(S_j)| - |S_j|) \geq 2d + k - 3.$$

Since each S_j is nonempty, applying Corollary 3.3 with $k = 1$ for each $j \neq i$,

$$|N_j(S_j)| - |S_j| \geq f_1(d-1) = 2d - 3.$$

Hence if $t \geq 3$, then

$$\sum_{j \in I_1 \setminus \{i\}} (|N_j(S_j)| - |S_j|) \geq 2(2d-3) = 4d - 6.$$

Since $d \geq 3$ and $k \leq d$, we have $4d - 6 \geq 2d + k - 3$.

If $t = 2$, suppose S_j ($j \neq i$) is the other nonempty subset of S. Note $m \geq 3$ implies there are at least four layers containing no elements of S. We obtain at least $2d - 3$ additional neighbors from the local neighborhood of S_j, and at least $|S_i| + |S_j| \geq (k-1) + 1 = k$ additional neighbors from the shadows of S_i and S_j in layers that contain no elements of S. Hence again we obtain the needed $2d + k - 3$ additional neighbors.

Case 4: *At most one subset satisfies the induction hypothesis and at least one subset is too big.*

From the proof of Lemma 3.5, any subset S_i which satisfies $|S_i| > g_{k-1}(d-1)$ has the property that

$$|N(S_i \cup S_\star) \cap (V(G_d^i) \cup V(G_d^\star))| - |S_i \cup S_\star| \geq \frac{1}{2}(2m)^{d-1} - (k-2) - |S_\star|,$$

where $S_\star \in \{S_{i-1}, S_{i+1}\}$. Out of all such sets S_i, let S_j be one such that $|S_\star|$ is minimum (where $S_\star \in \{S_{j-1}, S_{j+1}\}$). Define

$$\phi(m,k,d) = \frac{1}{2}(2m)^{d-1} - (k-2) - f_k(d) = \frac{1}{2}(2m)^{d-1} - 2dk + \frac{1}{2}(k^2+k) + 1.$$

If $|S_\star| \leq \phi(m,k,d)$, then $\frac{1}{2}(2m)^{d-1} - (k-2) - |S_\star| \geq f_k(d)$, and we are done. Otherwise, we may assume $|S_\star| > \phi(m,k,d)$. We claim that $|S_\star| \geq k-1$. Otherwise, if $|S_\star| < k-1$, then since $k \leq d$, $m \geq 3$ and $d \geq 3$,

$$|S_\star| - \phi(m,k,d) < k - 1 - \phi(m,k,d) < 0,$$

a contradiction.

Hence, we may assume that $|S_\star| \geq k-1$ and also that $|S_\star| > \phi(m,k,d)$. By the minimality of $|S_\star|$ and our assumption that at most one set satisfies the induction hypothesis, every S_i other than $S_\star$ has $|S_i| > g_{k-1}(d-1)$. We show that this situation leads to a contradiction regarding $|S|$. Observe

$$\begin{aligned} |S| &> (2m-1) \cdot g_{k-1}(d-1) + \phi(m,k,d) \\ &= \frac{1}{2}(2m)^d + m(-4d(k-1) + k^2 + 3k - 4) - 2d + 3 - k. \end{aligned}$$

Since $k^2 + 3k - 4 \geq 0$ and $k \leq d$,

$$\begin{aligned} |S| &> \frac{1}{2}(2m)^d - m(4d(d-1)) - 3d + 3 \\ &= \frac{1}{4}(2m)^d + \frac{1}{4}(2m)^d - m(d-1)\left(4d + \frac{3}{m}\right). \end{aligned}$$

Since $m \geq 3$,

$$\frac{1}{4}(2m)^d - m(d-1)\left(4d + \frac{3}{m}\right) \geq m\left(\frac{1}{2}(2m)^{d-1} - (d-1)(4d+1)\right) > 0$$

when $d \geq 4$. When $d = 3$, we must have $k = 2$ or $k = 3$, and it is straightforward to verify that $\frac{1}{4}(2m)^d + m(-4d(k-1) + k^2 + 3k - 4) - 2d + 3 - k > 0$ when $m \geq 3$. In both cases, this contradicts $|S| \leq (2m)^d/4$.

Having established $|N(S)| - |S| \geq f_k(d)$ in all cases, this completes the proof. □

4 Extendability of k-suitable matchings in G_d

We are now ready to prove our Main Theorem. Recall that X and Y are the partite sets of G_d, and $|X| = |Y| = (2m)^{d-1}$.

Proof of Theorem 1.2. When $m = 2$, the result is true by Theorem 1.1. Hence, we may assume $m \geq 3$. Suppose M is a k-suitable matching in G_d with at most $f_k(d)$ edges, $1 \leq k \leq d$, and let U_M be the endpoints of the edges in M. By Hall's Theorem, it suffices to show that $G_d - U_M$ satisfies Hall's Condition.

Let S be an arbitrary subset of a single partite set in $G_d - U_M$. Assume by symmetry that $S \subseteq X$. Let $N(S)$ be the neighborhood of S in G_d, and let $N_U(S)$ be the neighborhood of S in $G_d - U_M$. If $|S| < k$, then by the definition of k-suitable, $|N_U(S)| \geq |S|$. If $k \leq |S| \leq g_k(d)$, then by the Main Lemma, $|N(S)| - |S| \geq f_k(d)$, hence $|N(S)| \geq |S| + f_k(d)$. Since S has at most $f_k(d)$ neighbors in U_M, this implies $|N_U(S)| \geq |S|$.

Finally, consider $|S| > g_k(d)$. Take a subset S' with $|S'| = g_k(d)$. By the Main Lemma, $|N(S')| - |S'| \geq f_k(d)$, hence

$$|N(S)| \geq |N(S')| \geq |S'| + f_k(d) = g_k(d) + f_k(d) = \frac{(2m)^d}{2} - (k-1).$$

Hence $|Y - N(S)| < k$, and setting $T = Y - N(S) - U_M$ and applying Proposition 3.6, we see that $|N_U(S)| - |S| \geq |N_U(T)| - |T|$. Since M is k-suitable and $|T| < k$, $|N_U(T)| - |T| \geq 0$, and it follows that $|N_U(S)| - |S| \geq 0$ as needed. □

The theorem below shows that a slight modification of the construction used to prove sharpness in the hypercube result extends to this family of graphs. Note that both the construction and its verification follow the hypercube result closely.

Theorem 4.1. *For all integers k and d such that $1 \leq k \leq d$, there exists a k-suitable matching of size $f_k(d) + 1$ that does not extend to a perfect matching in $G(2m)_d$, unless $d = 1$ and $m = 2$.*

Proof. Recall any matching is trivially 1-suitable. If $d = 1$ and $m = 2$, then $G_d = G(4)_1 = C_4$, $k = 1$, and any matching of size $f_1(1) + 1 = 2$ is already a perfect matching, establishing the exceptional case. Furthermore, for any $m \geq 3$, any matching of size 2 that covers the neighborhood of a vertex does not extend to a perfect matching in $G(2m)_1 = C_{2m}$.

For the remainder of the proof, suppose $d \geq 2$. Moreover, the case $m = 2$ is established by Theorem 1.1, so we shall assume $m \geq 3$. When $k = 1$, any matching that has size $2d$ and covers the neighborhood of a fixed vertex cannot extend to a perfect matching. Since $f_1(d) + 1 = 2d$, the result is true. Therefore, we may suppose that $k \geq 2$. Let $S = \{e_1, \ldots, e_k\}$ and define the matching:

$$M = \{\{v, v - e_1\} : v \in N(S) \setminus \{e_1 + e_i : 1 \leq i \leq k\}\} \cup \{\{2e_1, 2e_1 - e_2\}\}.$$

By definition, each edge in M has one endpoint in $N(S)$ and the other endpoint in $V(G_d) \setminus (N(S) \cup S)$. Additionally, $|M| = |N(S)| - (k-1)$, and by Proposition 3.4, $|N(S)| = f_k(d) + k$, so $|M| = f_k(d) + 1$. Finally, M leaves only $k-1$ vertices in $N(S)$ uncovered, those of the form $e_1 + e_i$ where $2 \le i \le k$. Since the k vertices in S will have only $k-1$ uncovered neighbors, M will not extend to a perfect matching.

It remains to verify that M is k-suitable. Let U_M be the endpoints of the edges in M, and let T be an arbitrary subset of $V(G_d) \setminus U_M$ containing at most $k-1$ vertices. We must verify $|N(T)| \ge |T|$ in $G_d - U_M$. Note first that we may assume T is contained entirely within a single partite set, $X \setminus U_M$ or $Y \setminus U_M$, in $G_d - U_M$, since otherwise we can verify the result separately for the vertices in each partite set. We define an injection $\phi : T \to V(G_d) \setminus U_M$ such that $\phi(v) \in N(v) \setminus U_M$, establishing $|N(T)| \ge |T|$ in $G_d - U_M$. Without loss of generality, suppose $S \subset X \setminus U_M$. Consider two cases:

Case 1: $T \subset X \setminus U_M$. As $|T| < |S|$, at least one element of S is not in T. If $e_1 \notin T$, then define $\phi(v) = v + e_1$ for all $v \in T$. If $e_1 \in T$, then for some $i \in \{2, \ldots, k\}$, $e_i \notin T$. In this case define $\phi(v) = v + e_1$ for all $v \in T \setminus \{e_1\}$ and $\phi(e_1) = e_1 + e_i$. ϕ is clearly an injection into $N(T)$. We must verify $\phi(v) \notin U_M$. If $\phi(v) \in U_M$ and $v \ne e_1$, then $v + e_1 = u$ for some $u \in N(S) \setminus \{e_1 + e_i : 2 \le i \le k\} \cup \{2e_1\}$. But then either $v = u - e_1 \in U_M$, or $u = 2e_1$ meaning $v = e_1$, both contradictions. Lastly, by definition of M, $e_i \in S$ implies $\phi(e_1) = e_1 + e_i \notin U_M$.

Case 2: $T \subset Y \setminus U_M$. If $T = \{3e_1 - e_2\} \cup \{4e_1 - e_2 - e_i : 3 \le i \le k\}$, then $|T| = k-1$ and define $\phi(v) = v + e_2$ for all $v \in T$. Then $\phi(T) = \{3e_1\} \cup \{4e_1 - e_i : 3 \le i \le k\}$. Note $\phi(T) \cap U_M = \varnothing$ when $m \ge 3$.

Otherwise, since $|T| \le k-1$, there exists $u \in \{3e_1 - e_2\} \cup \{4e_1 - e_2 - e_i : 3 \le i \le k\}$ such that $u \notin T$. If $3e_1 - e_2 \notin T$ then define $\phi(v) = v - e_1$ for all $v \in T$. Otherwise, $u = 4e_1 - e_2 - e_i$ for some $i \in \{3, \ldots, k\}$, and define $\phi(v) = v - e_1$ for all $v \in T \setminus \{3e_1 - e_2\}$ and define $\phi(3e_1 - e_2) = 3e_1 - e_2 - e_i$. Again $\phi(T) \cap U_M = \varnothing$.

In all cases, ϕ defines an injection. Hence $G_d - U_M$ has no deficient sets of size less than k, thus M is k-suitable. □

Having completed the proof, we return to a discussion of the hypothesis in the Main Lemma and the Main Theorem that $k \le d$. This contrasts with the analog in the hypercube result from Theorem 1.1 in which $k \le 2d - 3$ is sufficient. We should note that all of our arguments throughout the paper easily extend if we replace the hypothesis with $k \le 2d - 3$, with the exception of Case 2 in the proof of the Main Lemma. While we believe the result holds for $k \le 2d - 3$ (or at least $k \le 2d - c$ for some small constant c), we did not feel the slightly stronger result merited the additional case analysis required to establish that case.

Finally, as mentioned in the introduction, although it is true that in Q_d (and therefore in $G(4)_{d/2}$), every induced matching extends to a perfect matching, this is not true for $G(2m)_d$ when $m > 2$. Observe that the matching

$$M = \{\{e_i, 2e_i\} : 1 \leq i \leq d\} \cup \{\{-e_i, -2e_i\} : 1 \leq i \leq d\}$$

is an induced matching. However, all neighbors of the vertex 0^d are covered by the matching, so M does not extend to a perfect matching.

5 Open questions

This paper is an example of a natural extension of the study of k-suitable matchings in hypercubes. There are many other bipartite graphs for which this idea could be studied. Those with a high degree of symmetry are natural candidates. Accordingly, we close with a few questions about the extremal function $f_k(G)$.

1. Extendability of Cayley graphs has been studied, though not in the explicit context of k-suitability. Both hypercubes and the family studied in this paper are Cayley graphs on direct products of cyclic groups. The usage of suitability could potentially be useful for exploring extendability in other bipartite Cayley graph families. For example, many connected 4-regular Cayley graphs are isomorphic to a *pseudo-Cartesian product* of two cycles (see [15]), a generalization of the Cartesian graph product. Can we determine $f_k(G)$ where G is a 4-regular bipartite Cayley graph? Note this would also tie into the study of quadrangulations of the torus (see [4]).

2. If G is a Cayley graph, $G = \mathrm{Cay}(\mathcal{G}; S)$ and H is a quotient graph of G, $H = \mathrm{Cay}(\mathcal{G}/\mathcal{H}; \overline{S})$, how might $f_k(G)$ and $f_k(H)$ be related?

3. More generally, we propose the following for study. Suppose G is a bipartite graph. What lower bound on $f_k(G \,\square\, K_2)$ or $f_k(G \,\square\, G)$ can be determined if we know $f_k(G)$?

4. Extendability of graphs on surfaces has been well-studied, but to our knowledge, the idea of extending k-suitable matchings in these graphs has not been investigated. What, for example, can be said about $f_k(G)$ if G is a bipartite planar graph?

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Code availability

A Mathematica notebook called `NeighborhoodCheckerC6XC6.nb` is available here in notebook format and here as a PDF. This was used in the proof of Lemma 3.8.

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