AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **82(3)** (2022), Pages 353–365

Conditions on the regularity of balanced c-partite tournaments for the existence of strong subtournaments with high minimum degree*

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Abstract

We consider the following problem posed by Volkmann in 2007: How close to regular must a c-partite tournament be, to secure a strongly connected subtournament of order c? We give sufficient conditions on the regularity of balanced c-partite tournaments to ensure the existence of a strong maximal subtournament with minimum degree at least $\lfloor \frac{c-2}{4} \rfloor + 1$. We obtain this result as an application of counting the number of subtournaments of order c for which a vertex has minimum out-degree (respectively, in-degree) at most $q \geq 0$.

* Supported by PAPIIT-México under project IN108121; CONACYT under projects A1-S-12891 and 47510664.
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ISSN: 2202-3518

1 Introduction

Let c be a non-negative integer. A *c-partite or multipartite tournament* is a digraph obtained from a complete c-partite graph by orienting each edge. In 1999, Volkmann [3] developed the first contributions in the study of the structure of strongly connected subtournaments in multipartite tournaments. He proved that every almost regular c-partite tournament contains a strongly connected subtournament of order p for each $p \in \{3, 4, \ldots, c-1\}$. In the same paper he also proved that, if each partite set of an almost regular c-partite tournament has at least $\frac{3c}{2} - 6$ vertices, then there exists a strong subtournament of order c. In 2008 Volkmann and Winzen [5] proved that every almost regular c-partite tournament has a strongly connected subtournament of order c for $c \geq 5$. In 2011, Xu et al. [6] proved that every vertex of a regular c-partite tournament with $c \geq 16$ is contained in a strong subtournament of order p for every $p \in \{3, 4, \ldots, c\}$. The following problem was posed by Volkmann [4]:

Determine further sufficient conditions for (strongly connected) c-partite tournaments to contain a strong subtournament of order p, for some $4 \leq p \leq c$. How close to regular must a c-partite tournament be, to secure a strongly connected subtournament of order c?

In this direction, in [2] in 2016 we proved that for every (not necessarily strongly connected) balanced c-partite tournament T of order $n \geq 6$, if the global irregularity of T is at most $\dfrac{c}{\sqrt{3c+26}}$, then T contains a strongly connected tournament of order c. A c-partite tournament is balanced if all partite sets contain the same number of vertices.

We follow all the definitions and notation of [1]. Let G be a c-partite tournament of order n with partite sets $\{V_i\}_{i=1}^{c}$. We denote by $G_{r,c}$ a balanced c-partite tournament satisfying $|V_i| = r$ for every $i \in [c]$, where $[c] = \{1, \ldots, c\}$. Throughout this paper $|V_i| = r$ for each $i \in [c]$.

Let G be a c-partite tournament. For $x \in V(G)$ and $i \in [c]$, the *out-neighborhood of x in V_i* is $N_i^+(x) = V_i \cap N^+(x)$; the *in-neighborhood of x in V_i* is $N_i^-(x) = V_i \cap N^-(x)$; $d_i^+(x) = |N_i^+(x)|$; $d_i^-(x) = |N_i^-(x)|$ and $\delta(G) = \min_{x \in V(G)} \{d^-(x), d^+(x)\}$.

For an oriented graph D, the *global irregularity of D* is defined as

$$i_g(D) = \max_{x,y \in V(D)} \left(\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} \right).$$

If $i_g(D) = 0$ ($i_g(D) \leq 1$, respectively) D is regular (almost regular, respectively). For our study we introduce another irregularity parameter, namely the *local partite irregularity of D*, which is defined as

$$\mu(D) = \max_{x \in V(D)} \max_{i \in [c]} |d_i^+(x) - d_i^-(x)|.$$

Observe that, for a balanced c-partite tournament $G_{r,c}$, we have $\mu(G_{r,c}) \geq \frac{i_g(G_{r,c})}{c-1}$.

In this paper we consider Volkmann's problem for balanced c-partite tournaments. We give sufficient conditions on its regularity to ensure the existence of a strong subtournament with minimum degree at least $\left\lfloor \frac{c-2}{4} \right\rfloor + 1$. We obtain this result as an application of counting the number of subtournaments of order c for which a vertex has minimum out-degree (respectively, in-degree) at most $q \geq 0$.

Our main result is the following.

Theorem *Let $G_{r,c}$ be a balanced c-partite tournament, with $r \geq 2$, such that $\delta(G_{r,c}) \geq \lfloor \frac{c-2}{4} \rfloor (r + \mu(G_{r,c})) \frac{c-1}{c-2}$. Then $G_{r,c}$ contains a strongly connected tournament T of order c such that $\delta(T) \geq \lfloor \frac{c-2}{4} \rfloor + 1$, whenever*

(i) $i_g(G_{r,c}) \leq \frac{r}{2}$ and $c \geq 13$, $c \notin \{14, 15, 18\}$, or

(ii) $i_g(G_{r,c}) \leq r$ and $c \geq 17$, $c \notin \{18, 19, 22\}$, or

(iii) $i_g(G_{r,c}) \leq \frac{3r}{2}$ and $c \geq 21$, $c \notin \{22, 23, 26\}$.

2 Maximal tournaments for which a vertex has minimum degree at most q

The aim of this section is to give sufficient conditions on the minimum degree, local partite irregularity and global irregularity to obtain a bound on the number of maximal tournaments in a balanced c-partite tournament $G_{r,c}$ in which a given vertex $x \in V(G_{r,c})$ has out-degree (in-degree respectively) at most q, for some given $q \geq 0$.

Let $x \in V(G_{r,c})$. We may asume that $x \in V_c$. A maximal tournament of $G_{r,c}$ containing the vertex x can be constructed by choosing a vertex from each partite set V_i for $i \in [c-1]$. We assign a vector to each maximal tournament T containing the vertex x as follows: $\mathbf{h} = (h_1, h_2, \ldots, h_{c-1}) \in \{0,1\}^{c-1}$ such that $h_i = 1$, if and only if the vertex of V_i is an out-neighbor of x, see Figure 1. Clearly, different tournaments can have the same vector and for a given maximal tournament T, $\sum h_i = d^+_T(x)$.

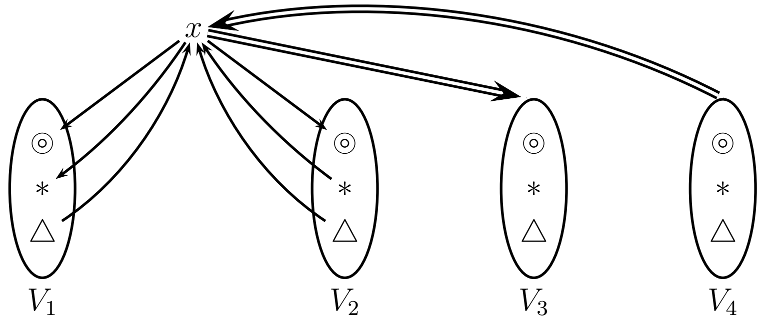

Figure 1: For $x \in V_5$. The vectors of the maximal subtournaments containing the vertex x induced by vertices $\circledcirc$, $*$ and $\triangle$ respectively are $h_\circledcirc = (1,1,1,0)$, $h_* = (1,0,1,0)$ and $h_\triangle = (0,0,1,0)$.

For each $0 \leq k \leq c-1$, let $\mathcal{H}^+_k(x)$ be the set of such vectors satisfying $\sum_{i=1}^{c-1} h_i = k$. Observe that if, for some $0 \leq i \leq c-1$, we have that $d^+_i(x) = r$, then $h_i = 1$ for every k and every $\mathbf{h} \in \mathcal{H}_k^+(x)$, analogously if $d_i^+(x) = 0$, then $h_i = 0$ for every k and every $\mathbf{h} \in \mathcal{H}_k^+(x)$.

The number of maximal tournaments for a fixed $\mathbf{h} = (h_1, h_2, \dots, h_{c-1})$ is i

$$\prod_{i=1}^{c-1} d_i^+(x)^{h_i} d_i^-(x)^{1-h_i}.$$

Thus we have the following remark.

Remark 1. *Let $G_{r,c}$ be a balanced c-partite tournament and let $x \in V_c$. The number of maximal tournaments of $G_{r,c}$ for which x has out-degree k is equal to*

$$\sum_{h \in \mathcal{H}_k^+(x)} \prod_{i=1}^{c-1} d_i^+(x)^{h_i} d_i^-(x)^{1-h_i}.$$

Let $x \in V(G_{r,c})$. For each $q \geq 0$, let $T_q^+(x)$ (respectively $T_q^-(x)$) be the number of maximal tournaments of $G_{r,c}$ for which x has out-degree (respectively in-degree) at most q. All the following results regarding $T_q^+(x)$ can be obtained for $T_q^-(x)$ in an analogous way.

Assume, without loss of generality, that $x \in V_c$. By Remark 1,

$$T_q^+(x) = \sum_{k=0}^{q} \sum_{h \in \mathcal{H}_k^+(x)} \prod_{i=1}^{c-1} d_i^+(x)^{h_i} d_i^-(x)^{1-h_i}.$$

In order to bound $T_q^+(x)$, for any integer $r \geq 2$, and $g_1, g_2, \dots, g_s$ real numbers such that $0 \leq g_i \leq r$, we define

$$M(g_1, \dots, g_s; k) = \sum_{h \in \mathcal{H}_k^s} \prod_{i=1}^{s} g_i^{h_i} (r - g_i)^{1-h_i},$$

where $\mathcal{H}_k^s$ is the set of s-vectors $(h_1, h_2, \dots, h_s) \in \{0, 1\}^s$ such that:

i) if $g_i = r$ then $h_i = 1$;

ii) if $g_i = 0$ then $h_i = 0$;

iii) $\sum_{i=1}^{s} h_i = k$.

Observe that if $s = c - 1$ and x is a vertex in a balanced c-partite tournament $G_{r,c}$ such that $d_i^+(x) = g_i$ for every $i \in [c-1]$, then $\mathcal{H}_k^+(x) = \mathcal{H}_k^s$.

Lemma 2.1. *Let $r \geq 2$ be an integer, and let $g_1, \dots, g_s$ be real numbers such that $0 \leq g_i \leq r$. Let $\Gamma = \max_{i \in [s]} g_i$ and $\gamma = \min_{i \in [s]} g_i$. If, for some integer $q \geq 1$, we have that $\sum_{i \in [s]} g_i \geq q(r + \Gamma - \gamma) - \Gamma$, then*

$$M(g_1, \dots, g_s; q) \geq M(g_1, \dots, g_s; q - 1).$$

Proof. Without loss of generality we may assume that there are integers t and p_r such that

i) $0 < g_i < r$ if and only if $i \in [t]$;

ii) $g_i = r$ if and only if $t+1 \leq i \leq t+p_r$;

iii) $g_i = 0$ if and only if $t+p_r+1 \leq i \leq s$.

Observe that for every $\mathbf{h} = (h_1, \dots, h_s) \in \mathcal{H}^s_{q-1}$ and for every $\mathbf{h'} = (h'_1, \dots, h'_s) \in \mathcal{H}^s_q$ we have that $h_i = h'_i = 1$ for $t+1 \leq i \leq t+p_r$ and $h_i = h'_i = 0$ for $t+p_r+1 \leq i \leq s$. Notice that, if $p_r \geq q$, then $\mathcal{H}^s_{q-1} = \emptyset$, which implies that $M(g_1, \dots, g_s; q-1) = 0$ and the lemma follows. Thus, we may assume that $q \geq p_r + 1$.

For each $\mathbf{h} = (h_1, \dots, h_s) \in \mathcal{H}^s_{q-1}$, let $F(\mathbf{h}) = \{(h'_1, \dots, h'_s) \in \mathcal{H}^s_q : h_i \leq h'_i \text{ for all } i \in [s]\}$ and let $a(\mathbf{h}) = \{j : h_j = 1 \text{ for } j \in [t]\}$. Observe that for every $\mathbf{h} \in \mathcal{H}^s_{q-1}$, $|a(\mathbf{h})| = q - 1 - p_r$.

By the definitions of $\mathcal{H}^s_q$ and $\mathcal{H}^s_{q-1}$, it follows that, given $\mathbf{h} \in \mathcal{H}^s_{q-1}$ and $\mathbf{h'} \in F(\mathbf{h}) \subseteq \mathcal{H}^s_q$, (there is a unique index $j_0 \in [t] \setminus a(\mathbf{h})$ such that $h'_{j_0} = h_{j_0} + 1$) and $h_i = h'_i$ for every $i \in [s] \setminus \{j_0\}$.

Thus,

$$\frac{\sum\limits_{\mathbf{h'} \in \mathbf{F}(\mathbf{h})} \prod\limits_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i}}{\prod\limits_{i=1}^{s} g_i^{h_i}(r-g_i)^{1-h_i}} = \sum_{\mathbf{h'} \in \mathbf{F}(\mathbf{h})} \frac{\prod\limits_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i}}{\prod\limits_{i=1}^{s} g_i^{h_i}(r-g_i)^{1-h_i}} = \sum_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r-g_j}. \tag{1}$$

Claim 1. $\sum\limits_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r-g_j} \geq q - p_r$.

Suppose that $\sum\limits_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r-g_j} < q - p_r$. Let $\gamma_t = \min\limits_{i \in [t]} g_i$. Thus, $\sum\limits_{j \in [t] \setminus a(\mathbf{h})} \frac{g_j}{r-\gamma_t} < q - p_r$ and therefore $\sum\limits_{j \in [t] \setminus a(\mathbf{h})} g_j < (r-\gamma_t)(q-p_r)$. On the other hand,

$$\sum_{j \in [s]} g_j = \sum_{j \in [t]} g_i + rp_r = \sum_{j \in [t] \setminus a(\mathbf{h})} g_i + \sum_{j \in a(\mathbf{h})} g_i + rp_r.$$

Hence, $\sum\limits_{j \in [t] \setminus a(\mathbf{h})} g_i = \sum\limits_{j \in [s]} g_j - \sum\limits_{j \in a(\mathbf{h})} g_i - rp_r$ which implies that

$$(r-\gamma_t)(q-p_r) > \sum_{j \in [s]} g_j - \sum_{j \in a(\mathbf{h})} g_i - rp_r$$

and therefore, after some easy calculations, we see that

$$rq + \sum_{j \in a(\mathbf{h})} g_i - \gamma_t(q-p_r) > \sum_{j \in [s]} g_j.$$

Let $\Gamma_t = \max_{i\in[t]} g_i$. Since $|a(\mathbf{h})| = q-1-p_r$, it follows that

$$rq + \Gamma_t(q-1-p_r) - \gamma_t(q-p_r) > \sum_{j\in[s]} g_j.$$

Since $\Gamma \geq \Gamma_t \geq \gamma_t \geq \gamma$ and $p_r \geq 0$, we see that

$$\begin{array}{rcl}\Gamma_t(q-1-p_r) - \gamma_t(q-p_r) & \leq & \Gamma(q-1-p_r) - \gamma(q-p_r) \\ & = & \Gamma(q-1) - \gamma q - p_r(\Gamma-\gamma) \\ & \leq & \Gamma(q-1) - \gamma q.\end{array}$$

Thus

$$\sum_{j\in[s]} g_j < q(r+\Gamma-\gamma) - \Gamma = rq + \Gamma(q-1) - \gamma q,$$

which, by hypothesis, is not possible, and the claim follows.

From Claim 1 and (1) it follows that for each $\mathbf{h} = (h_1, \ldots, h_s) \in \mathcal{H}^s_{q-1}$,

$$\sum_{\mathbf{h'}\in\mathbf{F(h)}} \prod_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i} \geq (q-p_r)\prod_{i=1}^{s} g_i^{h_i}(r-g_i)^{1-h_i}. \qquad (2)$$

Observe that, for every $\mathbf{h'} \in \mathcal{H}^{\mathbf{s}}_{\mathbf{q}}$, we have $|\{j : h'_j = 1 \text{ with } j \in [t]\}| = q - p_r$. Therefore, for every $\mathbf{h'} \in \mathcal{H}^{\mathbf{s}}_{\mathbf{q}}$, there are exactly $q-p_r$ elements $\mathbf{h} \in \mathcal{H}^s_{q-1}$ such that $\mathbf{h'} \in \mathbf{F}(\mathbf{h})$. Thus,

$$\sum_{\mathbf{h}\in H^s_{q-1}} \sum_{\mathbf{h'}\in\mathbf{F(h)}} \prod_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i} = (q-p_r)\sum_{\mathbf{h'}\in\mathbf{H^s_q}} \prod_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i}.$$

On the other hand, by (2) we see that

$$\sum_{\mathbf{h}\in H^s_{q-1}} \sum_{\mathbf{h'}\in\mathbf{F(h)}} \prod_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i} \geq \sum_{\mathbf{h}\in H^s_{q-1}} (q-p_r)\prod_{i=1}^{s} g_i^{h_i}(r-g_i)^{1-h_i}$$

implying that

$$\sum_{\mathbf{h'}\in\mathbf{H^s_q}} \prod_{i=1}^{s} g_i^{h'_i}(r-g_i)^{1-h'_i} \geq \sum_{\mathbf{h}\in H^s_{q-1}} \prod_{i=1}^{s} g_i^{h_i}(r-g_i)^{1-h_i}$$

which, by definition, is equivalent to $M(g_1, \ldots, g_s; q) \geq M(g_1, \ldots, g_s; q-1)$, and the lemma follows. □

Corollary 2.2. *Let $r \geq 2$, $c \geq 3$ and $G_{r,c}$ be a balanced c-partite tournament such that for some $q \geq 1$, $\delta(G_{r,c}) \geq q\,(r + \mu(G_{r,c}))$. Then, for every $x \in V(G_{r,c})$, the number of maximal tournaments in which x has out-degree q is at least equal to the number of maximal tournaments in which x has out-degree $q-1$.*

The following theorem gives a condition regarding the minimum degree and the local partite irregularity to obtain an upper bound on $T_q^+(x)$.

Theorem 2.3. *Let $r \geq 2$, $c \geq 5$ and $G_{r,c}$ be a balanced c-partite tournament such that for some $q \geq 0$, $\delta(G_{r,c}) \geq q\,(r + \mu(G_{r,c}))\,\frac{c-1}{c-2}$. Then, for every $x \in V(G_{r,c})$,*

$$T_q^+(x) \leq \sum_{k=0}^{q} \binom{c-1}{k} \left(\frac{d^+(x)}{c-1}\right)^k \left(\frac{d^-(x)}{c-1}\right)^{c-1-k}.$$

Proof. Let $x \in V(G_{r,c})$ and suppose that $x \in V_c$. By Remark 1, we see that

$$T_q^+(x) = \sum_{k=0}^{q} \sum_{h \in \mathcal{H}_k(x)} \prod_{i=1}^{c-1} d_i^+(x)^{h_i} d_i^-(x)^{1-h_i} = \sum_{k=0}^{q} M(d_1^+(x), \ldots, d_{c-1}^+(x); k).$$

For each $i \in [c-1]$, let $g_i = d_i^+(x)$, and, without loss of generality, assume that $g_{c-1} = \max_{i \in [c-1]} g_i = \Gamma$ and $g_{c-2} = \min_{i \in [c-1]} g_i = \gamma$. Let $g'_1, g'_2, \ldots, g'_{c-2}, g'_{c-1}$ be real numbers such that, for $i \in [c-3]$, $g'_i = g_i$; and $g'_{c-2} = g'_{c-1} = \frac{g_{c-2}+g_{c-1}}{2}$.

Claim 2. $\sum_{k=0}^{q} M(g_1, \ldots, g_{c-1}; k) \leq \sum_{k=0}^{q} M(g'_1, \ldots, g'_{c-1}; k)$.

If $q = 0$, $\sum_{k=0}^{q} M(g_1, \ldots, g_{c-1}; 0) = \prod_{i=1}^{c-1} (r - g_i)$. Since

$$(r - g_{c-2})(r - g_{c-1}) \leq (r - \frac{g_{c-2} + g_{c-1}}{2})^2,$$

the claim follows. Assume that $q \geq 1$. For the sake of readability, in what follows, $g_1, \ldots, g_{c-1}$ and $g_1, \ldots, g_{c-3}$ will be denoted as $g_{[c-1]}$ and $g_{[c-3]}$, respectively. Observe that

$$\begin{aligned} M(g_{[c-1]}; 0) &= M(g_{[c-3]}; 0) M(g_{c-2}, g_{c-1}; 0); \\ M(g_{[c-1]}; 1) &= M(g_{[c-3]}; 1) M(g_{c-2}, g_{c-1}; 0) + M(g_{[c-3]}; 0) M(g_{c-2}, g_{c-1}; 1) \end{aligned}$$

and for every $k \geq 2$,

$$M(g_{[c-1]}; k) = \sum_{j=0}^{2} M(g_{[c-3]}; k-j) M(g_{c-2}, g_{c-1}; j).$$

Therefore, for $q = 1$,

$$\begin{aligned} \sum_{k=0}^{q} M(g_{[c-1]}; k) &= M(g_{[c-3]}; 0) \left[M(g_{c-2}, g_{c-1}; 0) + M(g_{c-2}, g_{c-1}; 1) \right] \\ &\quad + M(g_{[c-3]}; 1) M(g_{c-2}, g_{c-1}; 0); \end{aligned}$$

and for $q \geq 2$,

$$\begin{aligned}\sum_{k=0}^{q} M(g_{[c-1]};k) &= \sum_{k=0}^{q-2} M(g_{[c-3]};k)\,[M(g_{c-2},g_{c-1};0)+M(g_{c-2},g_{c-1};1)\\ &\qquad + M(g_{c-2},g_{c-1};2)]\\ &\quad +M(g_{[c-3]};q-1)\,[M(g_{c-2},g_{c-1};0)+M(g_{c-2},g_{c-1};1)]\\ &\quad +M(g_{[c-3]};q)M(g_{c-2},g_{c-1};0).\end{aligned}$$

It is not hard to see that for any pair x, y of reals, $0 \leq x, y \leq r$,

$$M(x,y;0)=(r-x)(r-y);\ \ M(x,y;1)=r(x+y)-2xy \text{ and } M(x,y;2)=xy.$$

Therefore

$$M(x,y;2)+M(x,y;1)+M(x,y;0)=r^2.$$

Since $g'_i = g_i$ for $i \in [c-3]$ and $g_{c-2}+g_{c-1}=g'_{c-2}+g'_{c-1}$, we have, after some easy calculations, that

$$\begin{aligned}&\sum_{k=0}^{q} M(g'_{[c-1]};k)-\sum_{k=0}^{q} M(g_{[c-1]};k)=\\ &\qquad M(g_{[c-3]};q-1)\left[g_{c-2}g_{c-1}-g'_{c-2}g'_{c-1}\right]+M(g_{[c-3]};q)\left[g'_{c-2}g'_{c-1}-g_{c-2}g_{c-1}\right]\\ &\quad=\left(g'_{c-2}g'_{c-1}-g_{c-2}g_{c-1}\right)\left[M(g_{[c-3]};q)-M(g_{[c-3]};q-1)\right].\end{aligned}$$

Since $g'_{c-2}g'_{c-1} \geq g_{c-2}g_{c-1}$, it follows that $\sum_{k=0}^{q} M(g_{[c-1]};k) \leq \sum_{k=0}^{q} M(g'_{[c-1]};k)$, if and only if $M(g_{[c-3]};q-1) \leq M(g_{[c-3]};q)$.

Since $\sum_{i\in[c-1]} g_i = d^+(x) \geq \delta(G_{r,c}) \geq q\,(r+\mu(G_{r,c}))\,\frac{c-1}{c-2}$, it follows that

$$d^+(x)\frac{c-2}{c-1} \;=\; d^+(x)-\frac{d^+(x)}{c-1} \;\geq\; r+\mu(G_{r,c}).$$

Therefore, $d^+(x) \geq r+\mu(G_{r,c})+\frac{d^+(x)}{c-1}$. On the one hand, clearly, $\gamma \leq \frac{d^+(x)}{c-1}$ and $\mu(G_{r,c}) \geq \Gamma-\gamma$. It follows that $\sum_{i\in[c-1]} g_i = d^+(x) \geq q(r+\Gamma-\gamma)+\gamma$. Since $g_{c-1}=\Gamma$ and $g_{c-2}=\gamma$, we see that $\sum_{i\in[c-3]} g_i \geq q(r+\Gamma-\gamma)-\Gamma$. On the other hand, observe that $\Gamma \geq \Gamma^* = \max_{i\in[c-3]} g_i$ and $\gamma \leq \gamma^* = \min_{i\in[c-3]} g_i$. Since $q \geq 1$, it follows that $q(r+\Gamma-\gamma)-\Gamma \geq q(r+\Gamma^*-\gamma^*)-\Gamma^*$ which implies that $\sum_{i\in[c-3]} g_i \geq q(r+\Gamma^*-\gamma^*)-\Gamma^*$. Hence, by Lemma 2.1, $M(g_{[c-3]};q-1) \leq M(g_{[c-3]};q)$, and from here the claim follows.

Observe that $\Gamma \geq \Gamma' = \max_{i\in[c-1]} g'_i$ and $\gamma \leq \gamma' = \min_{i\in[c-1]} g'_i$. Since $\sum_{i\in[c-1]} g_i = \sum_{i\in[c-1]} g'_i$ it follows that $\sum_{i\in[c-1]} g'_i \geq q(r+\Gamma'-\gamma')\frac{c-1}{c-2}$, and clearly $0 \leq g'_i \leq r$. Hence, we can iterate this procedure, and by the way that g'_{c-2} and g'_{c-1} are defined, we see that the limit of the difference $\Gamma'-\gamma'$ by iterating this procedure is zero. Thus, by

Claim 2, it follows that $T_q^+(x)$ is bounded by $\sum_{k=0}^{q} M(\frac{d^+(x)}{c-1},\ldots,\frac{d^+(x)}{c-1};k)$. Finally, by definition, for each $k \in [q]$,

$$\begin{aligned} M\left(\tfrac{d^+(x)}{c-1},\ldots,\tfrac{d^+(x)}{c-1};k\right) &= \sum_{h\in\mathcal{H}_k^{c-1}} \prod_{i=1}^{s} \left(\tfrac{d^+(x)}{c-1}\right)^{h_i}\left(r-\tfrac{d^+(x)}{c-1}\right)^{1-h_i} \\ &= \sum_{h\in\mathcal{H}_k^{c-1}} \left(\tfrac{d^+(x)}{c-1}\right)^{k}\left(r-\tfrac{d^+(x)}{c-1}\right)^{c-1-k} \\ &= \tbinom{c-1}{k}\left(\tfrac{d^+(x)}{c-1}\right)^{k}\left(r-\tfrac{d^+(x)}{c-1}\right)^{c-1-k}, \end{aligned}$$

and it is not hard to see that $r-\frac{d^+(x)}{c-1}=\frac{d^-(x)}{c-1}$. From here the result follows. □

The following theorem gives a condition regarding the minimum degree, the local partite irregularity and the global irregularity to obtain an upper bound on $T_q^+(x)$.

Theorem 2.4. *Let $r \geq 2$, $c \geq 5$ and $G_{r,c}$ be a balance c-partite tournament. If for some $q \geq 0$, $\delta(G_{r,c}) \geq q\,(r+\mu(G_{r,c}))\,\frac{c-1}{c-2}$ and $i_g(G_{r,c}) = r(c-1)\beta$ with $0 \leq \beta < \frac{c-2q-2}{c}$, then, for every $x \in V(G_{r,c})$, we have that*

$$T_q^+(x) \leq \binom{c-1}{q+1}\left(\frac{r}{2}\right)^{c-1}\frac{(1+\beta)^{c-2-2q}\,(q+1)}{c(1-\beta)-2q-2}.$$

Proof. Let $x \in V(G_{r,c})$. By Theorem 2.3, it follows that

$$T_q^+(x) \leq \sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{c-1}\right)^{k}\left(\frac{d^-(x)}{c-1}\right)^{c-1-k}. \tag{3}$$

Observe that $\sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{c-1}\right)^{k}\left(\frac{d^-(x)}{c-1}\right)^{c-1-k} = \left(\frac{d^-(x)}{c-1}\right)^{c-1}\sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{d^-(x)}\right)^{k}$. For every q, with $0 \leq q \leq c-1$, let $g(q)=\sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{d^-(x)}\right)^{k}$. Observe that for $q < c-1$,

$$\begin{aligned} g(q+1) = 1+\sum_{k=1}^{q+1}\tbinom{c-1}{k}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k} &= 1+\sum_{k=0}^{q}\tbinom{c-1}{k+1}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k+1} \\ &= 1+\tfrac{d^+(x)}{d^-(x)}\sum_{k=0}^{q}\tbinom{c-1}{k+1}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k} \\ &= 1+\tfrac{d^+(x)}{d^-(x)}\sum_{k=0}^{q}\tbinom{c-1}{k}\tfrac{c-1-k}{k+1}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k}. \\ &\geq 1+\tfrac{d^+(x)}{d^-(x)}\sum_{k=0}^{q}\tbinom{c-1}{k}\tfrac{c-1-q}{q+1}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k} \\ &= 1+\tfrac{d^+(x)}{d^-(x)}\tfrac{c-1-q}{q+1}\sum_{k=0}^{q}\tbinom{c-1}{k}\left(\tfrac{d^+(x)}{d^-(x)}\right)^{k} \\ &> \tfrac{d^+(x)}{d^-(x)}\tfrac{c-1-q}{q+1}g(q). \end{aligned}$$

On the other hand, $g(q+1) = g(q) + \binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1}$. Therefore,

$$g(q) + \binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1} > \frac{d^+(x)}{d^-(x)}\frac{c-1-q}{q+1}g(q),$$

which implies that

$$\binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1} > \left(\frac{d^+(x)}{d^-(x)}\frac{c-1-q}{q+1} - 1\right) g(q). \tag{4}$$

Clearly, $\frac{d^+(x)}{d^-(x)} \geq \frac{\delta(G_{r,c})}{\Delta(G_{r,c})}$, and since $\Delta(G_{r,c}) = \frac{r(c-1)+i_g(G_{r,s})}{2}$, $\delta(G_{r,c}) = \frac{r(c-1)-i_g(G_{r,s})}{2}$, and $i_g(G_{r,c}) = r(c-1)\beta$, it is not hard to see that $\frac{\delta(G_{r,c})}{\Delta(G_{r,c})} = \frac{1-\beta}{1+\beta}$. Moreover, since $\beta < \frac{c-2q-2}{c}$, it follows that $\frac{1-\beta}{1+\beta} > \frac{2q+2}{2c-2q-2} = \frac{q+1}{c-q-1}$. Therefore $\frac{1-\beta}{1+\beta}\frac{c-1-q}{q+1} - 1 > 0$. Thus, $\frac{d^+(x)}{d^-(x)}\frac{c-1-q}{q+1} - 1 \geq \frac{1-\beta}{1+\beta}\frac{c-1-q}{q+1} - 1 = \frac{(1-\beta)(c-1-q)-(1+\beta)(q+1)}{(1+\beta)(q+1)} = \frac{c(1-\beta)-2q-2}{(1+\beta)(q+1)} > 0$. Hence, by (4),

$$\binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1} > \frac{c(1-\beta)-2q-2}{(1+\beta)(q+1)}g(q)$$

and then

$$\binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1}\frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2} > g(q).$$

Therefore, it follows that, for $q < c-1$,

$$\begin{aligned}\sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{c-1}\right)^k\left(\frac{d^-(x)}{c-1}\right)^{c-1-k} &= \left(\frac{d^-(x)}{c-1}\right)^{c-1}\sum_{k=0}^{q}\binom{c-1}{k}\left(\frac{d^+(x)}{d^-(x)}\right)^k \\ &= \left(\frac{d^-(x)}{c-1}\right)^{c-1} g(q) \\ &< \left(\frac{d^-(x)}{c-1}\right)^{c-1}\binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1}\frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2}.\end{aligned}$$

Thus, by (3),

$$T_q^+(x) < \left(\frac{d^-(x)}{c-1}\right)^{c-1}\binom{c-1}{q+1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1}\frac{(1+\beta)(q+1)}{c(1-\beta)-2q-2}.$$

Finally, observe that

$$\begin{aligned}\left(\frac{d^-(x)}{c-1}\right)^{c-1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1} &= \left(\frac{d^-(x)}{c-1}\right)^{c-1}\left(\frac{d^+(x)}{c-1}\right)^{q+1}\left(\frac{c-1}{d^-(x)}\right)^{q+1} \\ &= \left(\frac{d^-(x)}{c-1}\right)^{c-1-2q-2}\left(\frac{d^+(x)d^-(x)}{(c-1)^2}\right)^{q+1}.\end{aligned}$$

On the one hand, since $d^+(x) + d^-(x) = r(c-1)$, it follows that $\frac{d^+(x)d^-(x)}{(c-1)^2} \leq \frac{r^2(c-1)^2}{4(c-1)^2} = \frac{r^2}{4}$ and therefore $\left(\frac{d^+(x)d^-(x)}{(c-1)^2}\right)^{q+1} \leq \left(\frac{r}{2}\right)^{2q+2}$.

On the other hand, $d^-(x) \le \Delta(G_{r,c}) = \frac{r(c-1)+i_g(G_{r,c})}{2} = \frac{r(c-1)+r(c-1)\beta}{2} = \frac{r(c-1)(1+\beta)}{2}$ and therefore $\left(\frac{d^-(x)}{c-1}\right)^{c-1-2q-2} \le \left(\frac{r(1+\beta)}{2}\right)^{c-1-2q-2} = \left(\frac{r}{2}\right)^{c-1-2q-2}(1+\beta)^{c-1-2q-2}$. Thus,

$$\begin{aligned}\left(\frac{d^-(x)}{c-1}\right)^{c-1}\left(\frac{d^+(x)}{d^-(x)}\right)^{q+1} &\le \left(\frac{r}{2}\right)^{2q+2}\left(\frac{r}{2}\right)^{c-1-2q-2}(1+\beta)^{c-1-2q-2}\\ &= \left(\frac{r}{2}\right)^{c-1}(1+\beta)^{c-1-2q-2}\end{aligned}$$

and from here, the result follows. □

3 Maximal strong subtournament with minimum degree at least $\lfloor\frac{c-2}{4}\rfloor + 1$

As an application of Theorem 2.4, we give sufficient conditions for the existence of a maximal subtournament with minimum degree at least $\lfloor\frac{c-2}{4}\rfloor + 1$, in a balanced c-partite tournament. Note that, as a fairly simple consequence of its minimum degree, such a tournament is strong.

Theorem 3.1. *Let $G_{r,c}$ be a balanced c-partite tournament, with $r \ge 2$, such that $\delta(G_{r,c}) \ge \lfloor\frac{c-2}{4}\rfloor(r+\mu(G_{r,c}))\frac{c-1}{c-2}$. Then $G_{r,c}$ contains a strongly connected tournament T of order c such that $\delta(T) \ge \lfloor\frac{c-2}{4}\rfloor + 1$, whenever*

i) $i_g(G_{r,c}) \le \frac{r}{2}$ and $c \ge 13$, $c \notin \{14, 15, 18\}$, or

ii) $i_g(G_{r,c}) \le r$ and $c \ge 17$, $c \notin \{18, 19, 22\}$, or

iii) $i_g(G_{r,c}) \le \frac{3r}{2}$ and $c \ge 21$, $c \notin \{22, 23, 26\}$.

Proof. In order to prove this theorem, we first show the following.

Claim 3. *Let $r \ge 2$ and $c \ge 5$. Let $G_{r,c}$ be a balanced c-partite tournament with $\delta(G_{r,c}) \ge \lfloor\frac{c-2}{4}\rfloor(r+\mu(G_{r,c}))\frac{c-1}{c-2}$ and $i_g(G_{r,c}) \le \alpha r/2$ (where $\alpha \ge 0$). If*

$$2^{c-2} > \binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}\frac{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}(\lfloor\frac{c-2}{4}\rfloor+1)c}{c^{\frac{2c-2-\alpha}{2c-2}} - 2\lfloor\frac{c-2}{4}\rfloor - 2},$$

then $G_{r,c}$ contains a strongly connected tournament T of order c such that $\delta(T) \ge \lfloor\frac{c-2}{4}\rfloor + 1$.

Let $G_{r,c}$ be a balanced c-partite tournament as in the statement of the claim, and suppose that there is no tournament T of order c in $G_{r,c}$ such that $\delta(T) \ge \lfloor\frac{c-2}{4}\rfloor + 1$. Thus, each of those tournaments has minimal degree at most $\lfloor\frac{c-2}{4}\rfloor$, and since there are r^c tournaments of order c in $G_{r,c}$, it follows that

$$\sum_{x\in V(G_{r,c})}(T^+_{\lfloor\frac{c-2}{4}\rfloor}(x) + T^-_{\lfloor\frac{c-2}{4}\rfloor}(x)) \ge r^c.$$

Since $|V(G_{r,c})| = rc$, by Theorem 2.4, we see that

$$2rc\binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}\left(\frac{r}{2}\right)^{c-1}\frac{\left(1+\frac{i_g(G_{r,c})}{r(c-1)}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}(\lfloor\frac{c-2}{4}\rfloor+1)}{c\left(1-\frac{i_g(G_{r,c})}{r(c-1)}\right)-2\lfloor\frac{c-2}{4}\rfloor-2}\geq r^c,$$

and since $i_g(G_{r,c}) \leq \frac{\alpha r}{2}$, it follows that $\frac{i_g(G_{r,c})}{r(c-1)} \leq \frac{\alpha}{2(c-1)}$, and as an easy consequence we have that $1+\frac{i_g(G_{r,c})}{r(c-1)} \leq \frac{2c-2+\alpha}{2c-2}$ and $1-\frac{i_g(G_{r,c})}{r(c-1)} \geq \frac{2c-2-\alpha}{2c-2}$. Thus,

$$2rc\binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}\left(\frac{r}{2}\right)^{c-1}\frac{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}(\lfloor\frac{c-2}{4}\rfloor+1)}{c\frac{2c-2-\alpha}{2c-2}-2\lfloor\frac{c-2}{4}\rfloor-2}\geq r^c.$$

Multiplying both sides of the inequality by $\frac{2^{c-1}}{r^{c-1}}\frac{1}{2r}$, we obtain that

$$c\binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}\frac{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}(\lfloor\frac{c-2}{4}\rfloor+1)}{c\frac{2c-2-\alpha}{2c-2}-2\lfloor\frac{c-2}{4}\rfloor-2}\geq 2^{c-2}$$

and from here the claim follows.

Let $f_\alpha(c) = \binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}\frac{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}(\lfloor\frac{c-2}{4}\rfloor+1)c}{c\frac{2c-2-\alpha}{2c-2}-2\lfloor\frac{c-2}{4}\rfloor-2}$ and $g(c) = 2^{c-2}$. Notice that $f_\alpha(c) \leq f_{\alpha'}(c)$ for $0 \leq \alpha \leq \alpha'$ and $c \geq 2$.

If $i_g(G_{r,c}) \leq \frac{r}{2}$, it follows that $\alpha \leq 1$ and it is not hard to see that $f_1(c) < g(c)$ whenever $c \in \{13, 16, 19, 22\}$. Analogously, if $i_g(G_{r,c}) \leq r$, then $\alpha \leq 2$ and $f_2(c) < g(c)$ whenever $c \in \{17, 20, 23, 26\}$; and if $i_g(G_{r,c}) \leq \frac{3r}{2}$, then $\alpha \leq 3$ and $f_3(c) < g(c)$ whenever $c \in \{21, 24, 27, 30\}$.

To end the proof, we just need to show that, for $\alpha \in \{1, 2, 3\}$, if for some $c \geq 13$ we have that $f_\alpha(c) < g(c)$, then $f_\alpha(c+4) < g(c+4)$. For this we show that $\frac{f_\alpha(c+4)}{f_\alpha(c)} \leq \frac{g(c+4)}{g(c)}$. Clearly, for every $c \geq 13$, $\frac{g(c+4)}{g(c)} = 16$. On the other hand, it is not difficult to see that, for every $c \geq 13$,

$$\frac{\left(\frac{2(c+4)-2+\alpha}{2(c+4)-2}\right)^{(c+4)-2-2\lfloor\frac{c+2}{4}\rfloor}}{\left(\frac{2c-2+\alpha}{2c-2}\right)^{c-2-2\lfloor\frac{c-2}{4}\rfloor}} \leq \left(\frac{2c+6+\alpha}{2c+6}\right)^2 \leq \frac{6}{5},$$

and with some more effort it is possible to verify that, for $c \geq 13$,

$$\frac{\binom{(c+4)-1}{\lfloor\frac{(c+4)-2}{4}\rfloor+1}(\lfloor\frac{(c+4)-2}{4}\rfloor+1)(c+4)}{\binom{c-1}{\lfloor\frac{c-2}{4}\rfloor+1}(\lfloor\frac{c-2}{4}\rfloor+1)c}\frac{c\frac{2c-2-\alpha}{2c-2}-2\lfloor\frac{c-2}{4}\rfloor-2}{(c+4)\frac{2(c+4)-2-\alpha}{2(c+4)-2}-2\lfloor\frac{(c+4)-2}{4}\rfloor-2}\leq\frac{32}{3}.$$

Thus, for $c \geq 13$, $\frac{f_\alpha(c+4)}{f_\alpha(c)} \leq \frac{6}{5}\frac{32}{3} < 16 = \frac{g(c+4)}{g(c)}$ and the result follows. □

As we can observe from the proof of Claim 3, it is possible to obtain analogous results to Theorem 3.1 for greater values of global irregularity.

Acknowledgements

We thank the anonymous referees for their comments and suggestions that helped us to significantly improve the presentation of this paper.

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(Received 19 May 2021; revised 4 Jan 2022)