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A note on avoidance games on Steiner triple systems

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Abstract

In this note, we study a combinatorial avoidance game on Steiner triple systems introduced by Clark, Fisk and Goren in 2016. In particular, we prove that the first player wins the avoidance game on any 1-reverse Steiner triple system, which generalizes the results by Clark et al. (2016) and Johnson et al. (2017). Moreover, we completely determine the winners of the avoidance games on all Steiner triple systems of order at most fifteen.

1 Introduction

In the intersection of graph theory and combinatorial game theory, combinatorial achievement/avoidance games on graphs have been well-studied. Such games are inspired by Ramsey theory and so called Ramsey achievement/avoidance games. In particular, the Ramsey avoidance game originally comes from a simple pencil game,

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called Sim [13]. For known results and further studies of combinatorial achievement/avoidance games on graphs or other game boards, see several relatively new papers [3, 8, 14, 15] for example and books [1, 2]. In particular, [14] is a nice paper which gives many results on the computational complexity of the games and surveys the literature on Ramsey achievement/avoidance games.

On the other hand, very recently, an avoidance game on a Steiner triple system was introduced in [4], where the game was also called *anti-SET*. A *Steiner triple system* of order n, STS(n) for short, consists of an n-element set V whose elements are called *points*, and a set $\mathcal{B}$ of triples of V, called *blocks*, with the property that every pair of elements in V is contained in exactly one block. It is well known that an STS(n) exists if and only if $n \equiv 1$ or $3 \pmod 6$. A combinatorial avoidance game on Steiner triple systems is defined as follows.

Definition 1 (Avoidance game on Steiner triple systems)**.** There are two players, Alice and Bob, starting with Alice. For given an STS(n), $(V, \mathcal{B})$, the two players alternately color an uncolored point in V with their own color. The first player who makes a *monochromatic* block loses, whose points are colored by the same color, and the game ends in a draw if all points of V are colored without a monochromatic block.

Clark, Fisk and Goren [4] considered the avoidance game on a special class of STSs, called *affine* STSs. An *affine* STS is an STS(3^k) on the k-dimensional vector space $V = \mathbb{F}_3^k$ over the three-element field $\mathbb{F}_3$ with block set $\mathcal{B} = \{\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \in \binom{V}{3} : \mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}\}$. Thus, all the blocks are lines of the k-dimensional affine space AG(k, 3). This game is closely related to a famous mathematical problem, called the *cap set problem*, that is, the problem on determining the maximum size of a set, called a *cap set*, of points not containing any line of AG(k, 3). In particular, they proved that Alice can win the avoidance game on every affine STS. This result was also proved by Johnson, Leader and Walters in [9, Theorem 27] independently. Furthermore, Clark, Mancini and Van Hook [5] showed that Bob can win the avoidance game on every *projective* Steiner triple system with $2^k - 1$ points with $k \geq 3$. It is natural to study the problem on determining the spectrum of n such that there exists an STS(n) on which Alice/Bob can win the avoidance game, since no draw is possible in the avoidance game (Proposition 3) and STSs with some orders have both as winners (Proposition 9).

An *automorphism* of an STS(n) is a permutation σ on V which fixes the set of blocks, i.e., $\sigma(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$ where $\sigma(B) = \{\sigma(b) : b \in B\}$. In this note, we treat a 1-*reverse* STS, which admits an involution with exactly one fixed point as its automorphism. It is known that a 1-reverse STS(n) exists if and only if $n \equiv 1, 3, 9$ or $19 \pmod{24}$ [7, 12, 16]. An affine STS is clearly an example of 1-reverse STSs since the involution is given by $\sigma(\mathbf{x}) = -\mathbf{x}$ for any $\mathbf{x} \in \mathbb{F}_3^k \setminus \{\mathbf{0}\}$ and $\mathbf{0}$ is the unique fixed point. One of the purposes of this note is to show that Alice wins the avoidance game on any 1-reverse Steiner triple system, which generalizes the results in [4, 9] though the idea of our proof essentially comes from theirs. However, note that the authors in [4] (respectively, [9]) used the terminologies of STSs (respectively, automorphisms) but did not use the terminologies of automorphisms (respectively, STSs). Thus, we unify and generalize their ideas and results. Another purpose of this note is to determine the winners of the avoidance games on all nonisomorphic Steiner triple systems of order 13 and 15 by a computer.

2 Avoidance games on Steiner triple systems

First, we give the following fundamental property of an STS(n).

Lemma 2. *Let $n = 2m + 1$ be an odd integer with $n \geq 7$. Let V be the point set of an* STS(n)*. For any subsets X, Y of V with $Y = V \setminus X$, $|X| = m + 1$ and $|Y| = m$, there exists a block B such that $B \subseteq X$ or $B \subseteq Y$.*

Proof. Let $n = 2m + 1$ with $m \geq 3$. Suppose to the contrary that there is no block B such that $B \subseteq X$ or $B \subseteq Y$.

Let a (respectively, b) be the number of blocks containing two points in X and one point in Y (respectively, one point in X and two points in Y). Observe that the number of blocks in an STS(n) is $n(n-1)/6 = (2m+1)m/3$, and hence, $a + b = (2m+1)m/3$. On the other hand, the total number of pairs of points in X and Y is $m(m+1)/2$ and $m(m-1)/2$, respectively. Thus we have $a = m(m+1)/2$ and $b = m(m-1)/2$, that is, $(2m+1)m/3 = a + b = m(m+1)/2 + m(m-1)/2 = m^2$, which means that $m = 1$, a contradiction. □

The following important proposition follows from Lemma 2, which was also described in [4] but no proof was given.

Proposition 3. *For all Steiner triple systems S with at least seven points, the avoidance game on S never ends in a draw.*

Proof. At the end of the avoidance game on every STS(n) with an odd integer $n \geq 7$, the point set V of the STS can be decomposed into two sets X and Y colored by Alice and Bob, respectively; note that $V = X \cup Y$ and $X \cap Y = \emptyset$. Thus, there exists a monochromatic block by Lemma 2, which means the game did not end in a draw. □

Next, we give one lemma on a property of 1-reverse STSs, which was already mentioned in [12].

Lemma 4. *Let $(V, \mathcal{B})$ be a 1-reverse* STS(n) *with $n \geq 9$ having an involution σ with one fixed point ∞. Then any block containing ∞ has the form $\{\infty, x, \sigma(x)\}$ for some $x \in V$.*

Proof. Suppose to the contrary that a block has the form $B = \{\infty, x, y\}$ with $y \neq \sigma(x)$. Then the unique block containing x and $\sigma(x)$ has the form $B' = \{a, x, \sigma(x)\}$ with $a \neq \infty$. Furthermore, $\sigma(B') = \{\sigma(a), \sigma(x), x\}$ must be a block different from B' since $a \neq \sigma(a)$. Then there are two blocks containing both x and $\sigma(x)$, a contradiction. □

We now prove that Alice wins the game on every 1-reverse Steiner triple system. The idea of the following result can be found in [4, 9] but they deal only with affine Steiner triple systems.

Theorem 5. *Let* $(V, \mathcal{B})$ *be a* 1*-reverse* STS(n) *with* $n \geq 9$*. Then Alice wins the avoidance game on the Steiner triple system.*

Proof. Let σ be an involution with fixed point $\infty \in V$ as an automorphism of $(V, \mathcal{B})$. We show that Alice wins the game by using the following strategy.

1. First she colors $\infty \in V$.
2. If Bob colors $x \in V$, then she colors $\sigma(x) \in V$.

Suppose to the contrary that Alice loses by making a monochromatic block $B = \{x, y, z\} \in \mathcal{B}$.

Case 1 ($x = \infty \in B$): By Lemma 4, we have $z = \sigma(y)$. However, by Alice's strategy, either y or z must be colored by Bob, a contradiction.

Case 2 ($\infty \notin B$): Let x', y' and z' be points colored by Bob just before Alice colors x, y and z, respectively. Then we have

$$\begin{aligned}\{x', y', z'\} &= \{\sigma(x), \sigma(y), \sigma(z)\} \\ &= \sigma(\{x, y, z\}) \in \mathcal{B}.\end{aligned}$$

Therefore, Bob creates another monochromatic block $\{x', y', z'\} \in \mathcal{B}$ just before Alice creates B, that is, Bob loses just before Alice does, a contradiction. □

Recall that every affine Steiner triple system is 1-reverse. Thus, Theorem 5 leads to the following corollary.

Corollary 6 ([4])**.** *Alice wins the avoidance game on every affine* STS(3^k) *with* $k \geq 2$*.*

Furthermore, recalling that there exists a 1-reverse STS(n) for every $n \equiv 1, 3, 9, 19$ (mod 24), we have the following corollary.

Corollary 7. *For any positive integer* $n \equiv 1, 3, 9, 19$ (mod 24) *with* $n \geq 9$*, there exists an* STS(n) *on which Alice wins the avoidance game.*

It is known that Bob wins the avoidance game on the unique STS(7) [5] since the STS belongs to *projective* STS$(2^k - 1)$s. Hence, we next treat the avoidance games on STS(n) for $n = 13, 15$. It is well-known that there are exactly two non-isomorphic STS(13)s and eighty non-isomorphic STS(15)s. Note that it was already proved in [5] that Bob wins the game on the projective STS(15). We checked the following by computer[1].

[1]We generated all non-isomorphic STS(n)s for $n = 13, 15$ by using the function `SteinerLoop` in GAP [11]. We coded the program by using `C++`. For more details, see the Appendix.

Proposition 8. *Bob wins the avoidance games on both of the two non-isomorphic* STS(13)*s.*

Proposition 9. *There are two non-isomorphic* STS(15)*s on which Alice wins the avoidance games, and Bob wins the avoidance games on the remaining* 78 *non-isomorphic* STS(15)*s.*

The special two non-isomorphic STS(15)s on which Alice wins the games are listed below. The point set is $\{1, 2, \dots, 15\}$ and the block sets are

$$\begin{aligned}\mathcal{B}_1 = \{&\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},\{2,4,6\},\\ &\{2,5,7\},\{2,8,10\},\{2,9,11\},\{2,12,14\},\{2,13,15\},\{3,4,7\},\{3,5,6\},\{3,8,12\},\\ &\{3,9,14\},\{3,10,13\},\{3,11,15\},\{4,8,15\},\{4,9,12\},\{4,10,14\},\{4,11,13\},\\ &\{5,8,13\},\{5,9,10\},\{5,11,14\},\{5,12,15\},\{6,8,11\},\{6,9,15\},\{6,10,12\},\\ &\{6,13,14\},\{7,8,14\},\{7,9,13\},\{7,10,15\},\{7,11,12\}\};\\ \mathcal{B}_2 = \{&\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{1,12,13\},\{1,14,15\},\{2,4,6\},\\ &\{2,5,7\},\{2,8,10\},\{2,9,12\},\{2,11,14\},\{2,13,15\},\{3,4,7\},\{3,5,6\},\{3,8,11\},\\ &\{3,9,15\},\{3,10,13\},\{3,12,14\},\{4,8,12\},\{4,9,11\},\{4,10,15\},\{4,13,14\},\\ &\{5,8,15\},\{5,9,13\},\{5,10,14\},\{5,11,12\},\{6,8,14\},\{6,9,10\},\{6,11,13\},\\ &\{6,12,15\},\{7,8,13\},\{7,9,14\},\{7,10,12\},\{7,11,15\}\}.\end{aligned}$$

Proposition 9 implies that the player who wins the avoidance game on an STS(n) depends not only on n but also on its structure. In this section, we showed that for any $n \equiv 1, 3, 9, 19 \pmod{24}$ with $n \geq 9$ or $n = 15$, there exists an STS(n) on which Alice wins the avoidance game while the number of such STS(15)s is small in comparison to the total number of non-isomorphic STS(15)s. On the other hand, there is no STS(n) on which Alice wins the game for $n = 7, 13$. Thus, we pose the following problem. (A similar problem is proposed in [9] for the avoidance game on the vertex-transitive hypergraph.)

Problem 10. *For a fixed integer $n \not\equiv 1, 3, 9, 19 \pmod{24}$ with $n > 15$, does there exist an* STS(n) *on which Alice wins the game? More generally, determine the spectrum of n such that there exists an* STS(n) *on which Alice/Bob wins the game.*

Finally, we give some comments on avoidance games on other block designs. We can naturally extend the definition of the avoidance game on STSs to that on general t-(v, k, λ) designs. In particular, we determined by computer the winners of the avoidance games on 2-$(v, 3, 2)$ designs without repeated blocks for $v = 6, 7$ and 3-$(v, 4, 1)$ designs for $v = 8, 10$ as listed below. Note that it is known that these designs are unique [6, 10].

- Alice wins the avoidance game on the 2-$(6, 3, 2)$ design without repeated blocks.
- Bob wins the avoidance game on the 2-$(7, 3, 2)$ design without repeated blocks.

- The avoidance game on the 3-$(8, 4, 1)$ design is drawn, i.e., both players can force the game to end in a draw.
- Bob wins the avoidance game on the 3-$(10, 4, 1)$ design.

We conclude the paper by giving the following problem for future work.

Problem 11. *Find an infinite class of* t-(v, k, λ) *designs with* $t > 2$, $k > 3$ *or* $\lambda > 1$ *on which Alice (respectively, Bob) wins the avoidance game.*

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Appendix

We describe our computer program written in `C++` for checking which player wins on each of the 80 nonisomorphic STS(15)s. We can use a similar program also for STS(13) and SQS(10). The program and data sets are open to the public at `https://sites.google.com/view/naokimatsumoto/data`. GAP [11] was used to generate all STS(15), which was then input into the following algorithm.

Algorithm

The algorithm is based on a depth first search on the game tree of the avoidance game on each STS(15). We give a pseudo-code of the algorithm below. (Note that in fact, the program contains another function `check`. However, the function just determines whether or not there is a monochromatic block in the current state of the game. Thus, we omit the function in the following pseudo-code.)

By Proposition 3, it is guaranteed that the program `avoid_game_en` terminates with outputting "Alice returns to her first move" t times and "Bob loses!" t' times, where $t + t' = 15$. Observe that if the message "Bob loses!" appears, then Alice can win the avoidance game on the STS(15). Otherwise, Bob can win the game. Note that the variable f in the algorithm `search` is a global variable.

Algorithm 1 `avoid_game_en`

Set the label of all points $v_0, v_1, \dots, v_{n-1}$ of a given STS(15) to be zero.
for $i = 0$ to $n-1$ **do**
  Set $v_i = 1$ (this means that Alice's first move is v_i).
  Call `search`(Bob) (see Algorithm 2).
  Set $v_i = 0$.
end for

Algorithm 2 `search`(Player)

for $i = 0$ to $n-1$ **do**
  if v_i is a feasible move for Player, i.e., a move does not produce a monochromatic block **then**
    Set $v_i = 1$ (resp., 2) if Player is Alice (resp., Bob).
    Call `search`(X), where X is Alice (resp., Bob) if Player is Bob (resp., Alice)
    Set $v_i = 0$.
    if $f = 1$ **then**
      Set $f = 0$.
      return
    end if
  end if
end for
Set $f = 1$.
if The current state is Alice's second turn **then**
  Output "Alice returns to her first move".
else
  if The current state is Bob's first turn **then**
    Output "Bob loses!".
  end if
end if
return

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