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Blockers of pattern avoiding permutation matrices

Richard A. Brualdi*

Department of Mathematics
University of Wisconsin
Madison, WI 53706
U.S.A.
brualdi@math.wisc.edu

Lei Cao

Department of Mathematics
Nova Southeastern University
Ft. Lauderdale, FL 33314
U.S.A.
leicaomath@gmail.com

Abstract

We investigate $n \times n$ $(0,1)$-matrices A such that no permutation matrix $P \leq A$ belongs to a prescribed subset $\mathcal{Q}_n$ of the set $\mathcal{P}_n$ of all $n \times n$ permutation matrices. The subsets $\mathcal{Q}_n$ considered are those defined by avoiding a given pattern σ_k where σ_k is a permutation of $\{1, 2, \ldots, k\}$. This gives rise to consideration of (minimal) blockers which are certain subsets of the positions of an $n \times n$ matrix that intersect every permutation matrix that avoids the pattern σ. The classical case is that where $\mathcal{Q}_n = \mathcal{P}_n$ and thus our investigations can be viewed as a generalization of the well-known Frobenius-König theorem. By this theorem the positions of any $r \times s$ submatrix with $r + s = n + 1$ is a minimal blocker of $\mathcal{P}_n$; in particular rows and columns are not only minimal blockers but are minimum cardinality blockers; the maximum size of a minimal blocker occurs when r and s are (nearly) equal. The case $k = 3$ is considered in some detail. In case $k \geq 4$, we show that every minimum blocker for any given pattern σ_k with cardinality equal to n is a row or column.

* Corresponding author.

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1 Introduction

Let $\mathcal{P}_n$ be the set of $n \times n$ permutation matrices P_π corresponding to the set $\mathcal{S}_n$ of permutations $\pi = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$. For instance, when $n = 5$ and omitting 0's,

$$(3,5,2,1,4) \leftrightarrow \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & & & & 1 \\ \hline & 1 & & & \\ \hline 1 & & & & \\ \hline & & & 1 & \end{array}\right].$$

In general, we do not make any distinction between permutations in $\mathcal{S}_n$ and their corresponding permutation matrices in $\mathcal{P}_n$.

We formulate the following *General Problem* for a subset $\mathcal{Q}_n$ of $\mathcal{P}_n$:

(i) Investigate the set of all $n \times n$ $(0,1)$-matrices A such that A is *disjoint* from $\mathcal{Q}_n$, that is, every $n \times n$ permutation matrix $P \leq A$ (entrywise order) satisfies $P \notin \mathcal{Q}_n$; in terms of scalar product, $P \circ A < n$ for every $P \in \mathcal{Q}_n$. The primary interest is in those A that are *maximal* with respect to being disjoint from $\mathcal{Q}_n$, that is, if A' is an $n \times n$ $(0,1)$-matrix with $A \leq A'$ and $A \neq A'$, then there is some permutation matrix $Q \in \mathcal{Q}_n$ with $Q \leq A'$.

(ii) Our problem has an alternative formulation: Let B be the $n \times n$ $(0,1)$-matrix with $B = J_n - A$ (J_n is the $n \times n$ all 1's matrix). Then A is disjoint from $\mathcal{Q}_n$ if and only if each permutation matrix $Q \in \mathcal{Q}_n$ *intersects* B, that is, in terms of scalar product, $Q \circ B \geq 1$ for all $Q \in \mathcal{Q}_n$. The matrix B, or the set of positions of its 1's, is a *blocker* of $\mathcal{Q}_n$. The primary interest is in those blockers B that are *minimal* with respect to this intersection property, that is, if B is an $n \times n$ $(0,1)$-matrix with $B' \leq B$ and $B' \neq B$, then there there is some permutation matrix $Q \in \mathcal{Q}_n$ with $Q \circ B' = 0$.

The classical example of this problem is obtained when $\mathcal{Q}_n$ equals the entire set $\mathcal{P}_n$ of $n \times n$ permutation matrices. The solution is provided by the Frobenius-König theorem: In terms of (ii), *the minimal blockers of* $\mathcal{P}_n$ *are the* $n \times n$ $(0,1)$*-matrices* B *whose* 1*'s form an* $r \times s$ *submatrix with* $r + s = n + 1$*. Such a* B *has* rs 1*'s which has minimum value* n *when* $r = 1, s = n$ *or* $r = n, s = 1$*. In terms of* (i)*, the maximal* A*'s are those* $n \times n$ $(0,1)$*-matrices* A *such that* $J_n - A$ *is such a* B*.* For example with $n = 8$, displaying only the 1's in B and A:

$$B = \left[\begin{array}{c|c|c|c|c|c|c|c} & & & & & & & \\ \hline & & 1 & 1 & & 1 & 1 & \\ \hline & & 1 & 1 & & 1 & 1 & \\ \hline & & & & & & & \\ \hline & & 1 & 1 & & 1 & 1 & \\ \hline & & 1 & 1 & & 1 & 1 & \\ \hline & & & & & & & \\ \hline & & 1 & 1 & & 1 & 1 & \end{array}\right] \text{ and } A = \left[\begin{array}{c|c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & 1 & & & 1 \\ \hline 1 & 1 & & & 1 & & & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & 1 & & & 1 \\ \hline 1 & 1 & & & 1 & & & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & & & 1 & & & 1 \end{array}\right].$$

Our primary interest in this paper are subsets $\mathcal{Q}_n$ of $\mathcal{P}_n$, which avoid certain patterns. Let k be an integer with $2 \leq k \leq n$. Let $\sigma_k = (t_1, t_2, \ldots, t_k)$ be a permutation of $\{1, 2, \ldots, k\}$. A permutation $\pi_n = (i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ *contains* σ_k (also written as *contains an* $t_1t_2\cdots t_k$*-pattern*) provided π_n has a subsequence of length k in the same relative order as σ_k. The permutation π_n *avoids* σ_k (also written as σ_k*-avoiding or* $t_1t_2\ldots t_k$*-avoiding*) provided it does not contain σ_k. Thus, for instance, if $k = 3$ and $\sigma_3 = (3, 1, 2)$, then π_n is 312-avoiding provided that there do not exist integers $1 \leq p < q < r \leq n$ such that $i_p > i_r$ and $i_r > i_q$, that is, there does not exist a subsequence of π_n in the relative order $3, 1, 2$. More discussion on pattern avoidance in permutations and many references can be found in [2].

A $t_1t_2\cdots t_k$*-avoiding permutation matrix* is an $n \times n$ permutation matrix corresponding to a $t_1t_2\cdots t_k$-avoiding permutation of $\{1, 2, \ldots, n\}$. We carry over the above terminology to the corresponding permutation matrices. The number of 312-avoiding permutations in $\mathcal{S}_n$ (respectively, $n \times n$ 312-avoiding permutation matrices) is the classical Catalan number

$$\frac{\binom{2n}{n}}{n+1}.$$

In fact, this is the same number of σ_3-avoiding $n \times n$ permutation matrices for any permutation σ_3 of $\{1, 2, 3\}$. Up to reversal and complementation, there are only two such σ_3, namely $(1, 2, 3)$ and $(3, 1, 2)$. These and other facts can be found in [2].

In general, we denote the set of σ_k-avoiding permutations of $\{1, 2, \ldots, n\}$ (respectively, the set of σ_k-avoiding $n \times n$ permutation matrices) by $\mathcal{S}_n(\overline{\sigma_k})$ (respectively, $\mathcal{P}_n(\overline{\sigma_k})$). Let k and l be integers with $1 \leq k < l \leq n$, and let $\sigma_k = (t_1, t_2, \ldots, t_k)$, and $\sigma_l = (t_1, t_2, \ldots, t_k, t_{k+1}, \ldots, t_l)$. Then σ_l is an *extension* of σ_k. More generally, let σ_k be in the same relative order as a subsequence of σ_l, denoted as $\sigma_k \preceq^* \sigma_l$. Then if a permutation π_n contains the pattern σ_l, it also contains the pattern σ_k; thus if a permutation π_n is σ_k-avoiding, then π_n is also σ_l-avoiding. Thus being σ_l-avoiding is less restrictive than being σ_k-avoiding and hence $\mathcal{S}_n(\overline{\sigma_k}) \subseteq \mathcal{S}_n(\overline{\sigma_l})$, equivalently, $\mathcal{P}_n(\overline{\sigma_k}) \subseteq \mathcal{P}_n(\overline{\sigma_l})$. Since this is a crucial observation for our investigations, we formulate it as a lemma.

Lemma 1.1 *Let* $\sigma_k \preceq^* \sigma_l$*. Then* $\mathcal{P}_n(\overline{\sigma_k}) \subseteq \mathcal{P}_n(\overline{\sigma_l})$*; in words, the permutations that avoid* σ_k *also avoid* σ_l*.* □

Thus there are more σ_l-avoiding $n \times n$ permutation matrices than σ_k-avoiding $n \times n$ permutation matrices; informally, it is easier to avoid σ_l than to avoid σ_k when $\sigma_k \preceq^* \sigma_l$.

Our motivation for this paper stems primarily from the 1971 seminal paper of Fulkerson [5] which we now describe. Consider an unbounded, n-dimensional, convex polyhedron $\mathcal{A}$ described by an $m \times n$ nonnegative matrix A with rows $a^1, a^2, \ldots, a^m$:

$$\mathcal{A} := \{x \in \Re_n^+ : Ax \geq \mathbf{1}\}.$$

This polyhedron is the vector sum of the convex hull of its extreme points and the nonnegative orthant. Some rows of A may be redundant so that not all rows may represent facets of $\mathcal{A}$. By the Farkas lemma, *a row a^i does not represent a facet of $\mathcal{A}$ if and only if a^i is dominated entrywise by a convex combination of other rows of A*. The rows that do represent facets are the *essential rows* of A; all other rows can be deleted from A. We therefore assume that A is *proper*, that is, all rows of A are essential and thus represent facets. The *blocker* of $\mathcal{A}$ is

$$\mathcal{B} := \{x \in \Re^+_m : x\mathcal{A} \geq \mathbf{1}\}.$$

Let B be the matrix with rows given by the extreme points $b^1, b^2, \ldots, b^r$ of $\mathcal{B}$. Then B is the *blocking matrix* of A and is proper, and by Theorem 1 of Fulkerson [5],

$$\mathcal{A} = \{y \in \Re^+_n : \mathcal{B}y \geq \mathbf{1}\}$$

is the blocker of $\mathcal{B}$, that is, *the blocker of the blocker is the original.*

Consider the set $\mathcal{P}_n$ of all $n \times n$ permutation matrices and its corresponding $n! \times n^2$ incidence $(0,1)$-matrix A with rows corresponding to the permutation matrices and columns corresponding to the n^2 positions of an $n \times n$ matrix. Let $I, J \subseteq \{1, 2, \ldots, n\}$ with $|I| + |J| \geq n+1$, and let $B^{(I,J)} = [b^{(I,J)}_{kl}]$ be the $n \times n$ matrix with

$$b^{(I,J)}_{kl} = \frac{1}{|I| + |J| - n}, \text{if } (k,l) \in (I,J), \text{ and } b^{(I,J)}_{kl} = 0, \text{ otherwise.}$$

Then the matrices $B^{(I,J)}$, considered as 1 by n^2 vectors, are the essential rows of the blocker of A provided neither $I = \{1, 2, \ldots, n\}$ and $|J| > 1$ nor $J = \{1, 2, \ldots, n\}$ and $|I| > 1$ [5]. An example of such an essential row in terms of the corresponding matrix is the following 7×7 matrix

$$B^{(I,J)} = \frac{1}{2}\left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & & & \\ \hline 1 & 1 & 1 & 1 & & & \\ \hline 1 & 1 & 1 & 1 & & & \\ \hline 1 & 1 & 1 & 1 & & & \\ \hline 1 & 1 & 1 & 1 & & & \\ \hline & & & & & & \\ \hline & & & & & & \end{array}\right]$$

where $I = \{1, 2, 3, 4, 5\}$ and $J = \{1, 2, 3, 4\}$, with $|I| + |J| - n = 5 + 4 - 7 = 2$. In this case every 7×7 permutation matrix will have two 1's in the region given by the positions of the displayed 1's above. For general $n \times n$, the essential rows of the blocker in this case that correspond to $(0,1)$-matrices are thus those $n \times n$ $(0,1)$-matrices that have a $k \times l$ submatrix of all 1's with $k + l = n + 1$ and 0's elsewhere. These are the essential rows that arise from the Frobenius-König theorem and are a proper subset of the set of essential rows. These essential $(0,1)$-rows are minimal in the sense that replacing any 1 with a 0 no longer blocks all $n \times n$ permutation matrices.

In this paper we consider the more general problem with the full set $\mathcal{P}_n$ of $n \times n$ permutation matrices replaced with the subset $\mathcal{P}_n(\overline{\sigma_k})$ of σ_k-avoiding $n \times n$ permutation matrices for some permutation σ_k of $\{1, 2, \ldots, k\}$:

Investigate those $n \times n$ $(0,1)$-matrices A where A is disjoint from $\mathcal{P}_n(\overline{\sigma_k})$, equivalently, every permutation matrix $P \leq A$ contains a σ_k-pattern.

With each set $\mathcal{P}_n(\overline{\sigma_k})$ of σ_k-avoiding $n \times n$ permutation matrices, we associate the polyhedron

$$\Omega_n^*(\overline{\sigma_k}) = \{\Omega_n(\overline{\sigma_k}) + X : X \geq O\},$$

where $\Omega_n(\overline{\sigma_k})$ is the convex hull[1] of $\mathcal{P}_n(\overline{\sigma})$. Let J_n be the $n \times n$ matrix of all 1's. For the matrix A above, we take the $m \times n^2$ $(0,1)$-matrix whose rows are determined by the matrices $J_n - C$ (written as vectors of size n^2) as C ranges over all those $n \times n$ (0,1)-matrices such that for every permutation π with $P_\pi \leq C$, σ_k occurs as a subpattern of π. The essential rows of this matrix A correspond to $n \times n$ (0,1)-matrices C such that every permutation matrix $P_\pi \leq C$ has σ_k as a subpattern, the $n \times n$ *blockers*[2] of σ_k, but changing any 0 of C to a 1 results in a matrix C' such that there is a permutation matrix $P_\pi \leq C'$ which is σ_k-avoiding, the $n \times n$ *minimal blockers* of σ_k. (Note that if one takes such a maximal C and writes it as a convex combination of other such essential matrices (which are $(0,1)$-matrices), then all these matrices would have to have 1's where C has 1's and nowhere else. That is why it suffices to take only these maximal matrices C.)

We formulate our problem in a combination of matrix/set-theoretic language that is easier to understand. Let $\sigma_k \in \mathcal{P}_k$. Let $C(Y)$ be an $n \times n$ matrix whose n^2 positions are partitioned into two sets Y and the complement $\overline{Y}$ of Y and are labeled with a y and a $\overline{y}$, respectively. We denote this by leaving those positions labeled $\overline{y}$ empty. Then

(*) $C(Y)$ is a *σ_k-avoiding blocker* provided every σ_k-avoiding $n \times n$ permutation matrix P has a 1 in a position in Y, equivalently, $\overline{Y}$ does not completely contain all the positions of the 1's of any σ_k-avoiding $n \times n$ permutation matrix. Since every σ_k-avoiding $n \times n$ permutation matrix P contains a 1 in a position of Y, we also say that the set of positions of Y is a *σ_k-avoiding blocker*. The *size of the σ_k-avoiding blocker* $C(Y)$ is the cardinality of Y.

Another formulation of our problem is this: Let σ_k be a permutation of $\{1, 2, \ldots, k\}$ and let A be an $n \times n$ $(0,1)$-matrix. Suppose that every $n \times n$ permutation matrix $P \leq A$ *has* a σ_k-pattern; thus every $n \times n$ σ_k-avoiding permutation matrix Q has a 1 in at least one of the positions in which A has 0's. Investigate such matrices A and determine the maximum number of 1's (minimum number of 0's) A can have. However, some of our arguments are better written and understood using the matrices $C(Y)$ as described above.

The next lemma follows from Lemma 1.1.

[1]Thus $\Omega_n(\overline{\sigma_k})$ is the subpolytope of the polytope Ω_n of the $n \times n$ doubly stochastic matrices that are convex combinations of σ_k-avoiding $n \times n$ permutation matrices. We plan to discuss this polytope in a future paper.

[2]We really should say $(0,1)$-*blockers* but these are essentially the only blockers we consider. It is an interesting question to to consider, as with $\mathcal{P}_n$, what kind of essential non-(0,1) blockers occur for a set $\mathcal{P}_n(\overline{\sigma_k})$.

Lemma 1.2 *If $k > l$ and $\sigma_k \preceq^* \sigma_l$, then any σ_l-avoiding blocker is also a σ_k-avoiding blocker: a set of positions Y that meets every σ_l-avoiding permutation matrix also meets every σ_k-avoiding permutation matrix.*

Proof. Let Y be a σ_l-avoiding blocker set of positions. Thus every σ_l-avoiding permutation matrix contains a 1 in a position of Y. By Lemma 1.1, $\mathcal{P}_n(\overline{\sigma}_k) \subseteq \mathcal{P}_n(\overline{\sigma_l})$. Hence every σ_k-avoiding permutation matrix also contains a 1 in a position in Y. $\square$

For instance, permutations that contain a 1234-pattern also contain a 123-pattern, and thus permutations that avoid a 123-pattern also avoid a 1234-pattern. Hence if Y meets all 1234-avoiding permutations it must meet all 123-avoiding permutations.

For every σ_k, there are σ_k-avoiding blockers of size n, since the set of positions in any row or in any column contains a 1 of every $n \times n$ permutation matrix whether or not it is σ_k-avoiding.

To help clarify these ideas, we now consider some examples.

Example 1.3 Let $n = 8$ and $\sigma_3 = (1, 2, 3)$. If Y is the set of all the 8 positions in a row or column, then the corresponding matrix $C(Y)$ is a σ_3-avoiding blocker since *every* permutation contains a position in each row and each column. Now consider the matrix $C(Y)$ with Y as shown. (As already remarked, we usually only label the positions of Y with the remaining positions assumed to contain a $\overline{y}$.)

$$C(Y) = \left[\begin{array}{c|c|c|c|c|c|c|c}
 & & & y & y & y & y & y \\ \hline
 & & & & & & & y \\ \hline
 & & & & & & & y \\ \hline
 & & & & & & & y \\ \hline
 & & & & & & & \\ \hline
 & & & & & & & \\ \hline
 & & & & & & & \\ \hline
 & & & & & & &
\end{array}\right],$$

It is easy to verify that any 8×8 permutation matrix P that does not have a 1 in a position in Y contains a 123-pattern: such a permutation matrix must have a 1 in the first three positions of row 1, a 1 in the last four positions in column 8, and at least one 1 in the 3×4 submatrix below and to the left of the y's. Thus $C(Y)$ is a 123-avoiding blocker. Removing any y of $C(Y)$ results in a 123-avoiding permutation matrix P that is no longer blocked. Thus $C(Y)$ is a minimal 123-avoiding blocker. This argument generalizes to show the following: For any $n \geq 3$, if a set Y of n positions forms an upside-down L-shaped region in the upper-right corner, that is, the set of positions

$$\{(1, k), (1, k + 1), \ldots, (1, n), (2, n), (3, n), \ldots, (k, n),$$

Then $C(Y)$ is a minimal 123-avoiding blocker of size n.

Now let $\sigma = (3, 1, 2)$. Then

$$C(Y) = \begin{bmatrix} & & & & y & & & \\ & & & & y & & & \\ & & & & y & & & \\ & & & & y & & & \\ & & & & y & y & y & y \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$

is a 312-avoiding blocker of size 8. The reason is that any 8×8 permutation P that does not contain a position in Y must contain one of the four available positions in row 5 and one of the three available positions in column 5, and therefore one of the positions in the 4×3 submatrix in the upper-right, giving a 312-pattern. This argument also generalizes to show that for any $n \geq 3$, the set of n positions forming an L-shaped region in the upper-right corner, that is, the set of positions

$$\{(1, k), (2, k), \ldots, (k, k), (k, k + 1), (k, k + 2), \ldots, (k, n)$$

form a 312-avoiding blocker of size n. $\square$

Our motivation in this paper can be formulated as a search for analogues of the Frobenius-König theorem for certain subsets $\mathcal{P}'_n$ of the set $\mathcal{P}_n$ of permutation matrices, namely, the set of permutations of $\{1, 2, \ldots, n\}$ avoiding a specified permutation of $\{1, 2, \ldots, k\}$:

(a) There exists a permutation matrix $P \leq A$ with $P \in \mathcal{P}'_n$ if and only if there does not exist a (*to be determined for* $\mathcal{P}'_n$) .

(b) There does not exist a permutation matrix $P \leq A$ with $P \in \mathcal{P}'_n$ if and only if there exists a (*to be determined for* $\mathcal{P}'_n$).

In our situation of $\mathcal{P}_n(\overline{\sigma_k})$ of σ_k-avoiding permutation matrices, we will be primarily concerned with $(0, 1)$-blockers of σ_k-avoiding permutation matrices. But as with $\mathcal{P}_n$, there are other blockers as well.

Define a set Y of entries of an $n \times n$ matrix $C(Y)$ to be a (*minimal*) σ_k*-avoiding blocker of strength* t provided every $n \times n$ σ_k-avoiding permutation matrix contains at least t y's.

Example 1.4 Continuing with Example 1.3 with $n = 8$,

$$\begin{bmatrix} & & & y & y & y & y & y \\ & & & y & y & y & y & y \\ & & & & & & y & y \\ & & & & & & y & y \\ & & & & & & y & y \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \to \frac{1}{2} \begin{bmatrix} & & & 1 & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & 1 & 1 \\ & & & & & & 1 & 1 \\ & & & & & & 1 & 1 \\ & & & & & & 1 & 1 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}.$$

gives a 123-avoiding blocker of size 16 and strength 2. A blocker for an 8×8 $(0,1)$ matrix such that each 123-avoiding permutation matrix contains at least three positions occupied by y's, a 123-avoiding blocker of size 24 and strength 3, is given by

$$\left[\begin{array}{c|c|c|c|c|c|c|c} & & y & y & y & y & y & y \\ \hline & & y & y & y & y & y & y \\ \hline & & y & y & y & y & y & y \\ \hline & & & & & y & y & y \\ \hline & & & & & y & y & y \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \end{array}\right] \rightarrow \frac{1}{3}\left[\begin{array}{c|c|c|c|c|c|c|c} & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & & & & & 1 & 1 & 1 \\ \hline & & & & & 1 & 1 & 1 \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \end{array}\right].$$

$\square$

To conclude this introductory section, we contrast the difference between our investigations and the celebrated proof by Marcus and Tardos [6] of the Füredi-Hajnal conjecture; a good exposition of this is in [2].

Theorem 1.5 *Let σ_k be a permutation of $\{1, 2, \ldots, k\}$ with corresponding $k \times k$ permutation matrix P_{σ_k}. Let $f(n, P_{\sigma_k})$ be the maximum number of 1's that an $n \times n$ $(0,1)$-matrix A can have if A avoids P_{σ_k}. Then*

$$f(n, P_{\sigma_k}) \leq 2k^4 \binom{k^2}{k} n.$$

Thus Theorem 1.5 is concerned with avoiding the $k \times k$ permutation matrix P_{σ_k} in an $n \times n$ $(0,1)$-matrix A and not, as in our investigations, with avoiding $n \times n$ permutation matrices that contain σ_k in an $n \times n$ $(0,1)$-matrix A.

We now briefly summarize the content of this paper. In Section 2 we primarily consider blockers of $n \times n$ permutations σ_3 of length 3. In Section 3 we prove that in the case of $k \geq 4$, a σ_k-avoiding blocker of the minimal size n is actually a row or column and so blocks all $n \times n$ permutation matrices. In the final Section 4 we discuss some additional properties of blockers and some possible directions for further research.

2 Blockers

Let n and k be integers with $2 \leq k \leq n$, and let $\sigma_k = (t_1, t_2, \ldots, t_k)$ be a permutation of $\{1, 2, \ldots, k\}$ which we often write as $t_1 t_2 \cdots t_k$. We consider the class $\mathcal{P}_n(\overline{\sigma_k}) = \mathcal{P}_n(\overline{t_1 t_2 \cdots t_k})$ of $t_1 t_2 \ldots t_k$-avoiding $n \times n$ permutation matrices. We first consider $k = 2$ and $k = n$. For $\sigma = (1,2)$ and $(2,1)$, we have:

(a) There does not exist a permutation matrix $P \leq A$ with $P \in \mathcal{P}_n(\overline{12})$ (so corresponding to the unique permutation of length n with no increase, namely the permutation $(n, n-2, \ldots, 1)$) if and only if there does not exist a 1×1 zero submatrix of the form $a_{i,n+1-i} = 0$ for some $1 \leq i \leq n$. Thus the minimal (12)-avoiding blockers of $n \times n$ permutation have size 1 and consist of any position on the *Hankel diagonal* (the diagonal running from the upper right corner to the lower left corner).

(b) There does not exist a permutation matrix $P \leq A$ with $P \in \mathcal{P}_n(\overline{21})$ (so corresponding to the unique permutation of length n with no decrease, namely the identity permutation $\iota_n = (1, 2, \ldots, n)$) if and only if there exists a 1×1 zero submatrix of the form $a_{ii} = 0$ for some $1 \leq i \leq n$. Thus the minimal (21)-avoiding blockers of $n \times n$ permutation have size 1 and consist of any position on the main diagonal.

Now suppose that $k = n$ and so σ_n is a permutation of $\{1, 2, \ldots, n\}$ with corresponding permutation matrix P_{σ_n}. Consider first $\sigma_n = (1, 2, \ldots, n)$ with $P_{\sigma_n} = I_n$. So we want to avoid I_n. What are the $12\cdots n$-avoiding blockers in this case? We seek a set Y of positions labeled with y's of an $n \times n$ matrix such that every $12\cdots n$-avoiding permutation matrix (so every $n \times n$ permutation matrix $\neq I_n$) contains at least one of these positions. In addition to the rows and columns, there are two other obvious such sets Y; all the positions above the main diagonal and all the positions below the main diagonal; with $n = 6$, these are

$$C(Y) = \left[\begin{array}{c|c|c|c|c|c} & y & y & y & y & y \\ \hline & & y & y & y & y \\ \hline & & & y & y & y \\ \hline & & & & y & y \\ \hline & & & & & y \\ \hline & & & & & \end{array}\right] \text{ and } \left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline y & & & & & \\ \hline y & y & & & & \\ \hline y & y & y & & & \\ \hline y & y & y & y & & \\ \hline y & y & y & y & y & \end{array}\right].$$

These are clearly $12\cdots n$-avoiding blockers in general, since the only permutation matrix not having a 1 in a position labeled y is the identity matrix I_n; they are minimal $12\cdots n$-avoiding blockers, since removing a y from Y allows a permutation matrix without a 1 in a position labeled y different from the identity matrix. Thus these minimal $12\cdots n$-avoiding blockers have cardinality $\binom{n}{2}$. More generally, if we take $C(Y)$ to be any matrix where the y's form a triangle with row and column sums $1, 2, \ldots, n-1$ in some order and where the triangle does not contain a y on the main diagonal (so certain row and column permutations of the above $C(Y)$), then there is only one permutation matrix not containing a y which is therefore I_n. $C(Y)$ is then a minimal $12\cdots n$-avoiding blocker of size $\binom{n}{2}$. We call these blockers *n-triangular blockers*. Now there are other $12\cdots n$-avoiding blockers, namely, by the Frobenius-König theorem, any $r \times s$ submatrix Y of positions with $r + s = n + 1$, in particular the set of n positions of a row or column. These are also minimal $12\cdots n$-avoiding blockers since removing any position in such a Y will create a permutation matrix different from I_n.

Theorem 2.1 *Let $n \geq 3$. The maximum cardinality of a minimal $12\cdots n$-avoiding blocker is $\binom{n}{2}$ with equality if and only if the blocker is an n-triangular blocker.*

Proof. Consider a minimal $12\cdots n$-avoiding blocker Y. If this blocker contains a position on the main diagonal, then it blocks the entire set $\mathcal{P}_n$ of $n \times n$ permutation matrices. Then by the Frobenius-Kőnig theorem, it contains an entire $r \times s$ submatrix Y' of positions with $r, s \geq 1$ and $r + s = n + 1$. Since Y is a minimal blocker, $Y = Y'$ and hence the cardinality of Y satisfies

$$|Y| = rs \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil < \binom{n}{2}.$$

We now assume that Y does not contain a position on the main diagonal. Then for every pair of symmetrically opposite positions (i, j) and (j, i) with $1 \leq i < j \leq n$, Y contains at least one of the pair, for otherwise we have a non-identity permutation matrix avoiding Y. Hence

$$|Y| \geq \binom{n}{2}.$$

In order for equality to hold Y must contain exactly one of the positions (i, j) and (j, i) for each i and j with $1 \leq i < j \leq n$. Thus Y corresponds to the 1's of a so-called $n \times n$ tournament matrix. If the tournament matrix is not transitive, there would be a cycle of length 3 not meeting Y and thus with the $(n-3)$ complementary positions on the main diagonal, we would get a permutation matrix not meeting Y. It follows that equality implies that Y is an n-triangular block. $\square$

A similar analysis applies to any permutation σ_n of $\{1, 2, \ldots, n\}$ and corresponding permutation matrix P_{σ_n}.

As is clear from above, there is a substantial difference between a *minimal blocker* (in terms of set containment) and a *minimum blocker* (in terms of size).

We define the *Hankel-cyclic decomposition*[3] of the $n \times n$ matrix J_n of all 1's into n permutation matrices by starting with row 1 and cyclically permuting it as for circulant matrices but in a right-to-left fashion, rather than the left-to-right fashion. The Hankel-cyclic decomposition is illustrated for $n = 6$ using letters a, b, c, d, e, f below:

$$\begin{bmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e \end{bmatrix}.$$

In general, this gives a decomposition of J_n into 123-avoiding permutation matrices, since each permutation in the decomposition corresponds to a decreasing sequences followed by another decreasing subsequence (empty in one case). For that reason this is also a decomposition into 312-avoiding and 231-avoiding permutation matrices. We call the permutation matrices in the Hankel-cyclic decomposition the *Hankel-cyclic permutation matrices*; the diagonal running from the upper right to the lower left

[3] The term Hankel as used here comes from the fact that the resulting matrix is a Hankel matrix with the permutations cyclically constructed around the Hankel diagonal from the upper right to the lower left. These permutation matrices are the only permutation matrices that are Hankel matrices.

is the *Hankel diagonal*. Thus the Hankel-cyclic decomposition is a 123-*,* 312-*, and* 231*-avoiding decomposition of* J_n.

The *standard circulant decomposition*, illustrated again for $n = 6$:

$$\left[\begin{array}{c|c|c|c|c|c} a & b & c & d & e & f \\ \hline f & a & b & c & d & e \\ \hline e & f & a & b & c & d \\ \hline d & e & f & a & b & c \\ \hline c & d & e & f & a & b \\ \hline b & c & d & e & f & a \end{array}\right]$$

is a 132*-avoiding,* 213*-avoiding, and* 321*-avoiding decomposition.*

Lemma 2.2 *Let n be a positive integer. There is a unique decomposition $J_n = P_1 + P_2 + \cdots + P_n$ of the $n \times n$ matrix J_n of all* 1*'s into* 123*-avoiding permutation matrices, namely that given by the Hankel-cyclic $n \times n$ permutation matrices. The Hankel-cyclic decomposition of J_n is also the unique decomposition of J_n into* 312*-avoiding permutation matrices and into* 231*-avoiding permutation matrices.*

Proof. Consider a decomposition $J_n = P_1 + P_2 + \cdots + P_n$ into $n \times n$ 123-avoiding permutation matrices where P_k is the permutation matrix with a 1 in position $(1, k)$ for $k = 1, 2, \ldots, n$. Designate by a_k the positions of the 1's of the permutation matrix P_k. The submatrix of J_n obtained by deleting row and column 1, so equal to J_{n-1}, has only one permutation which is 12-avoiding, namely its Hankel diagonal. Thus these are the positions of P_1 and thus P_1 is the Hankel-cyclic permutation matrix with a 1 in position $(1, 1)$. Now let $k > 1$ and, proceeding by induction, we suppose that $P_1, \ldots, P_{k-1}$ are the Hankel-cyclic permutation matrices with 1's in the first $(k-1)$ positions of row 1 and making up the decomposition of J_n into 123-avoiding permutation matrices. Consider the permutation matrix P_k in this decomposition with a 1 in position $(1, k)$. Suppose that P_k had a 1 in one of the positions $(2, k+1), \ldots, (2, n-1)$. Then P_k has a 1 in the as yet unoccupied positions in column n creating a 123-pattern. Thus since row 2 already contains 1's from $P_1, \ldots, P_{k-1}$ in its first $(k-1)$ positions, P_k contains a 1 in position $(2, k-1)$. In a similar way we see that P_k contains 1's in all the positions $(k, 1), (k-1, 2), \ldots, (1, k)$. Now it follows that P_k must contain 1's in all the positions of the Hankel diagonal of the $(n-k) \times (n-k)$ submatrix determined by rows and columns $k+1, \ldots, n$. Hence P_k is the Hankel-cyclic submatrix with a 1 in position $(1, k)$. The first part of the theorem now follows by induction. Similar arguments work for 312-avoiding and 231-avoiding permutations.. □

Remark 2.3 A permutation $\sigma = i_1 i_2 \cdots i_n$ is 123-avoiding provided it can be partitioned into two decreasing subsequences [2]. An algorithm to check this and get such a partition when it exists is:

(a) Start with i_1 and iteratively choose the next element that is greater than the previously chosen one. This gives the sequence of *left-to-right minima* starting with i_1, and thus the longest decreasing subsequence beginning with i_1. Remove the sequence of left-to-right minima leaving a subsequence consisting of the elements not chosen.

or, alternatively,

(a') Start with $i_n = n$ and iteratively choose the previous element that is smaller than the previously chosen one. Remove the sequence of right-to-left minima starting with $i_k = n$ leaving a subsequence consisting of the elements not chosen.

In each case, every integer in $\{1, 2, \ldots, n\}$ is in one of the subsequences. Then σ is 123-avoiding if and only if the second subsequence of non-chosen elements is decreasing. Note that these two procedures may give different subsequences if $i_1 \neq n$; for example, with $n = 9$,

$$543987621 \to 54321 \text{ and } 98762, \text{ or } 543987621 \to 987621 \text{ and } 543.$$

That this algorithm decomposes a permutation into two decreasing subsequences when the permutation is 123-avoiding is easily checked: Starting with i_1, suppose the elements chosen are

$$a_1, \ldots, a_2, \ldots, a_3 \ldots, a_4, \ldots, a_5, \ldots$$

where $a_1 > a_2 > a_3 > a_4 > a_5 > \cdots$. Then, by choice, any x between a_1 and a_2 must satisfy $a_1 < x$, and the subsequence of elements between a_1 and a_2 must be decreasing, for otherwise we would get a 123-pattern. Similarly for a_2 and a_3, and a_3 and a_4, and so on. The complement of $a, b, c, d, \ldots$ must also be decreasing, because otherwise again we get a 123-pattern.

For example, given the permutation $(9, 7, 8, 5, 6, 4, 2, 3, 1)$ and using (a), we get, displaying both the permutation and permutation matrix:

$$978564231 \to 975421 \text{ and } 863 : \quad \begin{bmatrix} & & & & & & & & 1 \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & 1 & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ 1 & & & & & & & & \end{bmatrix}.$$

$\square$

Similar to Lemma 2.2, we have the following lemma.

Lemma 2.4 *Let n be a positive integer. There is a unique decomposition $J_n = P_1 + P_2 + \cdots + P_n$ of the $n \times n$ matrix J_n of all 1's into 321-avoiding permutation matrices, namely that given by the standard circulant decomposition. The standard circulant decomposition of J_n is also the unique decomposition of J_n into 213-avoiding permutation matrices, and 132-avoiding permutation matrices.*

Corollary 2.5 *Let $\sigma_3 = (t_1, t_2, t_3)$ be a permutation of $\{1, 2, 3\}$. A minimal $t_1t_2t_3$-avoiding blocker of $n \times n$ permutation matrices has cardinality at least n.*

Proof. This follows from the very useful fact that in each case at least one position from each of the permutation matrices in the Hankel-cyclic or standard circulant decompositions of J_n given in Lemmas 2.2 and 2.4 must be included in the specified blockers. □

Example 2.6 By Corollary 2.5 a 123-avoiding blocker of $n \times n$ permutation matrices has cardinality at least n. Besides the rows and columns, there are many such blockers of size n as the following (easy to verify) examples for $n = 6$ show:

$$\left[\begin{array}{c|c|c|c|c|c} & & y & y & y & y \\ \hline & & & & & \\ \hline & & & & y & \\ \hline & & & & & \\ \hline & & & y & & \\ \hline & & & & & \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c|c|c|c|c} & & & y & y & y \\ \hline & & & & & y \\ \hline & & & & & \\ \hline & & & & y & \\ \hline & & & & & \\ \hline & & & y & & \end{array}\right].$$

□

Let k and l be integers with $2 \leq k \leq l < n$, and let σ_k be a permutation of $\{1, 2, \ldots, k\}$ and σ_l a permutation of $\{1, 2, \ldots, l\}$. Recall that σ_l *contains* σ_k, written $\sigma_k \preceq^* \sigma_l$, provided σ_k is in the same relative order as a subsequence of σ_l. By Lemma 1.1, $\mathcal{P}_n(\overline{\sigma_k}) \subseteq \mathcal{P}_n(\overline{\sigma_l})$.

Let $\mathcal{B}_n(\overline{\sigma_l})$ be the set of $n \times n$ σ_l-avoiding blockers. If $\sigma_k \preceq^* \sigma_l$, then we have the following important lemma which asserts that any $n \times n$ σ_l-avoiding blocker is also a σ_k-avoiding blocker.

Lemma 2.7 *If $\sigma_k \preceq^* \sigma_l$, then $\mathcal{B}_n(\overline{\sigma_l}) \subseteq \mathcal{B}_n(\overline{\sigma_k})$.* □

Corollary 2.8 *For every permutation σ_k of $\{1, 2, \ldots, k\}$, the maximum cardinality of a minimal σ_k-avoiding blocker in $\mathcal{B}_n(\overline{\sigma_k})$ is at most $\binom{n}{2}$.*

Proof. This is an immediate consequence of Lemma 2.7 and Theorem 2.1. □

Example 2.9 Let $\sigma_{n-1} = (1, 2, \ldots, n-1)$. A σ_{n-1}-avoiding permutation of $\{1, 2, \ldots, n\}$ is one that does not have an increasing subsequence of length $n-1$; an example with $n = 6$ is the permutation $(1, 2, 3, 6, 5, 4)$. Since $(1, 2, \ldots, (n-1)) \preceq^* (1, 2, \cdots, n)$, by Lemma 2.7 a $12\cdots n$-avoiding blocker is also a $12\cdots(n-1)$-avoiding blocker. But e.g. with $n = 5$, we have the following 1234-avoiding blocker

$$\left[\begin{array}{c|c|c|c|c} & & y & y & y \\ \hline & & y & y & y \\ \hline & & & y & y \\ \hline & & & & y \\ \hline & & & & \end{array}\right].$$

The positions of the y's form a 1234-avoiding blocker, since within the complementary positions there are only two permutation matrices and both contain a 1234-pattern. This is not a 12345-avoiding blocker since it does not block the 12345-avoiding permutation $(2, 1, 3, 4, 5)$. This all generalizes easily for arbitrary n and $12\cdots(n-1)$-avoiding permutation matrices giving a $12\cdots(n-1)$-avoiding blocker of cardinality

$$(1 + 2 + \cdots + (n-2)) + (n-2) = \frac{(n-2)(n+1)}{2}.$$

□

The following theorem is important for the main result in the next section.

Theorem 2.10 *Let $n \geq 3$. If a 123-avoiding blocker of $n \times n$ permutation matrices contains the minimum number n of positions, then it must contain one of the positions $(1, n)$ and $(n, 1)$. If the blocker contains position $(1, n)$ (respectively, position $(n, 1)$), then it also contains either the position $(1, n-1)$ or the position $(2, n)$ (respectively, the position $(n, 2)$ or the position $(n-1, 1)$).*

Proof. Consider a 123-avoiding blocker with n positions and suppose it contains neither $(1, n)$ nor $(n, 1)$. Since the 123-avoiding blocker has size n, it must contain a unique position $(k, n+1-k)$ on the Hankel diagonal with $k \neq 1, n$. To formulate our proof, for ease of presentation and understanding, we refer to $n = 10$ but the general argument proceeds in the same way.

Label the positions of each Hankel-cyclic permutation matrix with a different symbol as in (1) for $n = 10$. A 123-avoiding blocker with n positions must use exactly one position on each Hankel-cyclic permutation matrix. Consider that position $(k, n+1-k)$ on the Hankel diagonal in the blocker and color it red. Then the 123-avoiding blocker must contain (i) one of the two positions $(k, n+2-k)$ immediately to its right and $(k-1, n+1-k)$ immediately above (positions with labels i and a and both colored green in (1)), and similarly (ii) one of the two positions $(k+1, n+1-k)$ immediately below and $(k, n-k)$ immediately to the left (also labeled i and a and both colored yellow). Otherwise, there are $(n-2)$ positions on the Hankel diagonal that then give a 123-avoiding permutation matrix; in case (ii), it would be the permutation matrix given by the positions colored yellow in (1).

$$\begin{bmatrix}
a & b & c & d & e & f & g & h & i & j \\
b & c & d & e & f & g & h & i & j & a \\
c & d & e & f & g & h & i & j & a & b \\
d & e & f & g & h & i & j & a & b & c \\
e & f & g & h & i & j & a & b & c & d \\
f & g & h & i & j & a & b & c & d & e \\
g & h & i & j & a & b & c & d & e & f \\
h & i & j & a & b & c & d & e & f & g \\
i & j & a & b & c & d & e & f & g & h \\
j & a & b & c & d & e & f & g & h & i
\end{bmatrix}. \tag{1}$$

Since the 123-avoiding blocker contains only n positions, each symbol can occur in it only once. This implies that the 123-avoiding blocker must have three consecutive positions centered at the red j; the positions i, j, a either vertically or horizontally. Without loss of generality, we assume they are the vertical positions, now all colored red in (2), since the arguments are similar in both instances.

Now consider the 3×3 submatrix C with the red j in its lower left corner in (2):

$$\begin{bmatrix}
a & b & c & d & e & f & g & h & i & j \\
b & c & d & e & f & g & h & i & j & a \\
c & d & e & f & g & h & i & j & a & b \\
d & e & f & g & h & i & j & a & b & c \\
e & f & g & h & i & j & a & b & c & d \\
f & g & h & i & j & a & b & c & d & e \\
g & h & i & j & a & b & c & d & e & f \\
h & i & j & a & b & c & d & e & f & g \\
i & j & a & b & c & d & e & f & g & h \\
j & a & b & c & d & e & f & g & h & i
\end{bmatrix}. \tag{2}$$

If this submatrix C contained a 3×3 123-avoiding permutation matrix then, with the yellow j's there would be a 123-avoiding permutation matrix in (2). The only 3×3 123-avoiding permutation matrix in C not already blocked is the one with color yellow in (3). To block it, we either block one of the two a in yellow or the h in yellow. Since our blocker already has a position labeled a (colored red in (3)), our blocker must contain the yellow h in C. We now have four consecutive positions in our blocker in column k, those labeled h, i, j, a in (3) below starting with the yellow h.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{3}$$

Repeating the above argument on the 3×3 matrix with the red j in the upper right corner, we get that the position labeled b in the column k of the h, i, j, a in (3) and colored yellow is also in our blocker so that now the five consecutive positions labeled h, i, j, a, b in column k are known to be in our blocker. Now proceeding recursively, choosing three consecutive positions in column k of which two are known to be in the blocker, we conclude that all the positions in column k are in the blocker. Thus in an $n \times n$ matrix, if a minimal 123-avoiding blocker of $n \times n$ permutation matrices contains exactly n positions, then it either blocks all $n \times n$ permutation matrices (that is, a row or a column) or it contains the $(1, n)$ or $(n, 1)$ positions.

Suppose it is the $(1, n)$ position that is in our 123-avoiding blocker. Then the blocker must contain either the position $(1, n-1)$ or $(2, n)$. Otherwise with these two positions and the $(n-2)$ positions of the Hankel diagonal in their complement, we have a 123-avoiding $n \times n$ permutation matrix not containing any entry of this blocker. A similar argument holds if the position $(n, 1)$ is in our blocker, $\square$

3 A Universal Theorem for $k \geq 4$

In contrast to $k = 3$ where there are blockers of size n not equal to the set of positions in a row or in a column, we use Theorem 2.10 to prove that for $4 \leq k \leq n$ and any permutation σ_k of $\{1, 2, \ldots, k\}$, the only σ_k-avoiding blockers of minimum cardinality n are those given by rows and columns and, as a result, not only do they block all σ_k-avoiding permutations, they actually block *all* permutations of $\{1, 2, \ldots, n\}$.

Theorem 3.1 *Let σ_k be a permutation of $\{1, 2, \ldots, k\}$ with $4 \leq k \leq n$, then any σ_k-avoiding blocker of size n is either a row or a column.*

The rest of this section is taken up with the proof of this theorem.

By Lemma 2.7, it suffices to prove Theorem 3.1 for $k = 4$, since any such σ_k contains at least one pattern of length 4. There are 24 permutations of $\{1, 2, 3, 4\}$ and using row reversal, column reversal, and rotation of a $(0, 1)$-matrix (which sends a blocker into a blocker of its image) we partition them into seven classes that are equivalent for our arguments. As a result we need to show that Theorem 3.1 holds for just one permutation of each class of (I)-(VII) listed below. We bold the representative of each class that we use.

(I) $\mathbf{1324}, 4231$

(II) $\mathbf{1234}, 4321$

(III) $\mathbf{4123}, 3214, 2341, 1432$

(IV) $\mathbf{1243}, 4312, 2143, 3421$

(V) $\mathbf{1423}, 4213, 4132, 3241, 2431, 1342, 3124, 2314.$

(VI) $\mathbf{2413}, 3142$

(VII) $\mathbf{3412}, 2143$

To give one instance of how these classes arise, if we have an $n \times n$ $(0,1)$-matrix that contains a 1 in the four positions $(i_1, k_1), (i_2, k_2), (i_3, k_3), (i_4, k_4)$ and $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ where (k_1, k_2, k_3, k_4) has the pattern $(4, 2, 3, 1)$, then by taking the columns in the reverse order we get 1's in the positions $(i_1, k_4), (i_2, k_3), (i_3, k_2), (i_4, k_1)$ where $(k_4, k_3, k_2.k_1)$ has the pattern $(1, 3, 2, 4)$. Notice that each of classes (I), (II), (III), (IV), and (V) contain the pattern 123, indeed each pattern within these classes contains either the pattern 123 or 321. Classes (VI) and (VII) contain neither the pattern 123 nor the pattern 321.

We first prove Lemmas 3.2 and Lemma 3.3 which imply that Theorem 3.1 holds for the classes which contain a permutation with a 123 pattern, that is, for the classes (I), (II), (III), (IV), and (V).

Lemma 3.2 *Let σ_k be a permutation of $\{1, 2, \ldots, k\}$ with $4 \leq k \leq n$ that contains one of the patterns* 1234, 4123, 1243, *and* 1423. *Then any σ_k-avoiding blocker of size n is either a row or a column.*

Proof. It suffices to prove that the lemma is true for $k = 4$, that is, for σ_4 equal to one of 1234, 4123, 1243, and 1423 where each of these patterns contains a 123 subpattern. By Theorem 2.10 a 123-avoiding blocker, and thus a σ_4-avoiding blocker of size n, must contain the $(1, n)$ position or the (n, n) position. Using symmetry, we suppose that the $(1, n)$ position is in the blocker and then again by Theorem 2.10, at least one of $(1, n-1)$ and $(2, n)$ is also in the blocker. Using symmetry again, we suppose that $(1, n-1)$ is also in the blocker. The positions $(1, n-1)$ and $(1, n)$ have labels i and j, respectively, and are colored red in (4) below. As is our general procedure, we use the $n = 10$ case to aid in describing and understanding our arguments. The letters in the 10×10 examples correspond to the Hankel-cyclic permutation matrices again.

If the blocker contains all the positions in row 1, we are done. If the element in position a in row 1 is not in the blocker, then the positions with color green in (4) give a σ_4-avoiding permutation matrix given by $(1, n-1, n-2, \ldots, 2, n)$ not meeting the blocker, a contradiction.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{4}$$

Thus we may assume that position $(1, 1)$ with label a is in our blocker. We now locate the first element from the left in row 1 which is not in the blocker, say in position k of row 1. In the illustration (5), it is the green e. So in this case, each of the red a, b, c, d and i, j is in the minimum blocker. Moreover, no more positions with labels a, b, c, d, i, j can be in the minimum blocker.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{5}$$

We now construct a σ_4-avoiding permutation matrix that does not meet the minimum blocker. We do this by identifying the position in row 1 immediately before its first non-blocker position (the position labeled d in this case of $n = 10$). We then take the position in the first row with label e and all the positions of the Hankel-cyclic permutation matrix with labels d except the position in the first and last row, where we take the position immediately preceding it (so in the same column of the first position of row 1 in the blocker):

$$\begin{bmatrix}
a & b & c & d & e & f & g & h & i & j \\
b & c & d & e & f & g & h & i & j & a \\
c & d & e & f & g & h & i & j & a & b \\
d & e & f & g & h & i & j & a & b & c \\
e & f & g & h & i & j & a & b & c & d \\
f & g & h & i & j & a & b & c & d & e \\
g & h & i & j & a & b & c & d & e & f \\
h & i & j & a & b & c & d & e & f & g \\
i & j & a & b & c & d & e & f & g & h \\
j & a & b & c & d & e & f & g & h & i
\end{bmatrix}$$

The result is a σ_4-avoiding permutation matrix given by $(k, k-2, \ldots, 1, n, n-1, \ldots, k+1, k-1)$. This contradiction show that our minimum blocker consists of the n positions of row 1. $\square$

Lemma 3.3 *Let σ_k be a permutation of $\{1, 2, \ldots, k\}$ with $4 \leq k \leq n$ that contains the pattern* 1324*. Then any σ_k-avoiding blocker of size n is either a row or a column.*

Proof. Since the pattern 1324 contains 123 as a subpattern, then as in the proof of Lemma 3.2 we may assume that the blocker contains the two positions $(1, n-1)$ and $(1, n)$. If the blocker contains all the positions in row 1 we are done. Otherwise, we locate the rightmost position in row 1 which is not in the blocker. We use the $n = 10$ case again to help clarify our proof. Referring to (6), we suppose f is this rightmost position so that the positions containing g, h, i, j in row 1 are in the blocker.

$$\begin{bmatrix}
a & b & c & d & e & f & g & h & i & j \\
b & c & d & e & f & g & h & i & j & a \\
c & d & e & f & g & h & i & j & a & b \\
d & e & f & g & h & i & j & a & b & c \\
e & f & g & h & i & j & a & b & c & d \\
f & g & h & i & j & a & b & c & d & e \\
g & h & i & j & a & b & c & d & e & f \\
h & i & j & a & b & c & d & e & f & g \\
i & j & a & b & c & d & e & f & g & h \\
j & a & b & c & d & e & f & g & h & i
\end{bmatrix}. \tag{6}$$

We construct a 1324-avoiding permutation matrix by first proceeding down the southeast diagonal starting from this rightmost position labeled f in (6) stopping just before we arrive at a position whose label is not among the labels of the positions in row 1 known to be in the blocker (b in this case). In general, we obtain positions with every other label in row 1 to the right of its non-blocker position as illustrated in (7) also for the case when the first non-blocker position in row 1 has label c. Using the letter of the position in row 1 above the last position on this southeast diagonal, we complete our permutation matrix using the positions with this letter in the remaining rows (h and f in the two cases illustrated).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a&b&c&d&e&f&g&h&i&j\\ \hline b&c&d&e&f&g&h&i&j&a\\ \hline c&d&e&f&g&h&i&j&a&b\\ \hline d&e&f&g&h&i&j&a&b&c\\ \hline e&f&g&h&i&j&a&b&c&d\\ \hline f&g&h&i&j&a&b&c&d&e\\ \hline g&h&i&j&a&b&c&d&e&f\\ \hline h&i&j&a&b&c&d&e&f&g\\ \hline i&j&a&b&c&d&e&f&g&h\\ \hline j&a&b&c&d&e&f&g&h&i \end{array}\right], \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a&b&c&d&e&f&g&h&i&j\\ \hline b&c&d&e&f&g&h&i&j&a\\ \hline c&d&e&f&g&h&i&j&a&b\\ \hline d&e&f&g&h&i&j&a&b&c\\ \hline e&f&g&h&i&j&a&b&c&d\\ \hline f&g&h&i&j&a&b&c&d&e\\ \hline g&h&i&j&a&b&c&d&e&f\\ \hline h&i&j&a&b&c&d&e&f&g\\ \hline i&j&a&b&c&d&e&f&g&h\\ \hline j&a&b&c&d&e&f&g&h&i \end{array}\right]. \tag{7}$$

If the right most position in the first row which is not in the blocker is the first position with label a, then our construction is illustrated in (8).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a&b&c&d&e&f&g&h&i&j\\ \hline b&c&d&e&f&g&h&i&j&a\\ \hline c&d&e&f&g&h&i&j&a&b\\ \hline d&e&f&g&h&i&j&a&b&c\\ \hline e&f&g&h&i&j&a&b&c&d\\ \hline f&g&h&i&j&a&b&c&d&e\\ \hline g&h&i&j&a&b&c&d&e&f\\ \hline h&i&j&a&b&c&d&e&f&g\\ \hline i&j&a&b&c&d&e&f&g&h\\ \hline j&a&b&c&d&e&f&g&h&i \end{array}\right]. \tag{8}$$

□

Corollary 3.4 *Theorem 3.1 holds for the classes* $(I), (II), (III), (IV)$*, and* (V).

None of permutations in classes (VI) and (VII) contains a 123-pattern. In the next two lemmas we prove that Theorem 3.1 holds for these two classes as well.

Lemma 3.5 *If a* 3412*-avoiding blocker or a* 2413*-avoiding blocker of cardinality* n *of* $n \times n$ *permutation matrices contains both the* $(1, 1)$ *and* $(1, n)$ *positions, then the blocker is the first row.*

Proof. Suppose there is a blocker containing both positions $(1, 1)$ and $(1, n)$, and it is not the first row. First, we locate the left most entry in the first row not in the blocker. In the 10×10 example shown, we suppose that d is this entry, implying that a, b and c in the first row are in the blocker and hence all other a's, b's, and c's are not in the blocker. Then we can construct a permutation matrix which avoids both 3412 and 2413 as in the following example:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]$$

. □

Corollary 3.6 *If a 3412-avoiding blocker or a 2413-avoiding blocker of cardinality n is not a row or a column, then it does not contain any pair of the following positions:*

1. $(1,1)$ *and* $(1,n)$
2. $(1,1)$ *and* $(n,1)$
3. $(1,n)$ *and* (n,n)
4. $(n,1)$ *and* (n,n)

Proof. Since both 3412-avoiding and 2413-avoiding are preserved by reflecting through the diagonal and Hankel diagonal and rotating 180°, Lemma 3.5 implies the corollary. □

Lemma 3.7 *Let $4 \leq k \leq n$ and σ_k be a permutation with length k containing either the pattern 2413 or the pattern 3412. Then any σ_k-avoiding blocker of cardinality n of $n \times n$ permutation matrices is the set of n positions in a row or column.*

Proof. It is sufficent to prove that the statement is true for $k = 4$. Since the proofs for the two patterns are very similar, we treat them simultaneously. We use induction on n.

For $n = 4$, suppose that there is a 2413-avoiding blocker (respectively, 3412-avoiding blocker) of size 4 which is not a row or column. It then follows from the Frobenius-König theorem that there exists some permutation matrix that does not use any position in the blocker. If this permutation is not $(2,4,1,3)$-avoiding (respectively, $(3,4,1,2)$-avoiding), then it must be the permutation matrix corresponding to the permutation $(2,4,1,3)$ (respectively, $(3,4,1,2)$), a contradiction. So we assume it is $(2,4,1,3)$ (respectively, $(3,4,1,2)$) and none of the positions

$(1,2),(2,4),(3,1),(4,3)$ (respectively, $(1,3),(2,4),(3,1),(4,2)$) is in the blocker, that is, the positions of the 0's in the respective matrices:

$$\left[\begin{array}{c|c|c|c} c & 0 & b & a \\ \hline e & a & d & 0 \\ \hline 0 & c & f & e \\ \hline f & b & 0 & d \end{array}\right] \text{ and } \left[\begin{array}{c|c|c|c} c & f & 0 & a \\ \hline d & e & a & 0 \\ \hline 0 & b & c & d \\ \hline b & 0 & f & e \end{array}\right].$$

For each of a,b,c,d,e,f at least one of the two occurrences must be in the blocker, for otherwise we get a 2413-avoiding (respectively, $(3,4,1,2)$-avoiding) permutation matrix using the two positions with the same letter. Thus a 2413-avoiding blocker (respectively, $(3,4,1,2)$-avoiding blocker) contains at least 6 positions, a contradiction.

From now on we use the word 'blocker' in referring to either a 2413-avoiding blocker or 3412-avoiding blocker. So suppose that the lemma is true for all $n \leq m$. Thus for any $n \leq m$, the only blockers of size n are either a row or a column.

We need to prove the statement is true for $n = m+1$. Note that none of the corner positions $(1,1),(1,n),(n,1)$ and (n,n) can be used to create a 2413 or 3412 pattern in a permutation. So if there is a position in the blocker in row 1, row n, column 1, or column n other than these four positions (that is, in one of the positions in yellow letters in (9)), we may put a 1 in one of these four corner positions as part of a permutation matrix and obtain an $m \times m$ submatrix by removing the row and the column containing that corner position. According to Corollary 3.6, at least one of the two corner positions in row 1, row n, column 1 and column n is not in the blocker and hence we can always find a position for a 1 in a possible permutation matrix that avoids our patterns.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{9}$$

Case 1: There is an element in our blocker B in one of the yellow positions, say, a position in the last column.

For example, if the red d in (10) is in the blocker B, then we consider the corner positions in the column of d, that is, the positions occupied by j and i in the last column. According to Corollary 3.6, at least one of these positions is not in the blocker. Suppose e.g. the position with an i is not in the blocker. Then we remove the last column and the last row to obtain an $m \times m$ matrix C in which by induction its corresponding blocker B' has cardinality at most m.

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c||c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline\hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{10}$$

If there is a σ-avoiding permutation matrix Q contained in C not meeting B, then with the 1 in position (n, n) we obtain a σ-avoiding permutation matrix. Otherwise, by the inductive hypothesis, the set B' of positions of the blocker B contained in C form either a row or column of C. Since the cardinality of B is $n = m + 1$, such a row or column of C cannot also contain the letter d. Thus B' is either the (i) row of C corresponding to the position of d in the column n or (ii) the column of C corresponding to the position of d in row n. If (i) then our blocker B is a row. Now suppose that (ii) holds, that is, B is given by the positions in red in (11).

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c||c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline\hline j & a & b & c & d & e & f & g & h & i \end{array}\right]. \tag{11}$$

Now consider the $m \times m$ matrix C' obtained by deleting the first row and last column. Since C' contains only $n - 2$ positions of B, the inductive assumption implies that C' contains a σ-avoiding permutation matrix which remains σ-avoiding by including the 1 in the position j in the upper right corner, a contradiction.

Case 2: No position in our blocker is in one of the non-corner positions in rows 1 and n, and columns 1 and n, that is, the yellow positions in (9).

We then choose the most northwest (NW) Hankel diagonal containing a position of the blocker (so by the assumptions in this case, it is not the first or second Hankel diagonal), and choose that position.

$$
\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{12}
$$

Suppose, as shown in (12), the red f is the chosen position in the blocker (note again that our assumptions in this case imply that it is not a position occupied by an a or b); thus all other positions containing an f are not in the blocker and all positions other than the $(1,1)$ position NW of this red f are possible. To construct the 2413-avoiding (resp. 3412-avoiding) matrix, we first take the longest Hankel diagonal in the left top corner above the position occupied by f as in (13), thus the Hankel diagonal (of a $t \times t$ principal submatrix) above this NW position (the positions occupied by e's in (12).

Consider the complementary principal $(n-t) \times (n-t)$ submatrix in the right bottom corner ($n-t=5$ in the illustration (13)). If there is a $(n-t) \times (n-t)$ 2413-avoiding (respectively, 3412-avoiding) permutation matrix contained in this submatrix, then with the green e's, we have an $n \times n$ 2413-avoiding (respectively, 3412-avoiding) permutation matrix (see (13)).

$$
\left[\begin{array}{c|c|c|c|c||c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline\hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{13}
$$

So suppose now that there does not exist a 2413-avoiding (respectively, 3412-avoiding) permutation matrix contained in this $(n-t) \times (n-t)$ submatrix on the bottom right. Then by induction there are at least $(n-t)$ positions of our blocker in this submatrix. In this case, take the Hankel diagonal of the $(n-t-1) \times (n-t-1$ principal submatrix in the lower right (all the positions of the f's in (14) and consider the $(t+1)\times(t+1)$ complementary submatrix D on the upper left in (14).

$$\left[\begin{array}{c|c|c|c|c|c||c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline\hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right]. \tag{14}$$

If we are able to construct a $(t+1)\times(t+1)$ 2413-avoiding (respectively, 3412-avoiding) permutation matrix in D, then with the Hankel diagonal in the lower $(n-t-1)\times(n-t-1)$ principal submatrix (the positions of the green f's in (13)), we have a 2413-avoiding (respectively, 3412-avoiding) permutation matrix of size n. Note that since the $(n-t-1)\times(n-t-1)$ submatrix on the bottom right contains at least $(n-t-1)$ positions of our blocker, the $(t+1)\times(t+1)$ submatrix D contains at most $(t+1)$ positions of our blocker. To block all 2413-avoiding permutation matrix in this $(n-t+1)\times(n-t+1)$ submatrix, these $(n-t+1)$ positions must form a row or column by induction. But that is a contradiction since there are no positions in our blocker in the positions of the yellow cells.

The proof of Theorem 3.1 is now complete. □

4 Coda

In this section we discuss a number of additional properties of blockers of 123-avoiding permutation matrices and possible directions for further development. Recall that the sequence $(C_n : n \geq 1)$ of Catalan numbers [7] is:

$$C_n = \frac{\binom{2n}{n}}{n+1}, (n \geq 0) : 1, 1, 2, 5, 14, 21, 42, 132, \ldots.$$

The Catalan numbers satisfy the recursion

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ where } C_0 = 1.$$

Let $H_n = [h_{ij}]$ be the $n\times n$ matrix where h_{ij} equals the number of 123-avoiding $n\times n$ permutation matrices with a 1 in position (i,j), $1 \leq i,j \leq n$. Thus the row and column sums of H_n equal the total number of 123-avoiding $n\times n$ permutation matrices, that is, the Catalan number C_n. The matrix H_n is symmetric, Hankel-symmetric, and centro-symmetric. Its rows and columns are blockers of the 123-avoiding $n\times n$ permutation matrices. (If there other blockers whose corresponding entries in H_n add to C_n for all n, then the positions of these entries have the property that every 123-avoiding $n \times n$ permutation matrix has a 1 in exactly one of these positions.)

Example 4.1 In the case $n = 4$, we have $C_4 = 14$ and

$$H_4 = \left[\begin{array}{c|c|c|c} 1 & 3 & 5 & 5 \\ \hline 3 & 4 & 2 & 5 \\ \hline 5 & 2 & 4 & 3 \\ \hline 5 & 5 & 3 & 1 \end{array}\right].$$

The entries in each row and column sum to $C_4 = 14$ but so do the entries on the Hankel diagonal also sum to $C_4 = 14$ (note that in this case these entries are also Catalan numbers). □

Theorem 4.2 *Let H_n be the $n \times n$ matrix whose (i, j)-entry equals the number of 123-avoiding $n \times n$ permutation matrices with a 1 in position (i, j), $1 \leq i, j \leq n$. Then the sum of the entries of H_n on its Hankel diagonal equals the Catalan number C_n.*

Proof. Consider a position $(i, n+1-i)$ on the Hankel diagonal of H_n and let $P = [p_{ij}]$ be a 123-avoiding $n \times n$ permutation matrix with $p_{i,n+1-i} = 1$. Then not both the $(i-1) \times (j-1)$ and $(n-i) \times (n-j)$ submatrices of P above, left and below, right can contain a 1, and then this implies both must be zero matrices. Hence the number of 123-avoiding $n \times n$ permutation matrices with a 1 in position $(i, n+1-i)$ equals $C_{i-1}C_{n-i}$. The recurrence relation for the Catalan numbers now completes the proof. □

A characterization of *minimal* 123-avoiding blockers would be of interest as would the *maximum* number of 0's in a minimal blocker of 123-avoiding $n \times n$ permutation matrices. Some examples of blockers of 123-avoiding permutation matrices are below, and these illustrate the many possibilities:

- $n = 6$

$$\left[\begin{array}{c|c|c|c|c|c} & & y & y & y & y \\ \hline & & & & & y \\ \hline & & & & & y \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & y & y & y & y \\ \hline & & & & y & y \\ \hline & & & & y & y \\ \hline & & & & & \\ \hline & & & & & \end{array}\right], \quad \left[\begin{array}{c|c|c|c|c|c} & & & & y & y \\ \hline & & & & & y \\ \hline & & & & & \\ \hline & y & y & y & y & y \\ \hline & & & & y & y \\ \hline & & & & & \end{array}\right].$$

- $n = 7$

$$\left[\begin{array}{c|c|c|c|c|c|c} & & & & & y & y \\ \hline & & & & & & y \\ \hline & & & & & & \\ \hline & y & y & & & y & y \\ \hline & & & & & y & y \\ \hline & & & & & & \\ \hline & & & & & & \end{array}\right].$$

- $n = 8$

$$\left[\begin{array}{c|c|c|c|c|c|c|c} & & & & & y & y & y \\ \hline & & & & & & & y \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & y & y & y & y & y & y \\ \hline & & & & & y & y & y \\ \hline & & & & & y & y & y \\ \hline & & & & & & & \end{array}\right]$$

For $k = 3$ and $\sigma_3 = (1, 2, 3)$, we now obtain a lower bound on the maximum size of a σ_3-avoiding blocker.

Theorem 4.3 *Let $n \geq 3$ and let $\sigma_3 = (1, 2, 3)$. Then there is a minimal σ_3-avoiding blocker B_n of $n \times n$ permutation matrices. of size*

$$\left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil .$$

This blocker B_n is a blocker of the entire set $\mathcal{P}_n$ of $n \times n$ permutation matrices.

Proof. As before there are only two possibilities to consider, namely, $\sigma_3 = 123$ or 132. We again illustrate the argument with $n = 10$ which easily generalizes for arbitrary n. We make use of the Hankel-cyclic decomposition of J_{10}. Let B_n be the set of positions in the lower right $\lfloor \frac{n+1}{2} \rfloor \times \lceil \frac{n+1}{2} \rceil$ submatrix designated with yellow below. By the Frobenius-Kőnig theorem, this set B_n of positions blocks all of $\mathcal{P}_n$:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c} a & b & c & d & e & f & g & h & i & j \\ \hline b & c & d & e & f & g & h & i & j & a \\ \hline c & d & e & f & g & h & i & j & a & b \\ \hline d & e & f & g & h & i & j & a & b & c \\ \hline e & f & g & h & i & j & a & b & c & d \\ \hline f & g & h & i & j & a & b & c & d & e \\ \hline g & h & i & j & a & b & c & d & e & f \\ \hline h & i & j & a & b & c & d & e & f & g \\ \hline i & j & a & b & c & d & e & f & g & h \\ \hline j & a & b & c & d & e & f & g & h & i \end{array}\right]$$

This set of positions is a minimal blocker of the 123-avoiding permutation matrices. This follows from the fact that given any position in B_n there is a 123-avoiding permutation matrix containing that position and no other position in B_n which otherwise contains only positions on the Hankel diagonal (labeled with j's) and the diagonal of length $(n-1)$ above it (labeled with i's):

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
a & b & c & d & e & f & g & h & i & j \\ \hline
b & c & d & e & f & g & h & i & j & a \\ \hline
c & d & e & f & g & h & i & j & a & b \\ \hline
d & e & f & g & h & i & j & a & b & c \\ \hline
e & f & g & h & i & j & a & b & c & d \\ \hline
f & g & h & i & j & a & b & c & d & e \\ \hline
g & h & i & j & a & b & c & d & e & f \\ \hline
h & i & j & a & b & c & d & e & f & g \\ \hline
i & j & a & b & c & d & e & f & g & h \\ \hline
j & a & b & c & d & e & f & g & h & i
\end{array}\right].$$

□

Conjecture 4.4 Let σ_3 be a permutation of $\{1,2,3\}$. The maximum cardinality of a minimal σ_3-avoiding blocker of $n \times n$ permutation matrices is

$$\left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Let A be an $n \times n$ $(0,1)$-matrix and σ_k a permutation of $\{1,2,\ldots,k\}$. If A does not contain a σ_k-avoiding permutation matrix, then its 0's determine a blocker of σ_k-avoiding $n \times n$ permutation matrices. This blocker contains at least one minimal blocker. Conversely, if A contains a σ_k-avoiding permutation matrix. Then its 0's do not contain a blocker. Thus: *A does not contain a σ_k-avoiding permutation matrix if and only if its zeros are in positions containing a minimal blocker.* So the analogue of the Frobenius-König theorem for σ_k-avoiding permutation matrices rests on determining all minimal blockers of σ_k-avoiding permutation matrices. In the proof of Theorem 4.3 we have exhibited only one such minimal blocker but conjectured to be of maximum size.

As a referee suggested, one can also consider blockers that simultaneously avoid several patterns (see e.g. [1]), mesh patterns (see e.g. [3]), and more general patterns (see e.g. [4]). This referee also raised the following very interesting question:

Question 4.5 Let B be an $n \times n$ $(0,1)$-matrix. Determine the maximal subset $\mathcal{Q}_n$ of the set $\mathcal{P}_n$ of $n \times n$ permutation matrices such that B blocks all permutation matrices in $\mathcal{Q}_n$.

The matrix in Example 2.6

$$\left[\begin{array}{c|c|c|c|c|c} & & y & y & y & y \\ \hline & & & & & \\ \hline & & & & y & \\ \hline & & & & & \\ \hline & & & y & & \\ \hline & & & & & \end{array}\right]$$

is a 123-avoiding blocker of 6×6 permutation matrices but it also blocks permutation matrices corresponding to permutations $(i_1, i_2, \ldots, i_6)$ where $i_1 \in \{3, 4, 5, 6\}, i_3 = 5$ or $i_5 = 4$. $\square$

We conclude with a discussion concerning patterns in derangements matrices $J_n - I_n$.

Example 4.6 Consider the derangement matrix $J_n - I_n$, e.g. with $n = 5$,

$$J_5 - I_5 = \left[\begin{array}{c|c|c|c|c} 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

Then every pattern of length $n-1$ occurs in some permutation matrix $P \leq J_n - I_n$ as argued below. Notice that by definition we get all patterns $i_1 i_2 \cdots i_n$ of length n other than those having at least one $i_j = j$ for some j.

Step 1. For a given pattern $1, 4, 3, 7, 6, 2, 5$ of size $n - 1 = 7$, construct the corresponding permutation matrix.

$$\left[\begin{array}{c|c|c|c|c|c|c} 1 & & & & & & \\ \hline & & & 1 & & & \\ \hline & & 1 & & & & \\ \hline & & & & & & 1 \\ \hline & & & & & 1 & \\ \hline & 1 & & & & & \\ \hline & & & & 1 & & \end{array}\right]$$

Step 2. Shifting the lower triangular part (excluding the diagonal, green in this example) down by 1 unit. The shifting does not change the pattern.

$$\left[\begin{array}{c|c|c|c|c|c|c} 1 & & & & & & \\ \hline 0 & & & 1 & & & \\ \hline & 0 & 1 & & & & \\ \hline & & 0 & & & & 1 \\ \hline & & & 0 & & 1 & \\ \hline & & & & 0 & & \\ \hline & 1 & & & & 0 & \\ \hline & & & & 1 & & 0 \end{array}\right]$$

Step 3. Add a column from the left and put a 1 in the row which does not have a 1. Note that the first row must have a 1 since it contains the first row of the $n-1$ by $n-1$ permutation completely.

$$\begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & & & 1 & & & \\ & & 0 & 1 & & & & \\ & & & 0 & & & & 1 \\ & & & & 0 & & 1 & \\ 1 & & & & & 0 & & \\ & & 1 & & & & 0 & \\ & & & & & 1 & & 0 \end{bmatrix}$$

This works in general: $J_n - I_n$ contains a permutation with any pattern of length $n-1$. $\square$

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References

[1] D. Bevan, The permutation classes $Av(1234), i2341)$ and $Av(1243, 2314)$, *Australas. J. Combin.* 64 (2016), 3–20.

[2] M. Bóna, *Combinatorics of Permutations*, 2nd ed., CRC Press 2012, Chapter 4.

[3] R. A. Brualdi and L. Cao, Pattern-avoiding $(0,1)$-matrices and bases of permutation matrices, *Discrete Appl. Math.* 304 (2021), 196–211.

[4] A. Claesson, Generalized pattern avoidance, *European J. Combin.* 22(7) (2001), 961–971.

[5] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Prog.* 1 (1971), 168–194.

[6] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley-Wilf Conjecture, *J.Combin. Theory Ser. A* 107 (2004), 153–160.

[7] R. P. Stanley, *Catalan Numbers*, 2nd ed., Cambridge University Press, Cambridge, 2015.

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