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Characterizing posets with more linear extensions than ideals

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Abstract

Two of the most important invariants associated with a poset P are the number of linear extensions, $e(P)$, and the number of order ideals, $i(P)$. Many important techniques to generate random linear extensions assume that $e(P) \geq i(P)$ and consequently choose to deal with ideals instead of linear extensions. However, this condition does not hold for every poset. In this paper we characterize when this condition holds for chain-irreducible posets, providing a complete list of posets where this fails. The proof is divided into three parts: for non-connected posets, for connected posets whose width exceeds 2, and for connected posets with width 2. We also give some applications of this result.

1 Introduction

Consider a finite partially ordered set (briefly, poset) $(P, \preceq)$. There are several invariants that can be associated with P, such as height, width and so on. Perhaps the two most important invariants in terms of mathematical properties and practical applications are the number of linear extensions and the number of (order) ideals.

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Counting linear extensions of a general poset is a $\#P$-complete problem [2], and the same is true for generating random linear extensions. For this reason, finding formulas for solving these problems for a particular family of posets is an interesting and relevant problem [3, 9, 11, 12, 13]. The same can be said for the number of ideals of a poset [4, 14, 23]. For example, if we consider the Boolean poset over a referential of n elements, it can be proved that the number of ideals coincides with the nth Dedekind number, and no simple formula is known to derive this number [15].

The difficulty of these problems is due to the fact that both the number of linear extensions and the number of ideals usually grow very fast when the cardinality of the poset increases. However, it seems that "in general", the number of linear extensions grows faster [16]. The relationship between the number of ideals and the number of linear extensions of a poset has been studied by using computational techniques (see [16]).

In this paper we characterize what "in general" means. More concretely, we determine which posets satisfy the property that the number of linear extensions exceeds the number of ideals.

Note that given a poset, if we add to this poset a chain, then the number of linear extensions remains the same while the number of ideals grows. Hence, any poset can be "turned" into a poset with more ideals than linear extensions. As this case is trivial, we have focused on the case of posets that cannot be written as sums of other posets and chains.

On the other hand, it seems that the width of the poset should have an influence on the answer to the question. Indeed, "in general," if the width is large, there are more linear extensions than ideals.

The main motivation of this research is the justification of many algorithms that generate random linear extensions. The existence of an easy bijection between linear extensions of P and maximal chains of the lattice of ideals of P is used by many algorithms to generate and simulate linear extensions. Consider for example the algorithm proposed in [16], that follows the following steps:

- Build the ideal lattice $\mathcal{I}(P)$ of P.
- Select randomly a maximal chain $\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{|P|} = P$ in $\mathcal{I}(P)$. We can do this by defining $I_{i+1} = I_i \cup \{x\}$, where $x \in \mathcal{MIN}(P \setminus I_i)$ is chosen at random. Equivalently, this can be done by selecting a path in the Hasse diagram between $\emptyset$ and P in $\mathcal{I}(P)$.
- Consider the corresponding linear extension ϵ given by $\epsilon(i) = I_{i+1} \setminus I_i$.

This algorithm has been applied in other papers such as [6, 17].

The other algorithm uses "conditional probabilities" of elements given an ideal, that represent the proportion of linear extensions satisfying that I appears in the first positions and x is assigned to the next position among all linear extensions such that I appears in the first positions. Then, starting with $I = \emptyset$, the algorithm selects minimal elements with probability given by the conditional probability, then adds the selected element to I and repeats this step until $I = P$. An application of this algorithm appears in [10] for generating points in the polytopes of 3-tolerant measures and other polytopes appearing in Decision Making with fuzzy measures.

These algorithms need to know in advance the set of ideals in order to randomly generate a linear extension. One could think that when the number of ideals is greater than the number of linear extensions, it is better to spend energy on counting linear extensions rather than ideals. We will see that the strategy of enumerating ideals in advance is well justified for a vast majority of posets.

The rest of the paper goes as follows. In the next section we introduce the notation and basic results that will be needed in the paper. In Section 3 we establish the main result of the paper, where we characterize the posets with more linear extensions than ideals. We have called these posets abundant posets, and the proof of this result is given in Section 5. In Section 4 we give several applications to other branches of mathematics.

2 Basic concepts

Let us begin with a short survey of Order Theory (see [7]) in order to introduce the notation that will be used in the paper. Let P be a finite set with p elements. Elements of P are denoted x, y and z and subsets of P are denoted by capital letters A, B, and so on. Over P we consider a binary relation $\preceq$ satisfying

(i) Reflexivity: $x \preceq x$, $\forall x \in P$.

(ii) Antisymmetry: If $x \preceq y$ and $y \preceq x$, then $x = y$, $\forall x, y \in P$.

(iii) Transitivity: If $x \preceq y$ and $y \preceq z$, then $x \preceq z$, $\forall x, y, z \in P$.

The pair $(P, \preceq)$ is a **partially order set** (or **poset** for short). With some abuse of notation, we will usually omit $\preceq$ and write P instead of $(P, \preceq)$ when referring to posets. For a poset P, we can define the **dual poset** $P^{\partial} = (P, \preceq_{\partial})$ such that $x \preceq_{\partial} y \Leftrightarrow y \preceq x$.

If $x \not\preceq y$ and $y \not\preceq x$, we write $x \parallel y$. We say that y **covers** x, denoted $x \lessdot y$, if $x \preceq y$ and there is no $z \in P \setminus \{x, y\}$ satisfying $x \preceq z \preceq y$.

A poset can be represented through *Hasse diagrams*. In Figure 1 we can see the Hasse diagram of two posets shaped like the letters "N" and "V" respectively, so we will name them after these letters.

If $x \in P$ satisfies that $x \not\succeq y$, for all $y \in P, y \neq x$, then x is a **minimal element**. The set of minimal elements of P is denoted by $\mathcal{MIN}(P)$. Similarly, if $x \in P$ satisfies $x \not\preceq y$, for all $y \in P, y \neq x$, then x is a **maximal element** and we denote the set of maximal elements of P by $\mathcal{MAX}(P)$.

A poset is a **chain** if $x \preceq y$ or $y \preceq x$, for all $x, y \in P$. We will denote the generic chain of n elements by $\boldsymbol{n}$; similarly, an **antichain** is a poset where $\preceq$ is given by

Figure 1: Hasse diagram of poset N (left) and V (right).

$x \preceq y$ if and only if $x = y$. We will denote the generic antichain of n elements by $\bar{\boldsymbol{n}}$. In this paper we admit the empty set as an antichain. We denote by $\mathcal{A}(P)$ the set of antichains of P and $a(P) := |\mathcal{A}(P)|$. A chain $C \subseteq P$ is said to be a **maximal chain** in P if there is no other different chain C' such that $C \subset C'$. Symmetrically, we can define **maximal antichains**. The **height** of P, denoted by $h(P)$, is defined as the cardinality of a longest chain in P. Similarly, the **width** of P, denoted by $w(P)$, is defined as the cardinality of a largest antichain in P.

Given an element x, we denote

$$\downarrow x := \{y : y \preceq x\}, \quad \uparrow x := \{y : x \preceq y\}, \quad \updownarrow x := \{y : x \preceq y \text{ or } y \preceq x\}.$$

An **ideal** or **down-set** I of P is a subset of P such that if $x \in I$, then $\downarrow x \subseteq I$. We will denote the set of all ideals of P by $\mathcal{I}(P)$ and $i(P) := |\mathcal{I}(P)|$. Symmetrically, a subset F of P is a **filter** or **up-set** if for any $x \in F$, then $\uparrow x \subseteq F$. We will assume that P and the empty set are both filters and ideals; therefore $\mathcal{I}(P)$ and $\mathcal{F}(P)$ have both maximum and minimum. One of the most important constructions in order theory is the **poset of ideals** ordered by inclusion, $(\mathcal{I}(P), \subseteq)$. It is easy to show that for a finite poset P,

$$i(P) = a(P), \tag{1}$$

via the bijective map $f : \mathcal{I}(P) \to \mathcal{A}(P)$ given by $f(I) = \mathcal{MAX}(I)$.

Two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ are **isomorphic** if there is a bijection $f : P \to Q$ such that $x \preceq_P y$ if and only if $f(x) \preceq_Q f(y)$, and this is denoted by $P \cong Q$ (or $P = Q$). If two posets are isomorphic, then their corresponding Hasse diagrams are the same up to differences in the names of the elements.

Now, let us introduce some important ways of defining new posets from old. Given two posets, $(P, \preceq_P)$, $(Q, \preceq_Q)$, their **ordinal sum**, denoted $P \oplus Q$, is a poset such that $x \preceq_{P \oplus Q} y$ for every $x \in P$ and $y \in Q$ and preserves the original orders on P and Q. We remark that the ordinal sum of posets is associative but not commutative (see Figure 2). A poset is **irreducible** if it cannot be written as an ordinal sum of two posets. For example, poset N in Figure 1 is irreducible, while poset V is reducible as it can be written as $V = \mathbf{1} \oplus \bar{\mathbf{2}}$.

Definition 2.1. Let P be a finite poset such that $P = P_1 \oplus \cdots \oplus P_k$ where P_i is an irreducible poset for $i = 1, \ldots, k$. We denote by $\Phi(P)$ the number of irreducible components isomorphic to the chain with one element, i.e. $P_i \cong \mathbf{1}$. We say that P is **chain-irreducible** if $\Phi(P) = 0$. We also define the **chain-irreducible reduction** of P as: $\mathfrak{R}(P) := \bigoplus_{\substack{i=1 \\ P_i \ncong \mathbf{1}}}^{n} P_i$.

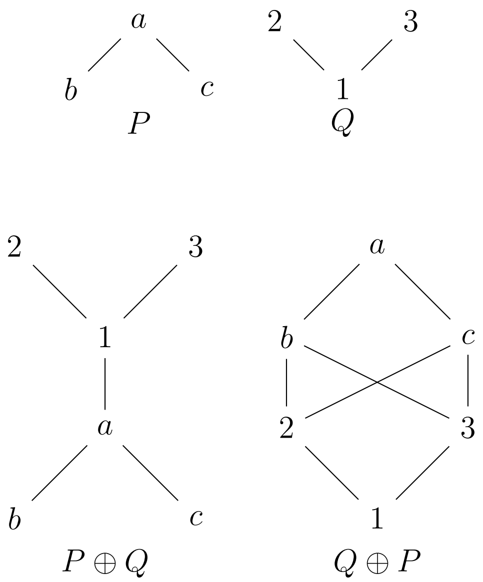


Figure 2: Ordinal sum of posets.

Note that a poset P is chain-irreducible if and only if every element of P is in some antichain with at least two elements. Obviously, if P is irreducible and $|P| > 1$, then P is chain-irreducible. The case $P = \mathbf{1}$ is trivially irreducible and chain-reducible. For example, V is chain-reducible (and then reducible), $\bar{\mathbf{2}} \oplus \bar{\mathbf{2}}$ is reducible and chain-irreducible, and poset N is irreducible (and then chain-irreducible).

Similarly, the **disjoint union** of two posets $(P, \preceq_P)$, $(Q, \preceq_Q)$, denoted $P \uplus Q$, is a poset $(P \cup Q, \preceq_{P \uplus Q})$ where $x \preceq_{P \uplus Q} y$ whenever $x, y \in P$ and $x \preceq_P y$, or $x, y \in Q$ and $x \preceq_Q y$. The disjoint union is commutative and associative (see Figure 3). A poset which cannot be written as the disjoint union of two posets is called **connected**. Obviously, the Hasse diagram of a connected poset is also a connected graph. A trivial property is that a non-connected poset is chain-irreducible.

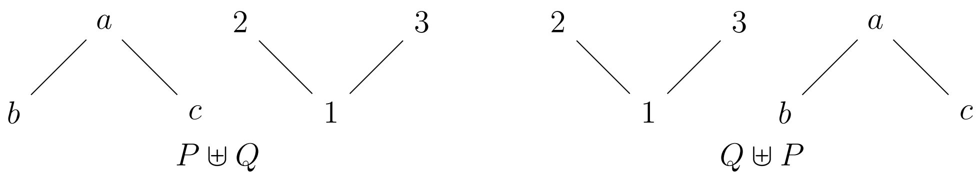


Figure 3: Disjoint union of posets.

Finally, we introduce a definition regarding the height of P. Remember that Dilworth's Theorem states that every poset P of width $w(P) = k$ can be split into k chains.

Theorem 2.1 (**Dilworth** [8]). *Let P be a finite poset of width $w(P) = k$. Then there exists a partition of P into k chains, that is, $P = C_1 \cup \cdots \cup C_k$ where C_i is a chain for all $i \in \{1 \ldots k\}$ and $C_i \cap C_j = \emptyset$, for all $i \neq j$.*

Definition 2.2. Let P be a finite poset with $w(P) = 2$, and consider all possible partitions of P into two chains:

$$\mathcal{CP}(P) := \{(C_1, C_2) : \text{ chain partition of } P \text{ where } |C_1| \geq |C_2|\}.$$

Let (C_1^*, C_2^*) be a partition in $\mathcal{CP}(P)$ where $|C_1^*|$ is a maximum among all the partitions (C_1, C_2). We define the **type 1 height** $h_1(P) := |C_1^*|$ and **type 2 height** $h_2(P) := |C_2^*|$.

Example 2.1. Consider the poset $Q \oplus P$ from Figure 2. Then, $Q \oplus P$ can be decomposed into chains $1{-}2{-}b$ and $3{-}c{-}a$. Another decomposition is $1{-}2{-}b{-}a$ and $3 - c$. This is indeed the decomposition (C_1^*, C_2^*) of Definition 2.2. Hence $h_1(P) = 4$ and $h_2(P) = 2$.

Note that heights $h_1(P)$ and $h_2(P)$ are well-defined and they do not depend on the chosen partition.

Definition 2.3. A **linear extension** of $(P, \preceq)$ is a sorting of the elements of P that is compatible with $\preceq$, i.e. $x \preceq y$ implies that x is before y in the sorting. In other words, if $|P| = n$, then a linear extension is an order-preserving bijection $\epsilon : P \to \boldsymbol{n}$.

We will denote by $\mathcal{L}(P)$ the set of all linear extensions of poset $(P, \preceq)$ and by $e(P) := |\mathcal{L}(P)|$. In a finite poset P, $e(P)$ equals the number of maximal chains of $(\mathcal{I}(P), \subseteq)$ [22]. This result is the starting point of some algorithms to randomly generate linear extensions [16]. The goal of this paper is to find conditions for a poset P to satisfy $i(P) \leq e(P)$.

The next lemma shows how $i(P)$ and $e(P)$ behave with respect to ordinal sum and disjoint union.

Lemma 2.1. [7, 22] *Let P and Q be two non-empty finite posets.*

(i) $i(P \oplus Q) = i(P) + i(Q) - 1$.

(ii) $i(P \uplus Q) = i(P) \cdot i(Q)$.

(iii) $e(P \oplus Q) = e(P) \cdot e(Q)$.

(iv) $e(P \uplus Q) = \binom{|P|+|Q|}{|P|} \cdot e(P) \cdot e(Q)$.

Let us now introduce some basic concepts about lattice theory. These concepts will be needed in the section on applications. Given a poset P, we can define

$$x \vee y := \min\{z \in P \mid z \succeq x,\ z \succeq y\}, \quad x \wedge y := \max\{z \in P \mid z \preceq x,\ z \preceq y\},$$

when these values exist. More generally, for a general subset $S \subseteq P$ we can define

$$\bigvee S := \min\{z \in P \mid z \succeq x, \forall x \in S\}, \quad \bigwedge S := \max\{z \in P \mid z \preceq x, \forall x \in S\},$$

when these values exist.

Definition 2.4. Let P be a non-empty poset. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a **lattice**.

Let L and K be lattices. A function $f : L \to K$ is a **lattice homomorphism** if

$$f(x \vee_L y) = f(x) \vee_K f(y), \quad f(x \wedge_L y) = f(x) \wedge_K f(y), \forall x, y \in L.$$

A bijective lattice homomorphism is a **lattice isomorphism**. An element x of a lattice L is said to be **join-irreducible** if x is not a minimum and $x = a \vee b$ implies $x = a$ or $x = b$. A **meet-irreducible** element is defined dually. The set of join-irreducible elements of a lattice L is denoted by $\mathcal{J}(L)$. A lattice L is said to be **distributive** if it satisfies the distributive law,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \ \forall x, y, z \in L.$$

Theorem 2.2 (**Birkhoff's representation theorem for finite lattices** [7]). *Let L be a finite distributive lattice. Then the map $\eta : L \to \mathcal{I}(\mathcal{J}(L))$, $x \mapsto \ \mathcal{J}(L) \cap \downarrow x$ is an isomorphism between L and $\mathcal{I}(\mathcal{J}(L))$.*

In this way, for any distributive lattice L, all the information is concentrated in the poset $\mathcal{J}(L)$. Note that the number of elements of $\mathcal{J}(L)$ is in general much lower than the cardinality of L.

Definition 2.5. We say that a finite poset P is **abundant** if $i(P) \leq e(P)$. Otherwise, we say that P is **deficient.**[1]

The set of abundant finite posets is denoted by $\mathfrak{A}$, and the set of deficient finite posets by $\mathfrak{D}$.

The first problem we face when characterizing abundant posets is the possibility of encountering a poset that can be written as a ordinal sum of a poset and a chain.

Theorem 2.3. *Let P be a finite poset. Then there exists $m \in \mathbb{N}$ such that $\boldsymbol{m} \oplus P$ is deficient.*

Proof. By Lemma 2.1 note that $e(\boldsymbol{m} \oplus P) = e(P)$ but $i(\boldsymbol{m} \oplus P) = i(P) + m$. It is enough to choose $m > e(P) - i(P)$. □

In this way, we can always add large enough chains to a poset such that the new poset is deficient. However, from a combinatorial point of view, adding or removing chains as ordinal summands does not change most of the combinatorial structure of the poset. For this reason we will focus on working with chain-irreducible posets.

[1] This notation is inspired by number theory. Remember that a number is said to be abundant if the sum of its divisors is greater than the number itself; otherwise it is said to be deficient.

3 Characterization of chain-irreducible abundant posets

In this section we summarize the main result of the paper, which characterizes chain-irreducible abundant posets. Let P be a non-connected finite poset. We will see that P is abundant if and only if P is not in the family $\mathfrak{CD}^1$, where:

$$\mathfrak{CD}^1 := \{\mathbf{1} \uplus \boldsymbol{m},\ \mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \bar{\mathbf{2}} \oplus \boldsymbol{m}_2),\ \mathbf{2} \uplus \mathbf{2},\ \mathbf{2} \uplus \mathbf{3}\}.$$

Now let us define the poset N_3 given in Figure 4.

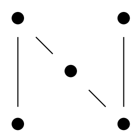


Figure 4: Hasse diagram of poset N_3.

We will see that this is the only deficient connected poset with $w(P) \geq 3$. Next, consider the family

$$\mathfrak{CD}^2 = \{CD_1^m, CD_2^m, CD_3, CD_4, CD_5, CD_6, CD_7, CD_8\},$$

given in Figure 5.

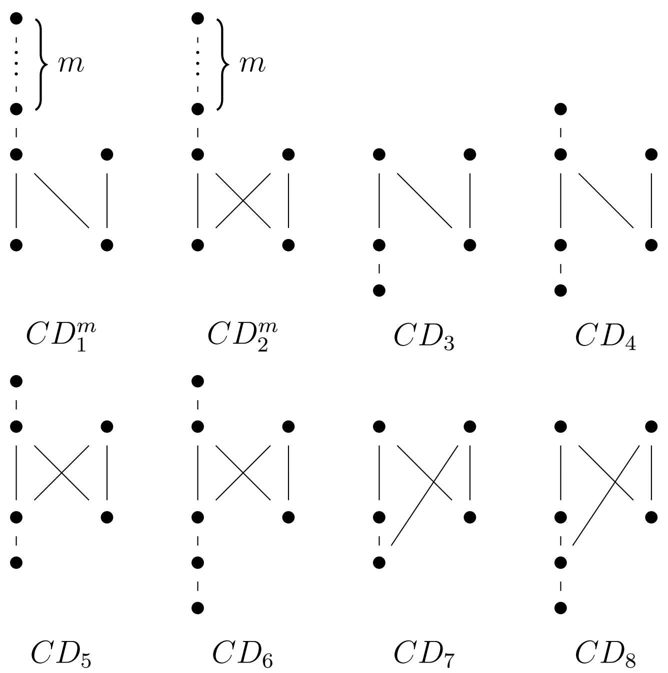


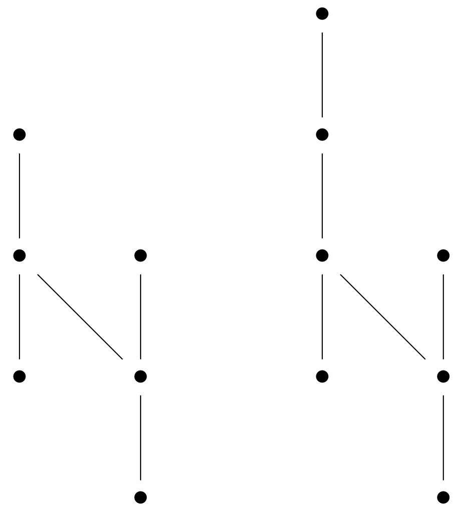
Figure 5: Connected deficient posets with $w(P) = 2$ and $h_2(P) = 2$ (modulo duality).

We will see that these (and their duals) are the only deficient connected posets with $w(P) = 2$ and $h_2(P) = 2$. Finally, we will denote $CD_9 = \bar{\mathbf{2}} \oplus \bar{\mathbf{2}} \oplus \bar{\mathbf{2}},\ CD_{10} = \bar{\mathbf{2}} \oplus N$

and CD_{11}, CD_{12} the posets of Figure 6. Let us denote

$$\mathfrak{CD}^3 := \{CD_9, CD_{10}, CD_{11}, CD_{12}\}.$$

It is easy to check that posets in $\mathfrak{CD}^3$ are chain-irreducible and deficient. Indeed, the pairs $(i(P), e(P))$ for CD_9, CD_{10}, CD_{11} and CD_{12} are $(10, 8)$, $(11, 10)$, $(12, 10)$ and $(14, 13)$ respectively. We will see that these (and their duals) are the only deficient connected posets with $w(P) = 2$ and $h_2(P) > 2$.



CD_{11} CD_{12}

Figure 6: CD_{11} and CD_{12} posets.

The main result in the paper is the following:

Theorem 3.1 (**Characterization of chain-irreducible abundant posets**). *Let P be a chain-irreducible finite poset. Then P is abundant if and only if P and P^∂ are not in $\mathfrak{CD}^*$, where:*

$$\mathfrak{CD}^* := \mathfrak{CD}^1 \cup \mathfrak{CD}^2 \cup \mathfrak{CD}^3 \cup \{N_3\}.$$

Proof. See Section 5. □

In other words, every chain-irreducible poset is abundant except for 17 exceptions.

Note that the set $\mathfrak{CD}^*$ is the set of chain-irreducible deficient posets modulo duality. We can remove the chain-irreducibility condition from the last result to get a more general one.

Theorem 3.2 (**General Ideal-Extension Inequality**). *Let P be a finite poset such that $\mathfrak{R}(P), \mathfrak{R}(P)^\partial \notin \mathfrak{CD}^*$. Then:*

$$i(P) \leq e(P) + \Phi(P).$$

Proof. Since the chain-irreducible reduction $\mathfrak{R}(P) \notin \mathfrak{CD}^*$, applying Theorem 3.1,

$$i(P) - \Phi(P) = i(\mathfrak{R}(P)) \leq e(\mathfrak{R}(P)) = e(P).$$

□

Corollary 3.1. *Let P be a chain-irreducible finite poset. Then $e(P) \geq |P|$. Moreover, $e(P) = |P|$ if and only if $P = \overline{\mathbf{2}} \oplus \overline{\mathbf{2}}$ or $P = \mathbf{1} \uplus \boldsymbol{m}$, where $\boldsymbol{m}$ is the chain of length $|P| - 1$.*

Proof. Suppose first that P is abundant. Consider the ideals of the form $\downarrow x$ and the empty ideal. Thus, we obtain $e(P) \geq i(P) \geq |P| + 1$ and we conclude that $e(P) \leq |P|$ is not possible in this case.

On the other hand, if P is not abundant, by Theorem 3.1 we know that $P \in \mathfrak{CD}^*$. It is straightforward to check that $e(P) \geq |P|$ for every $P \in \mathfrak{CD}^*$ and the equality holds just for the cases $P = \overline{\mathbf{2}} \oplus \overline{\mathbf{2}}$ and $P = \mathbf{1} \uplus \boldsymbol{m}$, where $\boldsymbol{m}$ is the chain of length $|P| - 1$. □

Therefore, the chain-irreducible poset with cardinal $n > 4$ with a minimum number of linear extensions is $P = \mathbf{1} \uplus (\boldsymbol{n} - \mathbf{1})$, having exactly n linear extensions and both $P = \mathbf{1} \uplus \mathbf{3}$ and $P = \overline{\mathbf{2}} \oplus \overline{\mathbf{2}}$ for $n = 4$.

4 Applications

As mentioned above, the main application of Theorem 3.1 is to offer a mathematical justification for enumerating ideals in algorithms for random generation of linear extensions. However, in this section we are going to see some further applications of the characterization of chain-irreducible abundant posets in different branches of mathematics.

4.1 Discrete Geometry

A convex polytope is a bounded convex polyhedron. The faces of a convex polytope $\mathcal{P}$ ordered by inclusion form a lattice $L(\mathcal{P})$ which is known as the face lattice of $\mathcal{P}$.

Let P be a poset. If there is a rank function $r : P \to \mathbb{N}$ such that $r(x) = 0$ for any minimal element x and $r(y) = r(x) + 1$ whenever $y \gtrdot x$, then P is called **graded** or **ranked** with rank r.

It is known that the face lattice of a polytope is always graded by the dimension of the face (see [22]). Let us also denote $L^*(\mathcal{P}) = L(\mathcal{P}) \setminus \{\emptyset, \mathcal{P}\}$.

Corollary 4.1. *Let $\mathcal{P}$ be a convex polytope with* $\dim(\mathcal{P}) > 1$*. Then $L^*(\mathcal{P})$ is chain-irreducible and abundant.*

Proof. Note that $L^*(\mathcal{P})$ is graded by the dimension and the only dimensions k such that the number of k-dimensional faces is 1 are $k = -1$ (the empty set) and

$k = \dim(\mathcal{P})$ (the whole polytope). This implies that $L^*(\mathcal{P})$ is chain-irreducible. Moreover, $L^*(\mathcal{P})$ is always connected and $w(L^*(\mathcal{P})) \geq 3$ if $\dim(\mathcal{P}) > 1$ because $\mathcal{P}$ has at least three vertices. By Theorem 3.1, the only deficient chain-irreducible connected poset with width greater than or equal to 3 is N_3, which is not associated with any polytope (because $\mathcal{P}$ must have at least three vertices). $\square$

Therefore, for every polytope different from a line segment, the number of sets of faces that are not related by inclusion (i.e. antichains) is smaller than the number of ways of ordering all the faces by inclusion (i.e. linear extensions).

4.2 Number Theory

Let $n \in \mathbb{N}$. The division lattice D_n of n is defined as the poset consisting in all the divisors of n ordered by divisibility: $a \preceq b \Leftrightarrow a$ divides b, $\forall a, b \in D_n$. D_n is a bounded distributive lattice (see [7]). Let us call the **pruned division lattice** $D_n^* := D_n \setminus \{1, n\}$.

Observe that the join-irreducible elements of D_n are the prime powers p^k dividing n. Therefore, if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, by Birkhoff's representation theorem, we get $D_n \cong \mathcal{I}(\boldsymbol{k}_1 \uplus \cdots \uplus \boldsymbol{k}_r)$. Using the relationship between the ideal lattice of the union and the product of posets (defined coordinatewise) we obtain [7]:

$$D_n \cong \mathcal{I}(\boldsymbol{k}_1 \uplus \cdots \uplus \boldsymbol{k}_r) \cong \mathcal{I}(\boldsymbol{k}_1) \times \cdots \times \mathcal{I}(\boldsymbol{k}_r) \cong (\boldsymbol{k_1} + \mathbf{1}) \times \cdots \times (\boldsymbol{k_r} + \mathbf{1}).$$

Theorem 4.1. *Let $n \geq 2$. The pruned division lattice D_n^* is abundant if and only if n is neither a prime power $n = p^k$ nor of the form $n = p_1^{k_1} p_2$ with $k_1 \leq 2$.*

Proof. If n is a prime power $n = p^k$, then $D_n \cong \boldsymbol{k} + \mathbf{1}$ is a chain, so D_n^* is also a chain and thus deficient. If $n = p_1 p_2$, then $D_n \cong \mathbf{2} \times \mathbf{2} \cong \mathbf{1} \oplus \bar{\mathbf{2}} \oplus \mathbf{1}$, and $D_n^* \cong \bar{\mathbf{2}}$ is also deficient. Also if $n = p_1^2 p_2$, then $D_n \cong \mathbf{3} \times \mathbf{2} \cong \mathbf{1} \oplus N \oplus \mathbf{1}$, and $D_n^* \cong N$ is also deficient.

Now suppose that $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is neither a prime power nor of the form $n = p_1^{k_1} p_2$ with $k_1 \leq 2$. It is clear that D_n^* is chain-irreducible. Indeed, for every element $d_1 = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ we can suppose without loss of generality that $s_1 < k_1,\ s_2 > 0$ and take $d_2 = p_1^{s_1+1} p_2^{s_2-1} \cdots p_r^{s_r}$ and $d_1 \parallel d_2$.

Now let us show that $D_n^* \notin \mathfrak{CD}^*$. If n has three different prime divisors, then the boolean lattice $B_3 = \mathbf{2} \times \mathbf{2} \times \mathbf{2}$ is a subposet of D_n, so $D_n^* \notin \mathfrak{CD}^*$. Therefore n should have at most two different prime divisors $n = p_1^{k_1} p_2^{k_2}$. If $k_1, k_2 \geq 2$ then the set $A = (p_1^2, p_1 p_2, p_2^2)$ is an antichain of three elements. Thus D_n^* is connected and $w(D_n^*) \geq 3$. By Theorem 3.1, the only posibility for D_n being deficient is $D_n^* \cong N_3$, which is impossible ($\mathbf{1} \oplus N_3 \oplus \mathbf{1}$ is not a product of chains). Finally, in the case $n = p_1^{k_1} p_2$ we get $D_n \cong 2 \times (\boldsymbol{k_1} + \mathbf{1})$, leading to $D_n^* \in \mathfrak{CD}^*$ if and only if $k_1 \leq 2$. $\square$

5 Proof of Theorem 3.1

In this section, we prove the main theorem of the paper. To shed light on this proof, we divide it into several cases.

5.1 Technical lemmas

Lemma 5.1. *Let P be a finite poset. Then the following inequalities hold:*

(*i*) $i(P) \leq |P| \cdot e(P) + 1$,

(*ii*) $2 \cdot i(P) \leq (1 + |P|) \cdot e(P)$, *if* $e(P) \geq 3$.

Proof. (*i*) For every non-empty ideal $I \in \mathcal{I}(P)$, there exists a linear extension $\epsilon \in \mathcal{L}(P)$ starting with the ideal I (note that two ideals I_1, I_2 may be related to the same linear extension if $I_1 \subset I_2$). Therefore, adding 1 for the empty ideal, we have $i(P) \leq |P| \cdot e(P) + 1$.

(*ii*) We consider two cases. Firstly, let us suppose that for every non-empy ideal $I \in \mathcal{I}(P)$, I or $P \setminus I$ is not a chain in P. Indeed, without loss of generality we can suppose that I is not a chain. Then, two different linear extensions $\epsilon_0, \epsilon_1 \in \mathcal{L}(P)$ starting with ideal $I \neq \emptyset$ exist. Moreover, since $e(P) \geq 3$, we can assign two different linear extensions δ_0, δ_1 to the empty ideal $I = \emptyset$. Hence $2 \cdot i(P) \leq (1 + |P|) \cdot e(P)$, and the result holds.

Assume now that a non-empty ideal I exists for which both sets I and $P \setminus I$ are chains in P. Since $e(P) > 1$, we have $I \neq P$, and the filter $P \setminus I$ is non-empty. Thus there exist $x, y \in P$ with $I = \downarrow x$ and $P \setminus I = \uparrow y$. Due to $(\uparrow x) \setminus \{x\} \subseteq P \setminus I$ and $(\downarrow y) \setminus \{y\} \subseteq I$, the poset P looks as in Figure 7.

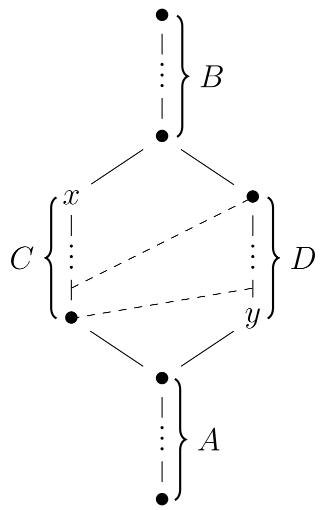


Figure 7: Poset P in proof of Lemma 5.1 *ii*).

In this Figure 7, P can be written as $P = A \oplus (C \uplus D) \oplus B$, where $\downarrow x = A \oplus C$, $B = (\uparrow x) \setminus \{x\}$, $\uparrow y = D \oplus B$, and $A = (\downarrow y) \setminus \{y\}$. Additionally, P contains an arbitrary number of edges from $(C \setminus \{x\}) \times (D \setminus \{y\})$ (represented by dotted lines connecting elements of C and D). Let us denote $a = |A|$, $b = |B|$, $c = |C|$ and $d = |D|$. Observe that $a, b \geq 0$ and $c + d \geq 3$ due to $e(P) \geq 3$.

There is a single linear extension of P with $\epsilon(y) = \epsilon(x) + 1$, i.e. y follows x in the linear extension ϵ. Moreover, for every element $z \in C \setminus \{x\}$ there exist at least d linear extensions with $\epsilon(y) = \epsilon(z) + 1$. In fact, we can take $\epsilon = (\downarrow z, y, C \setminus (\downarrow z \cup \{x\}), \dots)$ and we can place element x next or following any element in D, so we have at least d different linear extensions. For the same reason, there exist at least d linear extensions with element y following chain A. We conclude $e(P) \geq cd + 1$. Defining Q as poset P by erasing dashed lines, we have $i(P) \leq i(Q) = a + (c+1)(d+1) + b$. Joining these two facts, we get:

$$\begin{array}{rcl}(1+|P|) \cdot e(P) - 2 \cdot i(P) & \geq & (a+b+c+d+1) \cdot (cd+1) - 2 \cdot [a + (c+1)(d+1)+b] \\ & = & (a+b+c+d+1) \cdot (cd-1) - 2cd \\ & \geq & (c+d+1) \cdot (cd-1) - 2cd.\end{array}$$

The last expression is greater than or equal to zero for all pairs (c, d) with $c+d \geq 3$, so the inequality holds. $\square$

Lemma 5.2. *Let P be a finite irreducible poset. Then:*

(i) There exists $x \in \mathcal{MIN}(P)$ and $y \in \mathcal{MAX}(P)$ such that $P \setminus \{x\}$ and $P \setminus \{y\}$ are irreducible.

(ii) If $P \ncong \mathbf{1} \uplus Q$ for any poset Q and there is an antichain A of P such that $A \cap \mathcal{MIN}(P) \neq \emptyset$, $A \cap \mathcal{MAX}(P) \neq \emptyset$, then there exists $x \in \mathcal{MIN}(P) \setminus A$ satisfying $P \setminus \{x\}$ is irreducible.

Proof. (i) Consider a partition $\{M_i\}_{i=0,\dots,t}$ of P, where $M_i := \mathcal{MIN}\left(P \setminus \bigcup_{k=0}^{i-1} M_k\right)$. Note that $M_0 := \mathcal{MIN}(P)$ and $P = \bigcup_{i=1}^{t} M_i$ for some $t \in \mathbb{N}$. Now, for all $i \in \{1 \dots, t\}$ there exists $x_i^+ \in M_i$ and $x_{i-1}^- \in \bigcup_{k=0}^{i-1} M_k$ such that $x_{i-1}^- \parallel x_i^+$; otherwise $P = \left(\bigcup_{k=0}^{i-1} M_k\right) \oplus \left(P \setminus \bigcup_{k=0}^{i-1} M_k\right)$ which is a contradiction. Besides, $|M_0| \geq 2$ and we can choose some $x \in M_0$, $x \neq x_0^-$. We claim that $P \setminus \{x\}$ is irreducible. Indeed, if we define $\overline{M}_0 := M_0 \setminus \{x\}$ and $\overline{M}_i := M_i$ for $i \geq 1$, we get a partition for $P \setminus \{x\}$. Since the elements of each $\overline{M}_i$ form an antichain, they must be in the same irreducible component of $P \setminus \{x\}$. As $x_{i-1}^- \parallel x_i^+$, $\overline{M}_i$ is in the same irreducible component as $\overline{M}_{i-1}$ for all i and we conclude that the whole $P \setminus \{x\}$ is in just one irreducible component.

By duality, there is also $y \in \mathcal{MAX}(P)$ such that $P \setminus \{y\}$ is irreducible.

(ii) We remark that if $x \in \mathcal{MIN}(P) \cap \mathcal{MAX}(P)$, this implies that x is isolated and thus P can be written as $\mathbf{1} \uplus Q$. Hence, $\mathcal{MIN}(P) \cap \mathcal{MAX}(P) = \emptyset$. Let us see that there is a minimal element $x \notin A$. Otherwise, as $A \cap \mathcal{MAX}(P) \neq \emptyset$ and A is an antichain, there would exist some maximal element $z \in A$ such that $z \parallel y,\ \forall y \in \mathcal{MIN}(P)$, which is a contradiction.

Therefore, we can take $x \in \mathcal{MIN}(P) \backslash A$, $y \in \mathcal{MIN}(P) \cap A$ and $z \in \mathcal{MAX}(P) \cap A$. As $y \parallel z$, $P \setminus \{x\}$ is irreducible. □

Lemma 5.3. *Let P be a poset and $x \in P$ such that $w(P) = w(P \setminus \{x\}) = 2$. Consider a partition (C_1^*, C_2^*) of $P \backslash \{x\}$ into two chains such that $|C_1^*| = h_1(P \backslash \{x\})$. If $C_1^* \cup \{x\}$ or $C_2^* \cup \{x\}$ is a chain, then $h_1(P \setminus \{x\}) \leq h_1(P)$.*

Proof. If $C_1^* \cup \{x\}$ is a chain then $(C_1^* \cup \{x\}, C_2^*)$ is a partition of P into two chains. Hence,

$$h_1(P \setminus \{x\}) = |C_1^*| < |C_1^* \cup \{x\}| \leq h_1(P).$$

Now suppose $C_2^* \cup \{x\}$ is a chain. We can assume without loss of generality that $|C_1^*| > |C_2^*|$, otherwise we are in the conditions of the first case. Then $(C_1^*, C_2^* \cup \{x\})$ is a partition of P into two chains and we get

$$h_1(P \setminus \{x\}) = |C_1^*| \leq h_1(P).$$

□

Definition 5.1. Let P be a finite poset. We define the class of equivalence of P as:

$$[P] = \{Q\ :\ |P| = |Q|,\ i(P) = i(Q) \text{ and } e(P) = e(Q)\}.$$

In particular, P, $P^\partial \in [P]$. Obviously, if $Q \in [P]$ and $Q \in \mathfrak{A}$, it follows that $Q' \in \mathfrak{A}, \forall Q' \in [P]$.

Lemma 5.4. *Let P be a finite poset. Suppose there exists $Q \in [P]$ satisfying that there are $x, y \in \mathcal{MIN}(Q)$ such that $Q \setminus \{x\} \in \mathfrak{A}$ and $e(Q \setminus \{x, y\}) \geq 2$. Then, $P \in \mathfrak{A}$.*

Proof. Let P be a poset and consider Q, x, y satisfying the previous conditions. As $Q \setminus \{x\} \in \mathfrak{A}$, there exists an injective map $f : \mathcal{I}(Q \setminus \{x\}) \to \mathcal{L}(Q \setminus \{x\})$. Let us consider $F : \mathcal{I}(Q) \to \mathcal{L}(Q)$ given by

$$F(I) := \begin{cases} (x, f(I)) & \text{if } x \notin I \\ (x, f(I \setminus \{x\})) & \text{if } x \in I, x \notin \mathcal{MAX}(I) \\ (I \setminus \{x\}, x, f(I \setminus \{x\}) \setminus (I \setminus \{x\})) & \text{if } x \in I, x \in \mathcal{MAX}(I), I \neq \{x\} \\ (y, x, \hat{f}(\{y\}) \setminus \{y\}) & \text{if } I = \{x\} \end{cases}$$

where in the third case the elements of $I \setminus \{x\}$ are ordered in a compatible way. In the fourth case, we define $\hat{f}(\{y\})$ as a linear extension of $Q \setminus \{x\}$ such that $f(\{y\}) \backslash \{y\} \neq \hat{f}(\{y\}) \backslash \{y\}$. We remark that this is possible because $e(Q \backslash \{x, y\}) \geq 2$.

Let us first show that F is well-defined. As $f(I) \in \mathcal{L}(Q \setminus \{x\})$, it suffices to see that the inclusion of x does not violate the order. For this, as $x, y \in \mathcal{MIN}(Q)$, cases

1, 2 and 4 are straightforward. For the third case, note that $x \in \mathcal{MAX}(I)$, so that for $z \in I, z \neq x$, it cannot happen that $z \succeq x$. Besides, as $x \in \mathcal{MIN}(Q)$, for $z \notin I$, it cannot happen that $x \succeq z$.

Let us now see that F is injective. Let us show that F is injective within each case. For the first two cases, this holds because f is injective. The third case holds because it starts with $I \setminus \{x\}$.

Finally, let us see that F is injective between the different cases. For this, we have to compare the two first cases and the two last. For the two first cases, equality could arise if there exists $I \in \mathcal{I}(Q)$ such that $x \in I$, and $I \setminus \{x\} \in \mathcal{I}(Q)$. But this would imply that $x \in \mathcal{MAX}(I)$ and in the second case we have excluded this possibility. For the third and fourth cases, we could have equality for $I = \{x, y\}$ and $I' = \{x\}$. But as $f(\{y\}) \setminus \{y\} \neq \hat{f}(\{y\}) \setminus \{y\}$, injectivity holds.

Thus, $Q \in \mathfrak{A}$ and hence $P \in \mathfrak{A}$. $\square$

5.2 Characterizing non-connected abundant posets

Theorem 5.1 (**Characterization of non-connected abundant posets**). *Let P be a non-connected finite poset. Then P is abundant if and only if P is not in the family $\mathfrak{CD}^1$, where:*

$$\mathfrak{CD}^1 := \{\mathbf{1} \uplus \boldsymbol{m},\ \mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \bar{\mathbf{2}} \oplus \boldsymbol{m}_2),\ \mathbf{2} \uplus \mathbf{2},\ \mathbf{2} \uplus \mathbf{3}\}.$$

Proof. Let $P = P_1 \uplus P_2$ and $n := |P_1|,\ m := |P_2|$, with $n \leq m$. Applying Lemma 5.1 *i)* and Lemma 2.1 yields

$$\begin{aligned} i(P) &= i(P_1) \cdot i(P_2) \\ &\leq (n \cdot e(P_1) + 1) \cdot (m \cdot e(P_2) + 1) \\ &\leq (n+1) \cdot (m+1) \cdot e(P_1) \cdot e(P_2) \\ &= \frac{(n+1) \cdot (m+1)}{\binom{n+m}{m}} e(P). \end{aligned}$$

Therefore, the poset P is abundant if

$$(n+1) \cdot (m+1) \leq \binom{n+m}{n},$$

which is true for all combinations (n, m) with the exception of $(1, m), (2, 2)$, and $(2, 3)$. For the latter two alternatives, P must be one of the posets (see [18]):

$\mathbf{2} \uplus \mathbf{2},\ \mathbf{2} \uplus \overline{\mathbf{2}},\ \overline{\mathbf{4}},\ \mathbf{2} \uplus \mathbf{3},\ \mathbf{2} \uplus (\mathbf{1} \oplus \overline{\mathbf{2}}), \mathbf{2} \uplus (\overline{\mathbf{2}} \oplus \mathbf{1}),\ \mathbf{2} \uplus \overline{\mathbf{3}},\ \mathbf{2} \uplus (\mathbf{1} \uplus \mathbf{2}),\ \overline{\mathbf{5}}, \overline{\mathbf{2}} \uplus \mathbf{3},\ \overline{\mathbf{2}} \uplus (\mathbf{1} \oplus \overline{\mathbf{2}}),\ \overline{\mathbf{2}} \uplus (\overline{\mathbf{2}} \oplus \mathbf{1})$.

As we can see in Table 1, all of these posets are abundant except $\mathbf{2} \uplus \mathbf{2}$ and $\mathbf{2} \uplus \mathbf{3}$. Now let $n = 1$. According to Lemma 2.1, $P = \mathbf{1} \uplus P_2$ is abundant if and only if

$$i(P) = 2 \cdot i(P_2) \leq (1 + m) \cdot e(P_2) = e(P),$$

P	$i(P)$	$e(P)$		P	$i(P)$	$e(P)$
$\mathbf{2} \uplus \mathbf{2}$	9	6		$\mathbf{2} \uplus \overline{\mathbf{2}}$	12	12
$\overline{\mathbf{4}}$	16	24		$\mathbf{2} \uplus \mathbf{3}$	12	10
$\mathbf{2} \uplus (\mathbf{1} \oplus \overline{\mathbf{2}})$	15	20		$\mathbf{2} \uplus (\overline{\mathbf{2}} \oplus \mathbf{1})$	15	20
$\mathbf{2} \uplus \overline{\mathbf{3}}$	24	60		$\mathbf{2} \uplus (\mathbf{1} \uplus \mathbf{2})$	18	30
$\overline{\mathbf{5}}$	32	120		$\overline{\mathbf{2}} \uplus \mathbf{3}$	16	20
$\overline{\mathbf{2}} \uplus (\mathbf{1} \oplus \overline{\mathbf{2}})$	20	40		$\overline{\mathbf{2}} \uplus (\overline{\mathbf{2}} \oplus \mathbf{1})$	20	40

Table 1: Number of ideals and linear extensions for some non-connected posets.

which holds according to Lemma 5.1 (ii) for $e(P_2) \geq 3$. Finally, for the case $e(P_2) \leq 2$, poset P_2 is the chain $\boldsymbol{m}$ or isomorphic to $\boldsymbol{m}_1 \oplus \overline{\mathbf{2}} \oplus \boldsymbol{m}_2$ which lead to deficient posets. Indeed,

$$i(\mathbf{1} \uplus \boldsymbol{m}) = 2(m+1) > (m+1) = e(\mathbf{1} \uplus \boldsymbol{m})$$

and

$$i(\mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \overline{\mathbf{2}} \oplus \boldsymbol{m}_2)) = 2(m_1 + m_2 + 4) > 2(m_1 + m_2 + 2) = e(\mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \overline{\mathbf{2}} \oplus \boldsymbol{m}_2)).$$

So the result holds. □

5.3 Chain-irreducible connected abundant posets with $w(P) \geq 3$

We first treat the reducible case.

Theorem 5.2. *Let P be a finite, chain-irreducible, connected and reducible poset with $w(P) \geq 3$. Then $P \in \mathfrak{A}$.*

Proof. Let P be a poset in these conditions. Then, as P is reducible, we can write

$$P = P_1 \oplus P_2 \oplus \ldots \oplus P_k,$$

where each P_i is irreducible, $|P_i| > 1, k > 1$, and there exists i^* such that $w(P_{i^*}) \geq 3$.

If every P_i is equal to $\bar{\mathbf{2}}$ or $\bar{\mathbf{3}}$, then we can reorder the ordinal summands to get $Q \in [P]$ given by

$$Q := \bar{\mathbf{2}} \oplus \overset{k_1}{\ldots} \oplus \bar{\mathbf{2}} \oplus \bar{\mathbf{3}} \oplus \overset{k_2}{\ldots} \oplus \bar{\mathbf{3}},\ k_1 \geq 0,\ k_2 \geq 1,\ k_1 + k_2 \geq 2.$$

Then, by Lemma 2.1

$$i(Q) = 4k_1 + 8k_2 - (k_1 + k_2 - 1) = 3k_1 + 7k_2 + 1 \leq 2^{k_1} 6^{k_2} = e(Q), \forall k_1, k_2,$$

and $Q \in \mathfrak{A}$.

In the other case, let us make the proof by induction in $|P|$. There are no chain-irreducible, connected and reducible posets with $w(P) \geq 3$ and less than five elements. So let us prove the basis step for $|P| = 5$. The only posets with five elements in these conditions are (see [18]) $P = \bar{\mathbf{2}} \oplus \bar{\mathbf{3}}$ and $P^{\partial} = \bar{\mathbf{3}} \oplus \bar{\mathbf{2}}$, and hence $P, P^{\partial} \in \mathfrak{A}$.

Let us now assume $|P| > 5$ and suppose that the result holds until $|P| - 1$. We have to consider several cases.

Case 1: If $P_{i^*} = \bar{\mathbf{3}}$, by hypothesis there is some $j^* \neq i^*$ such that $P_{j^*} \not\cong \bar{\mathbf{2}}$. We can use Lemma 5.2 $i)$ to obtain some minimal element x of P_{j^*} with $P_{j^*} \setminus \{x\}$ irreducible. Hence, x is also minimal element of $Q := P_{j^*} \bigoplus_{i \neq j^*} P_i \in [P]$. Note that $Q \setminus \{x\} = (P_{j^*} \setminus \{x\}) \bigoplus_{i \neq j^*} P_i$ is chain-irreducible because $P_{j^*} \not\cong \bar{\mathbf{2}}$. Now, $w(Q) \geq w(P_{i^*}) = 3$. Applying induction, we conclude that $Q \setminus \{x\} \in \mathfrak{A}$. Finally, we can choose any minimal element $y \neq x$ of Q and we get $e(Q \setminus \{x, y\}) \geq 2$. Hence, by Lemma 5.4 the result holds.

Case 2: If $P_{i^*} = H \uplus \mathbf{1} \neq \bar{\mathbf{3}}$. Hence, as $w(P_{i^*}) \geq 3$, it follows $w(H) \geq 2$. Consequently, there is some antichain $\{h_1, h_2\} \in H$ such that $\{h_1, h_2, \mathbf{1}\}$ is an antichain in P_{i^*}. Moreover, since $P_{i^*} \neq \bar{\mathbf{3}}$, then $H \setminus \{h_1, h_2\} \neq \emptyset$ and we can take a minimal (or maximal) element $x \in H$ different from h_1 and h_2. Obviously, $P_{i^*} \setminus \{x\}$ is irreducible because it is not connected. Besides, $\{h_1, h_2, \mathbf{1}\} \subseteq P_{i^*} \setminus \{x\}$, so that $w(P_{i^*} \setminus \{x\}) \geq 3$.

Now, consider

$$Q := P_{i^*} \bigoplus_{i \neq i^*} P_i \in [P], \text{ (or } Q := P_{i^*}^{\partial} \bigoplus_{i \neq i^*} P_i \in [P], \text{ if } x \in \mathcal{MAX}(P)).$$

Hence, $Q \setminus \{x\}$ is reducible, chain-irreducible and $w(Q \setminus \{x\}) \geq w(P_{i^*} \setminus \{x\}) \geq 3$, so using the induction hypothesis, $Q \setminus \{x\} \in \mathfrak{A}$. Finally we can choose any minimal element $y \neq x$ of Q and we get $e(Q \setminus \{x, y\}) \geq 2$. Hence, by Lemma 5.4, $Q \in \mathfrak{A}$.

Case 3: Finally, assume $P_{i^*} \neq \bar{\mathbf{3}}$ and $P_{i^*} \neq H \uplus \mathbf{1}$. Let us see that we can find $Q \in [P]$ and $x \in \mathcal{MIN}(Q)$ such that $Q \setminus \{x\} \in \mathfrak{A}$. Let A be a 3-element antichain of P_{i^*}. If there is no minimal element in A we can apply Lemma 5.2 (i) to obtain some $x \in \mathcal{MIN}(P_{i^*})$ with $P_{i^*} \setminus \{x\}$ irreducible. Besides, $w(P_{i^*} \setminus \{x\}) \geq w(A) = 3$. Hence, considering $Q = P_{i^*} \bigoplus_{i \neq i^*} P_i \in [P]$, we conclude by induction that $Q \setminus \{x\} \in \mathfrak{A}$.

If there is some minimal element in A but there is no maximal element, we can apply Lemma 5.2 (i) to the dual $P_{i^*}^{\partial}$ and we obtain the same conclusions for $Q = P_{i^*}^{\partial} \bigoplus_{i \neq i^*} P_i \in [P]$.

Finally, if A has some minimal element and some maximal element, we can apply Lemma 5.2 (ii) to obtain a minimal element $x \notin A$ such that $P_{i^*} \setminus \{x\}$ is irreducible. Hence, considering $Q = P_{i^*} \bigoplus_{i \neq i^*} P_i \in [P]$, it follows that $Q \setminus \{x\}$ is reducible, chain-irreducible and $w(Q \setminus \{x\}) \geq 3$. We conclude by induction that $Q \setminus \{x\} \in \mathfrak{A}$.

Now, we can choose any minimal element $y \neq x$ of Q and we get $e(Q \setminus \{x, y\}) \geq 2$. Hence, by Lemma 5.4 the result holds. $\square$

Let us now generalize the last result to every chain-irreducible, connected poset P with $w(P) \geq 3$. In order to achieve this, let us consider a previous lemma.

Lemma 5.5. *Let P be a chain-irreducible connected poset with $w(P) \geq 3$ and $x \in \mathcal{MIN}(P)$.*

(i) If $|P| \geq 6$ and $P \setminus \{x\}$ is disconnected, then at least one of P or $P \setminus \{x\}$ is abundant.

(ii) If $P \setminus \{x\} \cong N_3$, then $P \in \mathfrak{A}$.

Proof. (i) If $P \setminus \{x\} \in \mathfrak{A}$ we are finished, so let us suppose that $P \setminus \{x\} \notin \mathfrak{A}$ and we are going to show that $P \in \mathfrak{A}$. Since $w(P) \geq 3$, this implies that $2 \leq w(P \setminus \{x\}) \leq 3$. Thus we need to distinguish two cases.

Case 1: If $w(P \setminus \{x\}) = 2$, as $P \setminus \{x\} \in \mathfrak{D}$, $P \setminus \{x\}$ is disconnected and $|P \setminus \{x\}| \geq 5$, we know by Theorem 5.1 that $P \setminus \{x\} \cong \mathbf{1} \uplus \boldsymbol{m}$, $m \geq 4$ or $P \setminus \{x\} \cong \mathbf{2} \uplus \mathbf{3}$.

If $P \setminus \{x\} \cong \mathbf{1} \uplus \boldsymbol{m}$, the Hasse diagram of P is given in Figure 8 (left) where it is clear that $w(P) = 2$, which is a contradiction.

If $P \setminus \{x\} \cong \mathbf{2} \uplus \mathbf{3}$, the only choices for P such that $w(P) \geq 3$ are depicted in Figure 8 (center and right). These two posets are abundant since their corresponding pairs $(i(P), e(P))$ are $(16, 26)$ and $(18, 35)$, respectively.

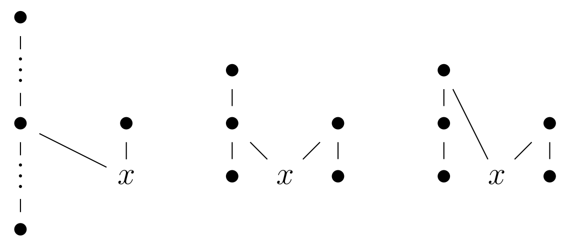


Figure 8: Hasse diagram of P when $P \setminus \{x\} \cong \mathbf{1} \uplus \boldsymbol{m}$ (left) and choices for P when $P \setminus \{x\} \cong \mathbf{2} \uplus \mathbf{3}$ (center and right).

Case 2: If $w(P \setminus \{x\}) = 3$, as $P \setminus \{x\} \in \mathfrak{D}$, $P \setminus \{x\}$ is disconnected and $|P \setminus \{x\}| \geq 5$, we know by Theorem 5.1 that $P \setminus \{x\} \cong \mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \bar{\mathbf{2}} \oplus \boldsymbol{m}_2)$.

Here we can also distinguish four possible cases for P (see [18]). These four cases (A, B, C and D) are depicted in Figure 9. Let us denote by P_{k_1,k_2,k_3} the posets belonging to families A and B, and by P_{k_1,k_2} the posets belonging to families C and D.

In Case A, we have P_{k_1,k_2,k_3}, $k_1 \geq 1$, (otherwise $P = \{x\} \oplus P_{1,}$), $k_2, k_3 \geq 0$. For counting ideals we use $i(P) = a(P)$ and hence we count the number of antichains of length 0, 1, 2 and 3. Hence

$$i(P_{k_1,k_2,k_3}) = 1 + (5 + k_1 + k_2 + k_3) + (4 + 2k_1 + k_2 + k_3) + 1 = 3k_1 + 2k_2 + 2k_3 + 11.$$

For counting $e(P_{k_1,k_2,k_3})$ we can apply the fact that for every poset P,

$$e(P) = \sum_{x \in \mathcal{MIN}(P)} e(P \setminus \{x\}).$$

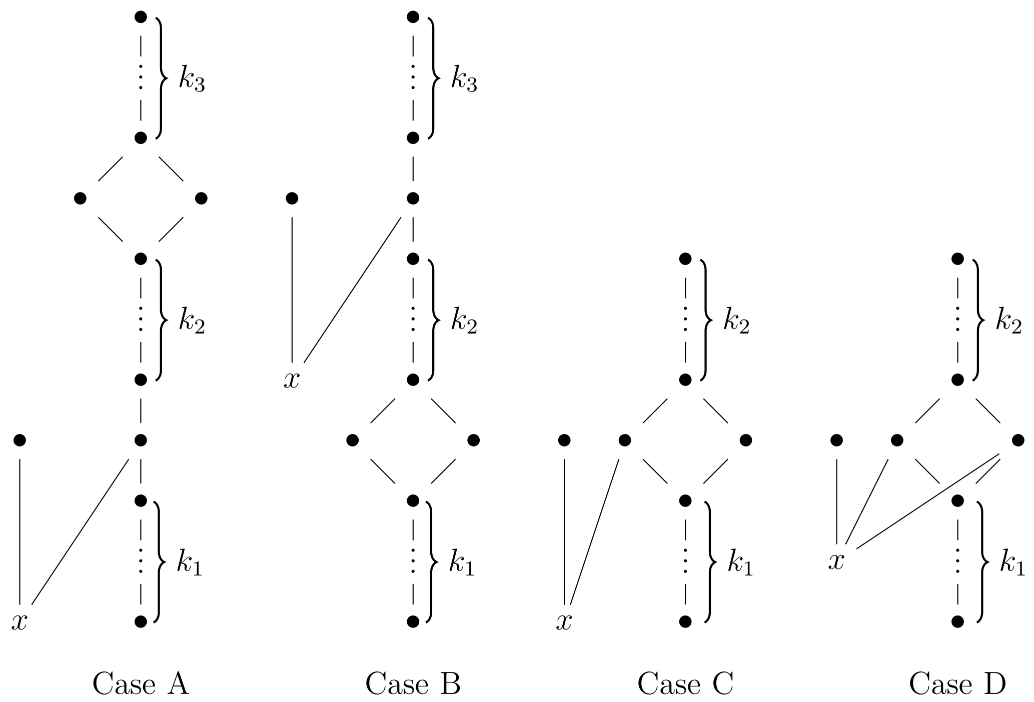


Figure 9: Different choices for P if $P \setminus x \cong \mathbf{1} \uplus (\boldsymbol{m}_1 \oplus \bar{\mathbf{2}} \oplus \boldsymbol{m}_2)$.

Next, there are $2(k_1 + k_2 + k_3 + 4)$ linear extensions in $P_{k_1,k_2,k_3} \setminus \{x\}$. Therefore,

$$e(P_{k_1,k_2,k_3}) = 2(k_1 + k_2 + k_3 + 4) + e(P_{k_1-1,k_2,k_3}).$$

If $k_1 = 1$, then $e(P_{1,k_2,k_3}) = 2(1 + k_2 + k_3 + 4) + 2(0 + k_2 + k_3 + 4)$. Thus

$$\begin{aligned} e(P_{k_1,k_2,k_3}) &= 2(k_1+1)(k_2+k_3+4) + 2\sum_{t=0}^{k_1} t \\ &= 2(k_1+1)(k_2+k_3+4) + k_1(k_1+1) \\ &= (k_1+1)(k_1+2k_2+2k_3+8), \end{aligned}$$

and $P_{k_1,k_2,k_3} \in \mathfrak{A}, \forall k_1, k_2, k_3$.

In Case B, we have $P_{k_1,k_2,k_3},\ k_1, k_2, k_3 \geq 0$. Proceeding as before,

$$i(P_{k_1,k_2,k_3}) = 3k_1 + 3k_2 + 2k_3 + 14.$$

And for $e(P)$, it can be seen proceeding as in Case A

$$e(P_{k_1,k_2,k_3}) = 2(k_1 + k_2 + 3) + e(P_{k_1,k_2,k_3-1}).$$

If $k_3 = 0$, then $e(P_{k_1,k_2,0}) = 2(k_1 + k_2 + 3) + 2\binom{k_1+k_2+4}{2}$. Thus

$$e(P_{k_1,k_2,k_3}) = 2(k_1 + k_2 + 3)(k_3 + 1) + (k_1 + k_2 + 4)(k_1 + k_2 + 3),$$

and $P_{k_1,k_2,k_3} \in \mathfrak{A}, \forall k_1, k_2, k_3$.

In Case C, we have $P_{k_1,k_2},\ k_1+k_2 \geq 1$. Counting ideals we get

$$i(P_{k_1,k_2}) = 3k_1 + 2k_2 + 10.$$

Observe that

$$e(P_{k_1,k_2}) = 2(k_1+k_2+3) + e(P_{k_1-1,k_2}).$$

If $k_1 = 0$ we obtain $e(P_{0,k_2}) = 2(k_2+3) + (k_2+2)$. Thus,

$$e(P_{k_1,k_2}) = \sum_{t=0}^{k_1} 2(t+k_2+3) + (k_2+2) = 2(k_1+1)(k_2+3) + (k_2+2) + k_1(k_1+1),$$

and $P_{k_1,k_2} \in \mathfrak{A}$ except for $k_1 = 0$ and $k_2 = 1$. However in this case $|P_{0,1}| = 5$, in contradiction with the hypothesis $|P_{k_1,k_2}| \geq 6xp$.

In Case D, we have $P_{k_1,k_2},\ k_1 \geq 1, k_2 \geq 0$. Counting ideals we get

$$i(P_{k_1,k_2}) = 3k_1 + 2k_2 + 9.$$

Observe that

$$e(P_{k_1,k_2}) = 2(k_1+k_2+3) + e(P_{k_1-1,k_2}).$$

If $k_1 = 0$ we obtain $e(P_{0,k_2}) = 2(k_2+3)$. Thus,

$$e(P_{k_1,k_2}) = 2(k_1+1)(k_2+3) + k_1(k_1+1),$$

so $P_{k_1,k_2} \in \mathfrak{A}, \forall k_1, k_2$.

(ii) As $x \in \mathcal{MIN}(P)$ and $P \setminus \{x\} \cong N_3$, we can consider all the possibilities for P being chain-irreducible, connected and with $w(P) \geq 3$. These alternatives depend on the number of elements of N_3 covering x. In Figure 10 we can see the different possible posets P (see [18]) and their corresponding pairs $(i(P), e(P))$ of ideals and linear extensions. As it can be checked, all of them are abundant, so the result holds. □

Theorem 5.3 (**Characterization of chain-irreducible connected abundant posets with** $w(P) \geq 3$)**.** *Let P be a chain-irreducible connected poset with $w(P) \geq 3$. Then $P \in \mathfrak{A}$ if and only if $P \ncong N_3$ (as defined in Figure 4).*

Proof. Start by noting that $N_3 \in \mathfrak{D}$ since $i(N_3) = 12$ and $e(N_3) = 11$.

Let us prove the other implication using induction on $|P|$.

There are no posets allowed by the conditions of the theorem with less than five elements and there are just four posets (modulo isomorphism and duality) with five elements (see [18]). These posets and their corresponding pairs $(i(P), e(P))$ are shown in Figure 11. As we can see these four posets are abundant.

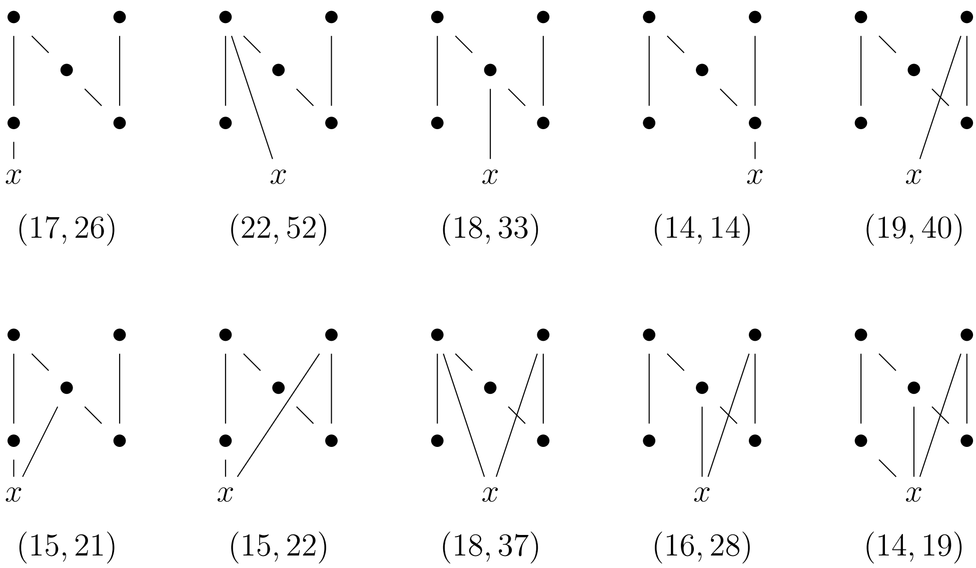


Figure 10: Choices for P such that $P \setminus \{x\} \cong N_3$ and pairs $(i(P), e(P))$.

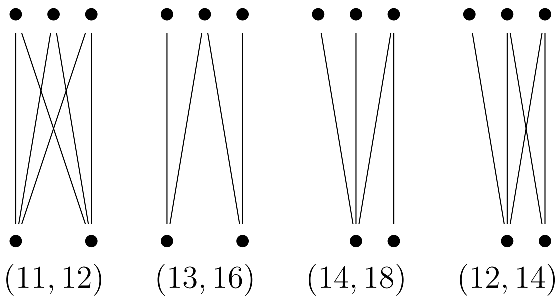


Figure 11: Chain-irreducible connected posets with $w(P) \geq 3$ and 5 elements.

Let us now prove the induction step. Let P be a poset with $|P| > 5$. By Theorem 5.2, if P is reducible then $P \in \mathfrak{A}$, so we can suppose that P is irreducible.

In the same way, by Lemma 5.5 (i), if there is some $x \in \mathcal{MIN}(P)$ such that $P \setminus \{x\}$ is non-connected, then $P \in \mathfrak{A}$ or $P \setminus \{x\} \in \mathfrak{A}$. If $P \in \mathfrak{A}$ we are done. If $P \setminus \{x\} \in \mathfrak{A}$, we can take some $y \in \mathcal{MIN}(P)$, $y \neq x$ such that $e(P \setminus \{x, y\}) \geq 2$. Otherwise, $P \setminus \{x, y\} \cong \boldsymbol{m}$ and as P is connected, this implies that either $P \cong (\boldsymbol{k}_1 \uplus \boldsymbol{1}) \oplus \boldsymbol{k}_2$ with $k_2 \geq 1$ or $P \cong (((\boldsymbol{k}_1 \uplus \boldsymbol{1}) \oplus \boldsymbol{k}_2) \uplus \boldsymbol{1}) \oplus \boldsymbol{k}_3$ with $k_3 \geq 1$, a contradiction since P is chain-irreducible. Hence, $P \in \mathfrak{A}$ by Lemma 5.4.

Next, by Lemma 5.5 (ii), if there is some minimal element x such that $P \setminus \{x\} = N_3$, then $P \in \mathfrak{A}$.

Therefore, we can suppose that P is irreducible and for every minimal element x (or maximal element by duality), $P \setminus \{x\}$ is connected and different from N_3. Since $w(P) \geq 3$, let A be a 3-element antichain of P. If there is no minimal element in A we can apply Lemma 5.2 (i) to obtain some minimal element x of P with $P \setminus \{x\}$ irreducible and $w(P \setminus \{x\}) \geq w(A) = 3$, so $P \setminus \{x\} \in \mathfrak{A}$ by induction. If there is some minimal element in A but there is no maximal element, we can apply Lemma 5.2 (i) to the dual P^∂ ($Q = P^\partial \in [P]$) and we obtain the same conclusions. Finally, if A has some minimal element and some maximal element then we can apply Lemma 5.2 (ii) to obtain a minimal element $x \notin A$ such that $P \setminus \{x\}$ is irreducible and $P \setminus \{x\} \in \mathfrak{A}$ by induction.

Finally, we can choose any minimal element $y \neq x$ of P and we get $e(P \setminus \{x, y\}) \geq 2$. Indeed, $A \setminus \{y\}$ has a 2-element antichain contained in $P \setminus \{x, y\}$. Therefore, by Lemma 5.4 the result holds. □

5.4 Chain-irreducible connected abundant posets with $w(P) = 2$

It remains to study the case $w(P) \leq 2$. Observe that the case $w(P) = 1$, i.e. chains, is trivial since every chain is deficient (and obviously is not chain-irreducible). So let us focus on the case $w(P) = 2$. We are going to divide the study of chain-irreducible connected abundant posets with $w(P) = 2$ into two cases: $h_2(P) \leq 2$ and $h_2(P) \geq 3$ (see Definition 2.2). Let us start with the case $h_2(P) \leq 2$. Observe that the case $h_2(P) = 1$ implies (modulo duality) $P \cong (\boldsymbol{k}_1 \uplus \mathbf{1}) \oplus \boldsymbol{k}_2$, $k_2 \geq 1$, which is always chain-reducible. Let us study the case $h_2(P) = 2$.

Theorem 5.4 (**Characterization of chain-irreducible connected abundant posets with** $w(P) = 2$ **and** $h_2(P) = 2$)**.** *Let P be a chain-irreducible connected poset with $w(P) = 2$ and $h_2(P) = 2$. Then P is abundant if and only if P and P^∂ are not in the family*

$$\mathfrak{CD}^2 = \{CD_1^m, CD_2^m, CD_3, CD_4, CD_5, CD_6, CD_7, CD_8\},$$

given in Figure 5.

Proof. As $w(P) = 2$ and $h_2(P) = 2$, P can be decomposed into one chain of length two and one longer chain. Since P should be connected there are just two possible choices for P or P^∂ given by Cases A and B in Figure 12. Let us denote by P_{m_1,m_2} and P_{m_1,m_2,m_3} the posets belonging to Case A and Case B, respectively.

In Case A, $m_2 \geq 1$, because if $m_2 = 0$ then $P = P_1 \oplus \mathbf{1}$, a contradiction. Moreover, counting antichains

$$a(P_{m_1,m_2}) = i(P_{m_1,m_2}) = 2m_1 + 3m_2 + 5.$$

For counting $e(P_{m_1,m_2})$ we use the fact that for every poset P,

$$e(P) = \sum_{x \in \mathcal{MIN}(P)} e(P \setminus \{x\}).$$

Therefore, $e(P_{m_1,m_2}) = (m_2 + 1) + e(P_{m_1-1,m_2})$.

Next, $e(P_{0,m_2}) = (m_2 + 1) + \binom{m_2+2}{2}$ and thus,

$$e(P_{m_1,m_2}) = (m_1 + 1)(m_2 + 1) + \binom{m_2 + 2}{2} = \frac{1}{2}(m_2 + 1)(2m_1 + m_2 + 4).$$

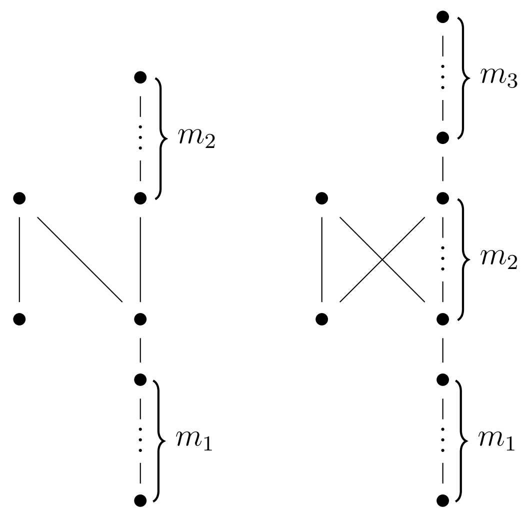


Figure 12: Possible chain-irreducible connected posets with $w(P) = 2$ and $h_2(P) = 2$.

If $m_2 = 1$, $i(P_{m_1,1}) > e(P_{m_1,1})$, so the poset is deficient and we get the dual of CD_1^m. If $m_2 = 2$, then $P_{m_1,2}$ is abundant for $m_1 > 1$ and deficient for $m_1 \leq 1$. The values $m_1 = 0$ and $m_1 = 1$ give us posets CD_3^∂ and CD_4^∂. It is straightforward to check that for $m_2 \geq 3$, $P_{m_1,m_2} \in \mathfrak{A}$.

In Case B, $m_1, m_3 \geq 0$ and $m_2 \geq 2$. Proceeding as before,

$$i(P_{m_1,m_2,m_3}) = 2m_1 + 3m_2 + 2m_3 + 1.$$

For $e(P_{m_1,m_2,m_3})$, it can be seen as in Case A that

$$e(P_{m_1,m_2,m_3}) = (m_2 + m_3) + e(P_{m_1-1,m_2,m_3}).$$

Moreover, $e(P_{0,m_2,m_3}) = (m_2 + m_3) + \binom{m_2}{2} + (m_2 - 1)(m_3 + 1)$. Therefore,

$$e(P_{m_1,m_2,m_3}) = (m_1 + 1)(m_2 + m_3) + \binom{m_2}{2} + (m_2 - 1)(m_3 + 1).$$

If $m_2 = 2$, we get a deficient poset if and only if $m_1 m_3 < 3$. So we get a deficient poset in the next cases: when $m_1 = 0$ (or $m_3 = 0$) we get poset CD_2^m (or its dual) and when $m_1 = 1$ and $1 \leq m_3 \leq 2$ (or $m_3 = 1$ and $1 \leq m_1 \leq 2$) we get CD_5 and CD_6 (or CD_6^∂). If $m_2 = 3$, we get a deficient poset if and only if $(m_1 + 1)(m_3 + 1) < 3$. So we get a deficient poset when $m_1 = 0$ and $m_3 \leq 1$ (or $m_3 = 0$ and $m_1 \leq 1$) and we get posets CD_7 and CD_8 (or their duals). Finally, if $m_2 \geq 4$ we always get an abundant poset, so the result holds. □

Lemma 5.6. *Let P be a chain-irreducible connected poset with $w(P) = 2$, $h_2(P) \geq 3$ and let $x \in \mathcal{MIN}(P)$.*

(*i*) *If $P \setminus \{x\}$ is disconnected, then at least one of P or $P \setminus \{x\}$ is abundant.*

(*ii*) *Let* $\overline{\mathfrak{CD}}^3 := \{CD_9, CD_{10}, CD_{10}^\partial, CD_{11}, CD_{12}, CD_{12}^\partial\}$. *If* $P \notin \overline{\mathfrak{CD}}^3$ *and* $P\setminus\{x\} \in \overline{\mathfrak{CD}}^3$, *then* $P \in \mathfrak{A}$.

Proof. (*i*) If $P \setminus \{x\} \in \mathfrak{A}$ then we are done, so let us suppose that $P \setminus \{x\} \notin \mathfrak{A}$. Hence, by Theorem 5.1, $P \setminus \{x\} \in \mathfrak{CD}^1$. On the other hand, as $w(P \setminus \{x\}) \leq 2$, it follows that the only possible cases are

$$P \setminus \{x\} \in \{\mathbf{1} \uplus \boldsymbol{m}, \mathbf{2} \uplus \mathbf{2}, \mathbf{2} \uplus \mathbf{3}\}.$$

Next, since $h_2(P) \geq 3$, this implies that $|P| \geq 6$ and i $|P \setminus \{x\}| \geq 5$. Therefore, $P \setminus \{x\} \ncong \mathbf{2} \uplus \mathbf{2}$.

If $P \setminus \{x\} \cong \mathbf{1} \uplus \boldsymbol{m}$, then P should be isomorphic to the poset displayed in Case 1 of Figure 13 and thus $h_2(P) = 2$, a contradiction.

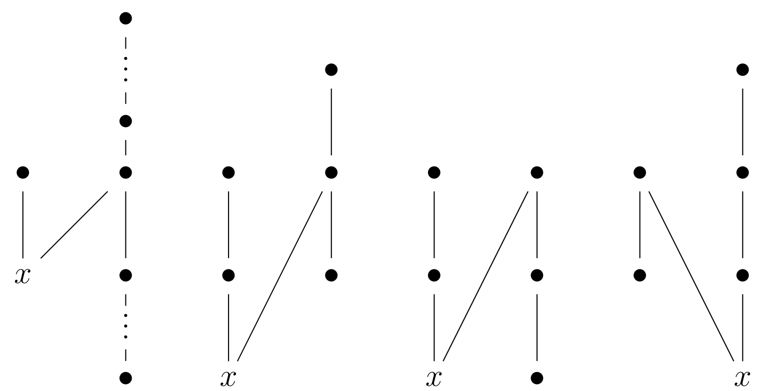


Figure 13: Different cases for P when $P \setminus \{x\} \in \{\mathbf{1} \uplus \boldsymbol{m}, \mathbf{2} \uplus \mathbf{3}\}$.

Finally, assume $P \setminus \{x\} \cong \mathbf{2} \uplus \mathbf{3}$. Then P should be isomorphic to one of the posets displayed in Cases 2, 3 and 4 of Figure 13. It is easy to check that Cases 2 and 3 are abundant with pairs $(i(P), e(P))$ given by $(14, 16), (15, 19)$, respectively. For Case 4, $h_2(P) = 2$, a contradiction.

(*ii*) Let us consider each case. If $P \setminus \{x\} = CD_9 = \bar{\mathbf{2}} \oplus \bar{\mathbf{2}} \oplus \bar{\mathbf{2}}$, then $P \cong (\mathbf{2} \uplus \mathbf{1}) \oplus \bar{\mathbf{2}} \oplus \bar{\mathbf{2}}$ which is abundant $(i(P), e(P)) = (12, 12)$. If $P \setminus \{x\} = CD_{10} = \bar{\mathbf{2}} \oplus N$, then $P \cong (\mathbf{2} \uplus \mathbf{1}) \oplus N$ which is also abundant $(i(P), e(P)) = (13, 15)$. If $P \setminus \{x\}$ is $CD_{10}^\partial, CD_{11}, CD_{12}$ or CD_{12}^∂ then the different cases with P irreducible and $w(P) = 2$ can be seen in the first, second, third and fourth rows of Figure 14, respectively. The pairs $(i(P), e(P))$ of each case are computed in Figure 14. We can observe that in all the possibilities, $P \in \mathfrak{A}$. □

Theorem 5.5 (**Characterization of chain-irreducible connected abundant posets with** $w(P) = 2$ **and** $h_2(P) \geq 3$). *Let P be a chain-irreducible connected poset with $w(P) = 2$ and $h_2(P) \geq 3$. Then $P \in \mathfrak{A}$ if and only if $P, P^\partial \notin \mathfrak{CD}^3$.*

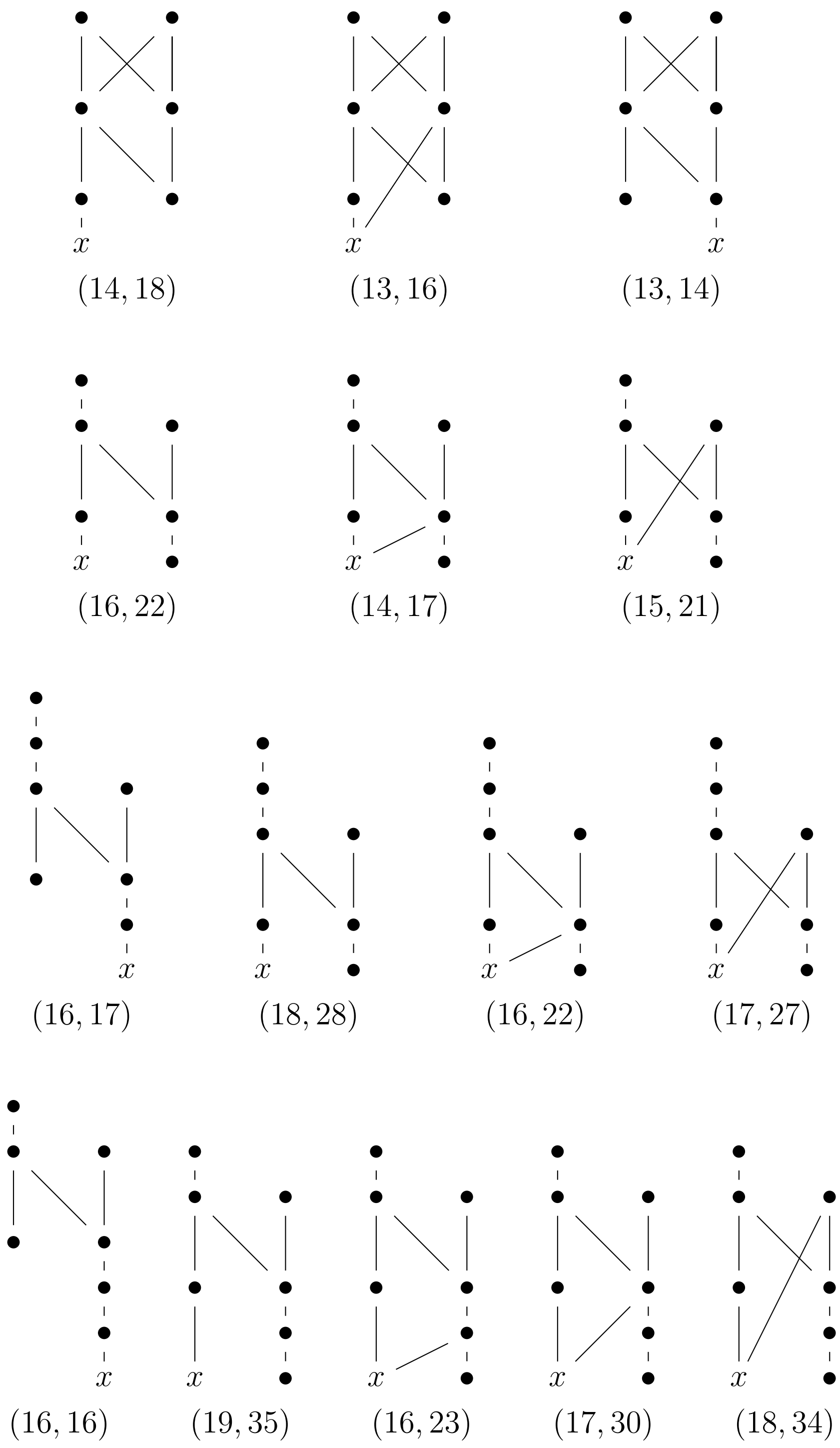
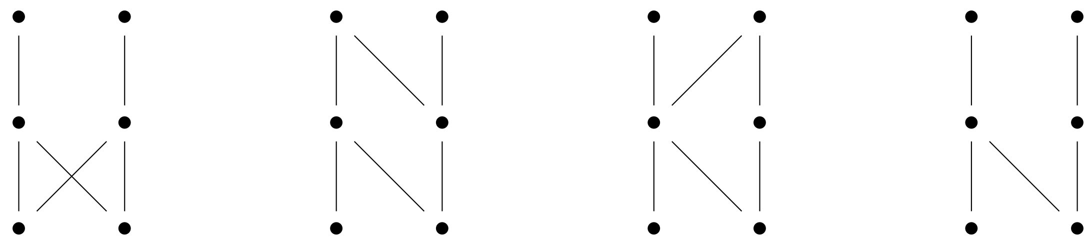


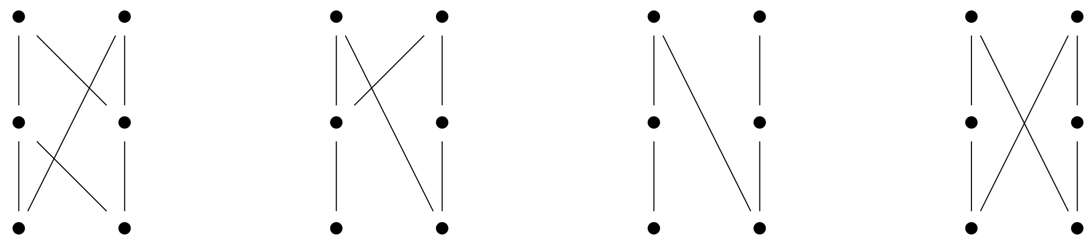
Figure 14: Different cases for P when $P \setminus \{x\} \in \overline{\mathfrak{CD}}^3$.

Proof. Observe that $P, P^\partial \notin \mathfrak{CD}^3$ if and only if $P \notin \overline{\mathfrak{CD}}^3$ (as defined in Lemma 5.6). We have already seen that $\overline{\mathfrak{CD}}^3 \subseteq \mathfrak{D}$. Hence, let us see that any other P in the conditions of the theorem is in $\mathfrak{A}$. We will prove this applying induction on $|P|$.

For $|P| = 6$, there are just eight posets (up to isomorphism) such that P is chain-irreducible, connected, $P \notin \overline{\mathfrak{CD}}^3$, $w(P) = 2$ and $h_2(P) \geq 3$ (see [18]). These posets and their corresponding pairs $(i(P), e(P))$ are shown in Figure 15. As it can be seen, these eight posets are abundant.



$(12, 12)$ $(13, 14)$ $(12, 12)$ $(14, 16)$



$(12, 13)$ $(13, 15)$ $(15, 19)$ $(14, 18)$

Figure 15: Posets in induction base with $|P| = 6$ and their corresponding pairs $(i(P), e(P))$.

Now let P be a chain-irreducible, connected poset such that $w(P) = 2$, $h_2(P) \geq 3$, $P \notin \overline{\mathfrak{CD}}^3$, $|P| > 6$, and assume the result holds until $|P| - 1$.

Let us first consider the case in which there exists $x \in \mathcal{MIN}(P)$ such that $P \setminus \{x\}$ is not connected. By Lemma 5.6 (i), this implies that $P \in \mathfrak{A}$ or $P \setminus \{x\} \in \mathfrak{A}$. If $P \in \mathfrak{A}$, then we are done.

Otherwise, $P \setminus \{x\} \in \mathfrak{A}$. Note that as P is chain-irreducible, there exists $y \in \mathcal{MIN}(P)$ such that $y \neq x$ and $e(P \setminus \{x, y\}) \geq 2$. Otherwise, $P \setminus \{x, y\}$ would be a chain and this would imply that either $P \cong (\boldsymbol{k}_1 \uplus \mathbf{1}) \oplus \boldsymbol{k}_2$ with $k_2 \geq 1$ or $P \cong (((\boldsymbol{k}_1 \uplus \mathbf{1}) \oplus \boldsymbol{k}_2) \uplus \mathbf{1}) \oplus \boldsymbol{k}_3$ with $k_3 \geq 1$, a contradiction since P is chain-irreducible. Hence, we can apply Lemma 5.4 and conclude that $P \in \mathfrak{A}$.

Thus, we can assume that $\forall x \in \mathcal{MIN}(P), P \setminus \{x\}$ is connected. If $P \setminus \{x\} \in \overline{\mathfrak{CD}}^3$, we can apply Lemma 5.6 (ii) to conclude that $P \in \mathfrak{A}$. Hence we can also assume that $P \setminus \{x\} \notin \overline{\mathfrak{CD}}^3$.

Note that as P is chain-irreducible and $w(P) = 2$, this implies that $w(P \setminus \{x\}) = 2$.

Otherwise, if $w(P \setminus \{x\}) = 1$, this would imply that $P \setminus \{x\}$ is a chain and thus

$$P = (\boldsymbol{k}_1 \uplus x) \oplus \boldsymbol{k}_2,\ k_2 \geq 1,$$

a contradiction to the fact that P is chain-irreducible.

In addition, we can assume that $P = P_k \oplus \overbrace{\bar{\mathbf{2}} \oplus \cdots \oplus \bar{\mathbf{2}}}^{k}$ where $P_k \ncong \bar{\mathbf{2}} \oplus P'_k$. If $P = \bar{\mathbf{2}} \oplus P_1$ we can take $Q = P_1 \oplus \bar{\mathbf{2}} \in [P]$. Now, if $P_1 \cong \bar{\mathbf{2}} \oplus P_2$ we can take $Q = P_2 \oplus \bar{\mathbf{2}} \oplus \bar{\mathbf{2}} \in [P]$. If we repeat this reasoning we have two choices: $P = \overbrace{\bar{\mathbf{2}} \oplus \cdots \oplus \bar{\mathbf{2}}}^{k}$ which is abundant since $k \geq 4$, or $Q = P_k \oplus \overbrace{\bar{\mathbf{2}} \oplus \cdots \oplus \bar{\mathbf{2}}}^{k} \in [P]$ where $P_k \ncong \bar{\mathbf{2}} \oplus P'_k$.

With the last considerations in mind, we have to now consider two different cases:

Case 1: $h_2(P) \geq 4$.

In this case, let us start by showing that there exists $x \in \mathcal{MIN}(P)$ (or $x \in \mathcal{MIN}(Q)$ where $Q \in [P]$) such that $P \setminus \{x\}$ is chain-irreducible and $h_1(P \setminus \{x\}) \leq h_1(P)$.

First, note that without loss of generality, the Hasse diagram of P is given as in Figure 16.

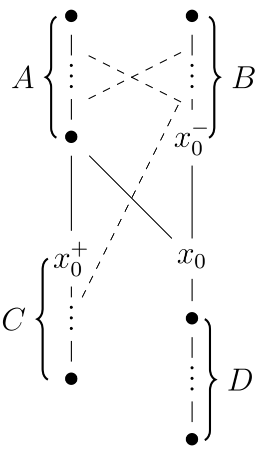


Figure 16: Hasse diagram for P (or P^∂) in Cases 1 and 2 of Theorem 5.5.

Consider a partition (C_1^*, C_2^*) of P into two chains such that $|C_1^*| = h_1(P)$, $|C_2^*| = h_2(P)$. As P is connected, there exist $a \in C_1^*, b \in C_2^*$ such that either $a \lessdot b$ or $b \lessdot a$. Let us consider (a, b) minimal in the sense that there does not exist a different pair $a' \in C_1^*, b' \in C_2^*$ satisfying $a' \lessdot b'$ or $b' \lessdot a'$ and such that $a' \preceq a, b' \preceq b$.

Given such (a, b), this allows the decomposition of P into several parts (namely A, B, C and D) as shown in Figure 16. In this figure, let us denote by $x_0 := \min\{a, b\}$. If $x_0 \in C_i^*$, let us denote by x_0^- the element in C_i^* covering x_0. Element x_0^- always exists. Otherwise, $B = \emptyset, A \neq \emptyset$ ($a \in A$ or $b \in A$) and hence $P = P_1 \oplus A$, and P would be chain-reducible. Finally, note that $|C| \geq 1$ (otherwise $|C| = 0$ and $P = D \oplus P_1$, so that P would be chain-reducible). Hence, we denote by x_0^+ the maximum of chain C.

By construction, an element in D is not related to an element in C. Note however that more relations between some other different parts of P are possible. This is depicted in Figure 16 as dashed lines.

Now consider a minimal element x in P. Note that $w(P \setminus \{x\}) = 2$. Obviously, $x \in C,\ x \in D$ or $x = x_0$ (if $|D| = 0$).

Suppose $|C| > 1$ and $x \in C$. Consider a partition $(C_1^{*'}, C_2^{*'})$ of $P \setminus \{x\}$ such that $|C_1^{*'}| = h_1(P \setminus \{x\})$. If $x_0^+ \in C_i^{*'}$, then $C_i^{*'} \cup \{x\}$ is a chain in P. Hence, by Lemma 5.3, $h_1(P \setminus \{x\}) \leq h_1(P)$. It remains to see that $P \setminus \{x\}$ is chain-irreducible, but this holds because the elements of C are not related to x_0.

Suppose $D \neq \emptyset$ and $x \in D$. Consider a partition $(C_1^{*'}, C_2^{*'})$ of $P \setminus \{x\}$ such that $|C_1^{*'}| = h_1(P \setminus \{x\})$. If $x_0 \in C_i^{*'}$, then $C_i^{*'} \cup \{x\}$ is a chain in P. Hence, again by Lemma 5.3, $h_1(P \setminus \{x\}) \leq h_1(P)$. It remains to be checked that $P \setminus \{x\}$ is chain-irreducible, but this holds because the elements of D are not related to x_0^+.

Finally, let us suppose $C = \{x_0^+\}$ and $D = \emptyset$. Take $x = x_0$. Consider a partition $(C_1^{*'}, C_2^{*'})$ of $P \setminus \{x\}$ such that $|C_1^{*'}| = h_1(P \setminus \{x\})$ and suppose $x_0^- \in C_i^{*'}$. If $x_0^+ \preceq x_0^-$ then $P = \bar{\mathbf{2}} \oplus P_1$ with P_1 some poset, a contradiction. Thus $x_0^+ \npreceq x_0^-$ and $C_i^{*'} \cup \{x\}$ is a chain in P. Hence, again by Lemma 5.3, $h_1(P \setminus \{x\}) \leq h_1(P)$. Moreover, as $x_0^- \parallel x_0^+$, then $P \setminus \{x\}$ is chain-irreducible.

Therefore we know that there exists $x \in \mathcal{MIN}(P)$ (or $x \in \mathcal{MIN}(Q)$ where $Q \in [P]$) such that $P \setminus \{x\}$ is chain-irreducible and $h_1(P \setminus \{x\}) \leq h_1(P)$. Observe that

$$h_2(P \setminus \{x\}) = |P| - 1 - h_1(P \setminus \{x\}) \geq |P| - 1 - h_1(P) = h_2(P) - 1 \geq 3.$$

Therefore, we can use induction to get $P \setminus \{x\} \in \mathfrak{A}$.

Finally, we have already seen that there exists $y \in \mathcal{MIN}(P),\ y \neq x$, such that $e(P \setminus \{x, y\}) \geq 2$. Hence we can apply Lemma 5.4 and conclude that $P \in \mathfrak{A}$.

Case 2: $h_2(P) = 3$.

As in the previous case, the possibilities for P are given in Figure 16. Moreover, we can decompose P into two chains (C_1^*, C_2^*) such that $|C_1^*| = h_1(P),\ |C_2^*| = h_2(P) = 3$ and we assume (taking duals $Q = P^\partial \in [P]$ and relabeling parts if necessary) that C_2^* is the chain (D, x_0, B). Let us take $x \in \mathcal{MIN}(P)$. Then $x \in C,\ x \in D$ or $x = x_0$ (if $D = \emptyset$).

Suppose $|C| > 1$ and let us choose $x \in C$. Then, as in the case for $h_2(P) > 3$, consider a partition $(C_1^{*'}, C_2^{*'})$ of $P \setminus \{x\}$ such that $|C_1^{*'}| = h_1(P \setminus \{x\})$. Now, $x_0^+ \in C_1^{*'}$, then $C_1^{*'} \cup \{x\}$ is a chain in P. Hence, by Lemma 5.3, $h_1(P \setminus \{x\}) \leq h_1(P)$. Moreover, since the chain containing x is $C_1^{*'}$, we obtain $h_1(P \setminus \{x\}) + 1 \leq h_1(P)$ (see the proof of Lemma 5.3). Note that the left chain $(C \setminus \{x\}, A)$ in $P \setminus \{x\}$ has length $|C| - 1 + |A| \geq 3$ (as $|P| > 6$) and is longer than or equal to the right one (with just three elements). Thus $h_1(P \setminus \{x\}) \geq h_1(P) - 1$. Therefore $h_1(P \setminus \{x\}) = h_1(P) - 1$ and $h_2(P \setminus \{x\}) = h_2(P) = 3$.

Besides, $P \setminus \{x\}$ is chain-irreducible because the chain C is not related to x_0.

Therefore $P \setminus \{x\} \in \mathfrak{A}$ by the induction hypothesis, and we have already seen that there exists $y \in \mathcal{MIN}(P)$, $y \neq x$, such that $e(P \setminus \{x, y\}) \geq 2$. Hencex, we can apply Lemmar 5.4 and conclude that $P \in \mathfrak{A}xp$.

Consider now the case $|C| = 1$. Since P is chain-irreducible and $h_2(P) = 3$, the length of D is bounded, $|D| \leq 1$. Suppose $|D| = 1$. In this case, there are just two possibilities for P that are depicted in the first row of Figure 17. In these cases $m \geq 2$ because $|C_1^*| \geq 4$. Moreover, $m \geq 3$, because for $m = 2$ these posets are CD_{12} and CD_6^∂, respectively. Now, for the first possibility, we get $i(P) = 2m + 10 \leq 3m + 7 = e(P)$, so it is abundant. For the second possibility we get $i(P) = 2m + 9 \leq 3m + 6 = e(P)$ so it is again abundant.

Now consider the last case where $|C| = 1$ and $|D| = 0$. Here we can take $Q = P^\partial \in [P]$ to choose $x \in \mathcal{MAX}(C_1^*)$. It holds that $h_2(P \setminus \{x\}) = 3$. By induction, if $P \setminus \{x\}$ is chain-irreducible, then $P \setminus \{x\} \in \mathfrak{A}$ and there exists $y \neq x$ such that $e(P \setminus \{x, y\}) \geq 2$, and we can use Lemma 5.4 to get $P \in \mathfrak{A}$. So we only have to consider the cases where $P \setminus \{x\}$ is not chain-irreducible for x being the maximum of the longest chain in P. As $h_1(P) \geq 4$, there are only three cases (see Figure 17, second row). These three families of posets are abundant. In the first case, $i(P) = 2m + 14 \leq 4m + 16 = e(P)$. For the second case, $i(P) = 2m + 13 \leq 4m + 14 = e(P)$, and finally for the third case, $i(P) = 2m + 12 \leq 4m + 12 = e(P)$. So the result holds. □

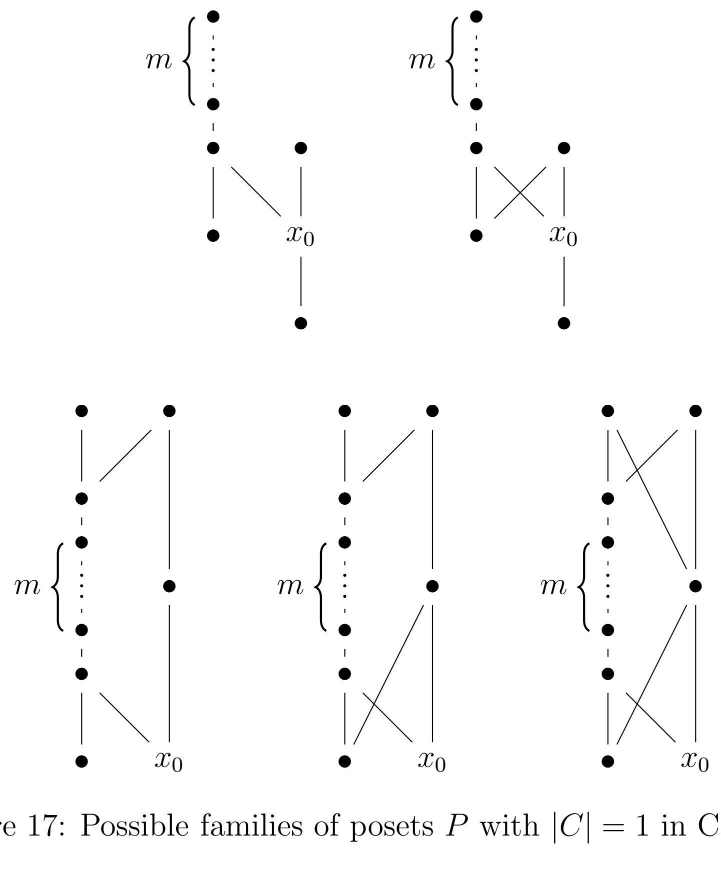


Figure 17: Possible families of posets P with $|C| = 1$ in Case 2.

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