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# Simultaneous coloring of vertices and incidences of graphs

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**Abstract**

An $n$-subdivision of a graph $G$ is a graph constructed by replacing each edge of $G$ with a path of length $n$ and an $m$-power of $G$ is a graph with the same vertices as $G$ and any two vertices of $G$ at distance at most $m$ are adjacent. The graph $G^{\frac{m}{n}}$ is the $m$-power of the $n$-subdivision of $G$. Iradmusa and Mozafari-Nia (2021) conjectured that the chromatic number of $\frac{3}{3}$-power of graphs with maximum degree $\Delta \geq 2$ is at most $2\Delta + 1$. In this paper, we introduce the simultaneous coloring of vertices and incidences of graphs and show that the minimum number of colors for simultaneous proper coloring of vertices and incidences of $G$, denoted by $\chi_{vi}(G)$ and called the $vi$-simultaneous chromatic number of $G$, is equal to the chromatic number of $G^{\frac{3}{3}}$. Also by determining the exact value or the upper bound for the $vi$-simultaneous chromatic number, we investigate the correctness of the conjecture for some classes of graphs such as $k$-degenerate graphs, cycles, forests, complete graphs, and regular bipartite graphs. In addition, we investigate the relationship between this new chromatic number and the other parameters of graphs.

## 1 Introduction

All graphs we consider in this paper are simple, finite and undirected. For a graph $G$, we denote its vertex set, edge set, and face set (if $G$ is planar) by $V(G)$, $E(G)$, and $F(G)$, respectively. Maximum degree, independence number and maximum size of cliques of $G$ are denoted by $\Delta(G)$, $\alpha(G)$ and $\omega(G)$, respectively. Also, for a vertex $v \in V(G)$, $N_G(v)$ is the set of neighbors of $v$ in $G$ and any vertex of degree $k$ is called a *$k$-vertex*. From now on, we use the notation $[n]$ instead of $\{1, \ldots, n\}$. We mention

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some definitions that are referred to throughout this paper and for other necessary definitions and notation we refer the reader to a standard text-book [3].

A mapping $c$ from $V(G)$ to $[k]$ is a *proper k-coloring* of $G$ if $c(v) \neq c(u)$ for any two adjacent vertices. A minimum integer $k$ for which $G$ has a proper $k$-coloring is the *chromatic number* of $G$ and is denoted by $\chi(G)$. Instead of the vertices, we can color the edges of the graph. A mapping $c$ from $E(G)$ to $[k]$ is a *proper edge-k-coloring* of $G$ if $c(e) \neq c(e')$ for any two adjacent edges $e$ and $e'$ ($e \cap e' \neq \varnothing$). A minimum integer $k$ for which $G$ has a proper edge-$k$-coloring is the *chromatic index* of $G$ and is denoted by $\chi'(G)$.

Another coloring of graphs is the coloring of incidences of graphs. The concepts of incidence, incidence graph, and incidence coloring were introduced by Brualdi and Massey in 1993 [5]. In a graph $G$, any pair $i = (v, e)$ is called an *incidence* of $G$ if $v \in V(G)$, $e \in E(G)$, and $v \in e$. Also in this case the elements $v$ and $i$ are called *incident.* For any edge $e = \{u, v\}$, we call $(u, e)$ the first incidence of $u$, and $(v, e)$ the second incidence of $u$. In general, for a vertex $v \in V(G)$, the sets of the first incidences and the second incidences of $v$ are denoted by $I_1^G(v)$ and $I_2^G(v)$, respectively. Also let $I_G(v) = I_1^G(v) \cup I_2^G(v)$ , $I_1^G[v] = \{v\} \cup I_1^G(v)$, and $I_G[v] = \{v\} \cup I_G(v)$. Sometime we remove the index $G$ for simplicity.

Let $I(G)$ be the set of the incidences of $G$. The *incidence graph* of $G$, denoted by $\mathcal{I}(G)$, is a graph with vertex set $V(\mathcal{I}(G)) = I(G)$ such that two incidences $(v, e)$ and $(w, f)$ are adjacent in $\mathcal{I}(G)$ if (i) $v = w$; or (ii) $e = f$; or (iii) $\{v, w\} = e$ or $f$. Any proper $k$-coloring of $\mathcal{I}(G)$ is an *incidence k-coloring* of $G$. The *incidence chromatic number* of $G$, denoted by $\chi_i(G)$, is the minimum integer $k$ such that $G$ is incidence $k$-colorable. An incidence $k$-coloring of $G$ is an incidence $(k, s)$-coloring of $G$ if for any vertex $v$, the number of colors used for coloring $I_2(v)$ is at most $s$. We denote by $\chi_{i,s}(G)$ the smallest number of colors required for an incidence $(k, s)$-coloring of $G$. Note that $\chi_i(G) \leq \chi_{i,s}(G)$ for every graph $G$ and every $s \in \mathbb{N}$.

Total coloring is one of the first simultaneous colorings of graphs. A mapping $c$ from $V(G) \cup E(G)$ to $[k]$ is a *proper total-k-coloring* of $G$ if $c(x) \neq c(y)$ for any two adjacent or incident elements $x$ and $y$. A minimum integer $k$ for which $G$ has a proper total-$k$-coloring is the *total chromatic number* of $G$ and is denoted by $\chi''(G)$ [2]. In 1965, Behzad conjectured that $\chi''(G)$ never exceeds $\Delta(G)+2$. This conjecture is known as the Total Coloring Conjecture (TCC).

Another simultaneous coloring began in the mid-1960s with Ringel [14], who conjectured that the vertices and faces of a planar graph may be colored with six colors such that every two adjacent or incident ones are colored differently. In addition to total coloring defined for any graph, there are three other types of simultaneous colorings of a planar graph $G$, depending on using at least two of the sets $V(G)$, $E(G)$, and $F(G)$ in the coloring. These colorings of graphs have been studied extensively in the literature and there are many results and many open problems. For further information see [4, 6, 16, 17].

Inspired by the total coloring of a graph $G$ and its connection with the fractional power of graphs introduced in [13], we define a new kind of simultaneous coloring of graphs. In this coloring, we color simultaneously the vertices and the incidences of a graph.

**Definition 1.1** Let $G$ be a graph. A *vi-simultaneous proper k-coloring* of $G$ is a coloring $c : V(G) \cup I(G) \longrightarrow [k]$ in which any two adjacent or incident elements in the set $V(G) \cup I(G)$ receive distinct colors. The *vi-simultaneous chromatic number*, denoted by $\chi_{vi}(G)$, is the smallest integer $k$ such that $G$ has a $vi$-simultaneous proper $k$-coloring.

**Example 1.2** As a first example, we consider the cycle graphs $C_3$ and $C_4$. We know that $\chi(C_3) = \chi'(C_3) = 3$ and $\chi''(C_3) = \chi_i(C_3) = 3$. But four colors are not enough for $vi$-simultaneous proper coloring of $C_3$ and one can easily show that $\chi_{vi}(C_3) = 5$. For the cycle of order 4, we have $\chi(C_4) = \chi'(C_4) = 2$ and $\chi''(C_4) = \chi_i(C_4) = 4$. In addition, Figure 1 shows a $vi$-simultaneous proper 4-coloring of $C_4$ and so $\chi_{vi}(C_4) = 4$.

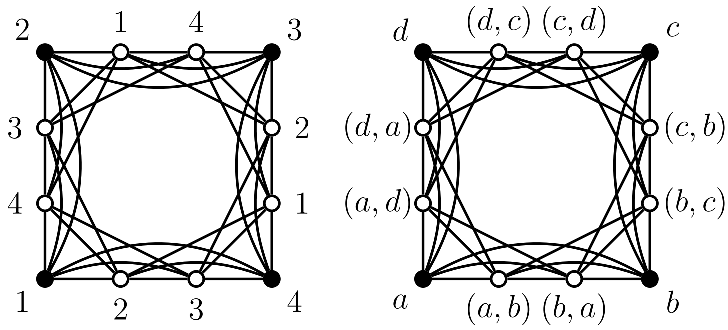


Figure 1: (Right) Black and white vertices are corresponding to the vertices and the incidences of $C_4$, respectively. The incidence $(u,\{u,v\})$ is denoted by $(u,v)$. Each edge represents an incidence or adjacency between two elements. (Left) A $vi$-simultaneous proper 4-coloring of $C_4$.

Similar to incidence coloring, we can define some special kind of $vi$-simultaneous coloring of graphs according to the number of colors appearing on the incidences of each vertex.

**Definition 1.3** A $vi$-simultaneous proper $k$-coloring of a graph $G$ is called a $vi$-simultaneous $(k,s)$-coloring of $G$ if for any vertex $v$, the number of colors used for coloring $I_2(v)$ is at most $s$. We denote by $\chi_{vi,s}(G)$ the smallest number of colors required for a $vi$-simultaneous $(k,s)$-coloring of $G$.

For example, the $vi$-simultaneous coloring of $C_4$ in Figure 1 is a $vi$-simultaneous $(4,1)$-coloring and so $\chi_{vi,1}(C_4) = 4$. Observe that $\chi_{vi,1}(G) \geq \chi_{vi,2}(G) \geq \cdots \geq \chi_{vi,\Delta}(G) = \chi_{vi}(G)$ for every graph $G$ with maximum degree $\Delta$.

### 1.1 Fractional power of graphs

For the edge coloring and total coloring of any graph $G$, two corresponding graphs are defined. In the *line graph* of $G$, denoted by $\mathcal{L}(G)$, the vertex set is $E(G)$ and two vertices $e$ and $e'$ are adjacent if $e \cap e' \neq \varnothing$. In the *total graph* of $G$, denoted by $\mathcal{T}(G)$, the vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they are adjacent or incident in $G$. According to these definitions, we have $\chi'(G) = \chi(\mathcal{L}(G))$ and $\chi''(G) = \chi(\mathcal{T}(G))$. So edge coloring and total coloring of graphs can be converted to vertex coloring of graphs.

Motivated by the concept of the total graph, the fractional power of a graph was first introduced in [13]. Let $G$ be a graph and $k$ be a positive integer. The *$k$-power of $G$*, denoted by $G^k$, is defined on the vertex set $V(G)$ by adding edges joining any two distinct vertices $x$ and $y$ with distance at most $k$. Also the *$k$-subdivision* of $G$, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge $xy$ of $G$ with a path of length $k$ with the vertices $x = (xy)_0, (xy)_1, \ldots, (xy)_{k-1}$, and $y = (xy)_k$. Note that the vertex $(xy)_l$ has distance $l$ from the vertex $x$, where $l \in \{0, 1, \ldots, k\}$. Also, $(xy)_l = (yx)_{k-l}$, for any $l \in \{0, 1, \ldots, k\}$. The vertices $(xy)_0$ and $(xy)_k$ are called terminal vertices and the others are called internal vertices. We refer to these vertices, in short, as $t$-vertices and $i$-vertices of $G^{\frac{1}{k}}$, respectively. Now the fractional power of a graph $G$ is defined as follows.

**Definition 1.4** Let $G$ be a graph and $m, n \in \mathbb{N}$. The graph $G^{\frac{m}{n}}$ is defined to be the $m$-power of the $n$-subdivision of $G$. In other words, $G^{\frac{m}{n}} = (G^{\frac{1}{n}})^m$.

The sets of terminal and internal vertices of $G^{\frac{m}{n}}$ are denoted by $V_t(G^{\frac{m}{n}})$ and $V_i(G^{\frac{m}{n}})$, respectively. It is worth noting that $G^{\frac{1}{1}} = G$ and $G^{\frac{2}{2}} = \mathcal{T}(G)$.

By Definition 1.4, one can show that $\omega(G^{\frac{2}{2}}) = \Delta(G) + 1$, and the Total Coloring Conjecture can be reformulated as follows.

**Conjecture 1.5** *For any simple graph $G$, $\chi(G^{\frac{2}{2}}) \leq \omega(G^{\frac{2}{2}}) + 1$.*

In [13], the chromatic number of some fractional powers of graphs was first studied and it was proved that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ where $n = m + 1$ or $m = 2 < n$. Also it was conjectured that $\chi(G^{\frac{m}{n}}) = \omega(G^{\frac{m}{n}})$ for any graph $G$ with $\Delta(G) \geq 3$ when $\frac{m}{n} \in \mathbb{Q} \cap (0, 1)$. This conjecture was disproved by Hartke, Liu, and Petrickova [9] who proved that the conjecture is not true for the cartesian product $C_3 \square K_2$ (triangular prism) when $m = 3$ and $n = 5$. However, they claimed that the conjecture is valid except when $G = C_3 \square K_2$. In addition, they proved that the conjecture is true when $m$ is even.

It can be easily seen that $G$ and $\mathcal{I}(G)$ are isomorphic to the induced subgraphs of $G^{\frac{3}{3}}$ by $V_t(G^{\frac{3}{3}})$ and $V_i(G^{\frac{3}{3}})$, the sets of terminal and internal vertices of $G^{\frac{3}{3}}$, respectively. So $\chi_i(G) = \chi(G^{\frac{3}{3}}[V_i(G^{\frac{3}{3}})])$. Also, by considering the 3-subdivision of a graph $G$, two internal vertices $(uv)_1$ and $(uv)_2$ of the edge $\{u, v\}$ in $G^{\frac{3}{3}}$ correspond to the incidences of the edge $\{u, v\}$ in $G$. For convenience, we denote $(uv)_1$ and $(uv)_2$ by $(u, v)$ and $(v, u)$, respectively.

Similar to the equality $\chi''(G) = \chi(G^{\frac{2}{2}})$, we have the following basic theorem about the relation between $vi$-simultaneous coloring of a graph and vertex coloring of its $\frac{3}{3}$-power.

**Theorem 1.6** *For any graph $G$, $\chi_{vi}(G) = \chi(G^{\frac{3}{3}})$.*

Because of Theorem 1.6, we use the terms $\chi_{vi}(G)$ and $\chi(G^{\frac{3}{3}})$ interchangeably in the rest of the paper. We often use the notation $\chi_{vi}(G)$ to express the theorems and the notation $\chi(G^{\frac{3}{3}})$ in the proofs.

As mentioned in [13], one can easily show that $\omega(G^{\frac{3}{3}}) = \Delta(G)+2$ when $\Delta(G) \geq 2$, and $\omega(G^{\frac{3}{3}}) = 4$ when $\Delta(G) = 1$. So $\Delta(G)+2$ is a lower bound for $\chi(G^{\frac{3}{3}})$ and $\chi_{vi}(G)$, when $\Delta(G) \geq 2$. In [13], the chromatic number of fractional power of cycles and paths are considered, which can show that graphs with maximum degree 2 are $vi$-simultaneous 5-colorable (see Section 4). In [12, 15] it is shown that $\chi(G^{\frac{3}{3}}) \leq 7$ for any graph $G$ with maximum degree 3. Moreover, in [11] it is proved that $\chi(G^{\frac{3}{3}}) \leq 9$ for any graph $G$ with maximum degree 4. Also in [12] it is proved that $\chi(G^{\frac{3}{3}}) \leq \chi(G) + \chi_i(G)$ when $\Delta(G) \leq 2$, and $\chi(G^{\frac{3}{3}}) \leq \chi(G) + \chi_i(G) - 1$ when $\Delta(G) \geq 3$. In addition, in [5], it is shown that $\chi_i(G) \leq 2\Delta(G)$ for any graph $G$. Hence, for any graph $G$ of maximum degree $\Delta(G) \geq 2$, $\chi(G^{\frac{3}{3}}) = \chi_{vi}(G) \leq 3\Delta(G)$.

According to the results mentioned in the previous paragraph, the following conjecture is true for graphs with maximum degree at most 4.

**Conjecture 1.7** [11] *Let $G$ be a graph with $\Delta(G) \geq 2$. Then $\chi(G^{\frac{3}{3}}) \leq 2\Delta(G) + 1$.*

We know that $\chi(G^{\frac{3}{3}}) \geq \omega(G^{\frac{3}{3}}) = \Delta(G) + 2$ when $\Delta(G) \geq 2$. In addition, the Total Coloring Conjecture states that $\chi(G^{\frac{2}{2}}) \leq \Delta(G) + 2$. Therefore, if the Total Coloring Conjecture is correct, then the following conjecture is also true.

**Conjecture 1.8** [11] *Let $G$ be a graph with $\Delta(G) \geq 2$. Then $\chi(G^{\frac{2}{2}}) \leq \chi(G^{\frac{3}{3}})$.*

Similar to the graphs $\mathcal{L}(G)$, $\mathcal{T}(G)$, and $\mathcal{I}(G)$, for any graph $G$ we can define a corresponding graph, denoted by $\mathcal{T}_{vi,1}(G)$, such that $\chi_{vi,1}(G) = \chi(\mathcal{T}_{vi,1}(G))$.

**Definition 1.9** Let $G$ be a nonempty graph. The graph $\mathcal{T}_{vi,1}(G)$ is a graph with vertex set $V(G) \times [2]$ such that two vertices $(v, i)$ and $(u, j)$ are adjacent in $\mathcal{T}_{vi,1}(G)$ if and only if one of the following conditions holds:

- $i = j = 1$ and $d_G(v, u) = 1$,
- $i = j = 2$ and $1 \leq d_G(v, u) \leq 2$,
- $i \neq j$ and $0 \leq d_G(v, u) \leq 1$.

**Example 1.10** As an example, $\mathcal{T}_{vi,1}(C_6)$ is shown in Figure 2. Unlabeled vertices belong to $V(C_6) \times \{2\}$.

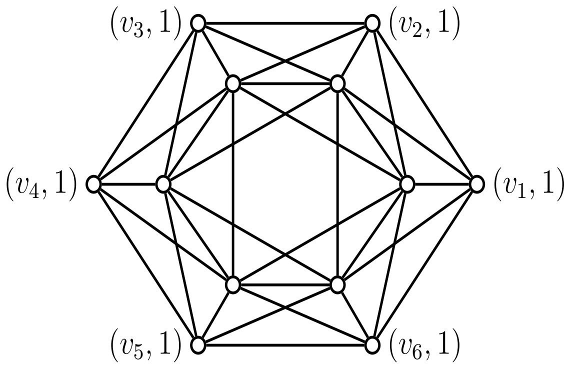


Figure 2: $\mathcal{T}_{vi,1}(C_6)$

**Theorem 1.11** *For any nonempty graph $G$, $\chi_{vi,1}(G) = \chi(\mathcal{T}_{vi,1}(G))$.*

For a graph $G$, the digraph $\overrightarrow{G}$ is obtained from $G$ by replacing each edge of $E(G)$ with two opposite arcs. Any incidence $(v, e)$ of $I(G)$, with $e = \{v, w\}$, can then be associated with the arc $(v, w)$ in $A(\overrightarrow{G})$. So an incidence coloring of $G$ can be viewed as a proper arc coloring of $\overrightarrow{G}$ satisfying $(i)$ any two arcs having the same tail vertex are assigned distinct colors and $(ii)$ any two consecutive arcs are assigned distinct colors.

Similar to incidence coloring, there is another equivalent coloring for proper coloring of the $\frac{3}{3}$-power of a graph or equivalently $vi$-simultaneous proper coloring.

**Definition 1.12** Let $G$ be a graph, let $S = S_t \cup S_i$ be a subset of $V(G^{\frac{3}{3}})$ such that $S_t \subseteq V_t(G^{\frac{3}{3}})$, $S_i \subseteq V_i(G^{\frac{3}{3}})$, and let $H$ be the subgraph of $G^{\frac{3}{3}}$ induced by $S$. Also let $A(S_i) = \{(u, v) \mid (uv)_1 \in S_i\}$ and $V(S_i) = \{u \in V(G) \mid I(u) \cap S_i \neq \varnothing\}$. The underlying digraph of $H$, denoted by $D(H)$, is a digraph with vertex set $S_t \cup V(S_i)$ and arc set $A(S_i)$. In particular, $D(G^{\frac{3}{3}}) = \overrightarrow{G}$.

Now any proper coloring of $G^{\frac{3}{3}}$ (or equivalently, any $vi$-simultaneous coloring of $G$) can be viewed as a coloring of vertices and arcs of $D(G^{\frac{3}{3}})$ satisfying: $(i)$ any two adjacent vertices are assigned distinct colors; $(ii)$ any arc and its head and tail are assigned distinct colors; $(iii)$ any two arcs having the same tail vertex (of the form $(u, v)$ and $(u, w)$) are assigned distinct colors; and $(iv)$ any two consecutive arcs (of the form $(u, v)$ and $(v, w)$) are assigned distinct colors.

A *star* is a tree with diameter at most 2. A *star forest* is a forest whose connected components are stars. The *star arboricity* $st(G)$ of a graph $G$ is the minimum number of star forests in $G$ whose union covers all edges of $G$. In [19] it was proved that $\chi_i(G) \leq \chi'(G) + st(G)$. Similar to this result, we can give an upper bound for $\chi_{vi}(G)$ in terms of the total chromatic number and star arboricity.

**Theorem 1.13** *For any graph $G$, $\chi_{vi}(G) \leq \chi(G^{\frac{2}{2}}) + st(G)$.*

This paper aims to find the exact value or upper bound for the $vi$-simultaneous chromatic number of some classes of graphs by coloring the vertices of $G^{\frac{3}{3}}$ and checking the truthfulness of Conjecture 1.7 for some classes of graphs. We show that Conjecture 1.7 is true for some graphs such as trees, complete graphs, and bipartite graphs. Also, we study the relationship between the $vi$-simultaneous chromatic number and other parameters of graphs.

### 1.2 Structure of the paper

After this introductory section where we have established the background, purpose, and some basic definitions and theorems of the paper, we divide the paper into four sections. In Section 2, we prove Theorems 1.6, 1.11, and 1.13, and some basic lemmas and theorems. In Section 3, we give an upper bound for the $vi$-simultaneous chromatic number of a $k$-degenerate graph in terms of $k$ and the maximum degree of the graph. In Section 4, we provide the exact value for the chromatic number of $\frac{3}{3}$-powers of cycles, complete graphs, and complete bipartite graphs, and we give an upper bound for the chromatic number of $\frac{3}{3}$-powers of bipartite graphs and conclude that Conjecture 1.7 is true for these classes of graphs.

## 2 Basic theorems and lemmas

First, we prove Theorems 1.6, 1.11, and 1.13.

**Proof of Theorem 1.6** First, suppose that $\chi(G^{\frac{3}{3}}) = k$ and that $c : V(G^{\frac{3}{3}}) \longrightarrow [k]$ is a proper coloring of $G^{\frac{3}{3}}$. We show that the following $vi$-simultaneous $k$-coloring of $G$ is proper.

$$c'(x) = \begin{cases} c(x), & x \in V(G) = V_t(G^{\frac{3}{3}}); \\ c((uv)_1), & x = (u,v) \in I(G). \end{cases}$$

Since $G$ is an induced subgraph of $G^{\frac{3}{3}}$ by the terminal vertices, $c$ is a proper coloring of $G$. So $c'$ assigns different colors to the adjacent vertices of $G$. Now suppose that $(u,v)$ and $(r,s)$ are adjacent vertices in $\mathcal{I}(G)$. There are three cases:

(i) $(r,s) = (v,u)$. Since $(vu)_1$ and $(uv)_1$ are adjacent in $G^{\frac{3}{3}}$, $c'((u,v)) = c((uv)_1) \neq c((vu)_1) = c'((r,s))$.

(ii) $r = u$. Since $d_{G^{\frac{1}{3}}}((uv)_1, (us)_1) = 2$, $(uv)_1$ and $(us)_1$ are adjacent in $G^{\frac{3}{3}}$. So in this case, $c'((u,v)) = c((uv)_1) \neq c((us)_1) = c'((u,s))$.

(iii) $r = v$. Since $d_{G^{\frac{1}{3}}}((uv)_1, (vs)_1) = 3$, $(uv)_1$ and $(vs)_1$ are adjacent in $G^{\frac{3}{3}}$. So in this case, $c'((u,v)) = c((uv)_1) \neq c((vs)_1) = c'((v,s))$.

Finally, suppose that $u \in V(G)$ and $(r,s) \in I(G)$ are incident. So $u = r$ or $u = s$. In the first case, we have $d_{G^{\frac{1}{3}}}(u, (rs)_1) = 1$, and in the second case we have $d_{G^{\frac{1}{3}}}(u, (rs)_1) = 2$, and $u$ and $(rs)_1$ are adjacent in $G^{\frac{3}{3}}$. So

$$c'(u) = c(u) \neq c((rs)_1) = c'((r,s)).$$

Similarly we can show that each proper $vi$-simultaneous $k$-coloring of $G$ gives us a proper $k$-coloring of $G^{\frac{3}{3}}$. Therefore, $\chi_{vi}(G) = \chi(G^{\frac{3}{3}})$. □

**Proof of Theorem 1.11** First, suppose that $\chi_{vi,1}(G) = k$, and $c : V(G) \cup I(G) \longrightarrow [k]$ is a $vi$-simultaneous $(k, 1)$-coloring of $G$. We show that the following $k$-coloring of $\mathcal{T}_{vi,1}(G)$ is proper.

$$c'(x) = \begin{cases} c(u) & x = (u, 1), \\ s & x = (u, 2), s \in c(I_2(u)). \end{cases}$$

Since $c$ is a $vi$-simultaneous $(k, 1)$-coloring, $|c(I_2(u))| = 1$ for any vertex $u \in V(G)$ and so $c'$ is well-defined. Now suppose that $(v, i)$ and $(u, j)$ are adjacent in $\mathcal{T}_{vi,1}(G)$.

- If $i = j = 1$, then $c'((v, i)) = c(v) \neq c(u) = c'((u, j))$.
- If $i = j = 2$ and $d_G(v, u) = 1$, then $c'((v, i)) = c(u, v) \neq c((v, u)) = c'((u, j))$.
- If $i = j = 2$ and $d_G(v, u) = 2$, then $c'((v, i)) = c(z, v) \neq c((z, u)) = c'((u, j))$ where $z \in N_G(v) \cap N_G(u)$.
- If $i = 1$, $j = 2$ and $v = u$, then $c'((v, i)) = c(v) \neq c((z, v)) = c'((u, j))$ where $z \in N_G(v)$.
- If $i = 1$, $j = 2$ and $d_G(v, u) = 1$, then $c'((v, i)) = c(v) \neq c((v, u)) = c'((u, j))$.

So $c'$ assigns different colors to the adjacent vertices of $\mathcal{T}_{vi,1}(G)$.

Now suppose that $\chi(\mathcal{T}_{vi,1}(G)) = k$, and $c' : V(\mathcal{T}_{vi,1}(G)) \longrightarrow [k]$ is a proper $k$-coloring of $\mathcal{T}_{vi,1}(G)$. One can easily show that the following $k$-coloring is a $vi$-simultaneous $(k, 1)$-coloring of $G$.

$$c(x) = \begin{cases} c'((x, 1)), & x \in V(G); \\ c'((v, 2)), & x = (u, v) \in I(G). \end{cases}$$

Thus $\chi_{vi,1}(G) = \chi(\mathcal{T}_{vi,1}(G))$. □

**Proof of Theorem 1.13** Let $G$ be a graph with star arboricity $st(G)$ and $s : E(G) \longrightarrow [st(G)]$ be a mapping such that $s^{-1}(i)$ is a star forest for any $i$, $1 \leq i \leq st(G)$. Also, suppose that $c$ is a total coloring of $G$ with colors $\{st(G)+1, \ldots, st(G)+\chi''(G)\}$. Now, to color $t$-vertices and $i$-vertices of the graph $G^{\frac{3}{3}}$, define the mapping $c'$ by $c'((u, v)) = s(\{u, v\})$ if $v$ is the center of a star in some forest $s^{-1}(i)$. If some star is reduced to one edge, we arbitrarily choose one of its end vertices as the center. Note that, for any edge $\{u, v\}$, one of the $t$-vertices $u$ or $v$ is the center of a star forest. It suffices to color the $t$-vertices and the other $i$-vertices of $G^{\frac{3}{3}}$.

Consider the subgraph of $G^{\frac{3}{3}}$ induced by the $t$-vertices and the uncolored $i$-vertices. It can easily be seen that the resulting graph, $G'$, is isomorphic to a subgraph of $G^{\frac{2}{2}}$. Now, assign the colors $c(u)$ and $c(\{u, v\})$ to the $t$-vertex $u$ and the $i$-vertex $(u, v)$ in $G'$, respectively. So we have $\chi(G^{\frac{3}{3}}) \leq \chi(G^{\frac{2}{2}}) + st(G)$. □

For any star forest $F$, we have $st(F) = 1$, $\chi(F^{\frac{2}{2}}) = \Delta(F) + 1$, and $\chi(F^{\frac{3}{3}}) = \Delta(F) + 2$. So the upper bound of Theorem 1.13 is tight.

The following lemmas will be used in the proofs of some theorems in the next sections. The set $\{c(a) \mid a \in A\}$ is denoted by $c(A)$ where $c : D \to R$ is a function and $A \subseteq D$.

**Lemma 2.1** *Let $G$ be a graph with maximum degree $\Delta$ and let $c$ be a proper $(\Delta+2)$-coloring of $G^{\frac{3}{3}}$ with colors from $[\Delta+2]$. Then $|c(I_2(v))| \leq \Delta - d_G(v) + 1$ for any $t$-vertex $v$. In particular, $|c(I_2(v))| = 1$ for any $\Delta$-vertex $v$ of $G$.*

**Proof:** Let $v$ be a $t$-vertex of $G^{\frac{3}{3}}$. Since all vertices in $I_1[v]$ are pairwise adjacent in $G^{\frac{3}{3}}$, there are exactly $d_G(v)+1$ colors in $c(I_1[v])$. Now, consider the vertices in $I_2(v)$. Since any vertex in $I_2(v)$ is adjacent to each vertex of $I_1[v]$, the only available colors for these $i$-vertices are the remaining colors from $[\Delta+2] \setminus c(I_1[v])$. So $|c(I_2(v))| \leq \Delta - d_G(v) + 1$. □

**Lemma 2.2** *Let $G$ be a graph, $e$ be a cut edge of $G$ and $C_1$ and $C_2$ be two components of $G-e$. Then $\chi_{vi,l}(G) = \max\{\chi_{vi,l}(H_1), \chi_{vi,l}(H_2)\}$ where $H_i = C_i + e$ for $i \in \{1,2\}$ and $1 \leq l \leq \Delta(G)$.*

**Proof:** Obviously $\chi_{vi,l}(H_1) \leq \chi_{vi,l}(G)$ and $\chi_{vi,l}(H_2) \leq \chi_{vi,l}(G)$. So $\max\{\chi_{vi,l}(H_1), \chi_{vi,l}(H_2)\} \leq \chi_{vi,l}(G)$. Now suppose that $\chi_{vi,l}(H_1) = k_1 \geq k_2 = \chi_{vi,l}(H_2)$. We show that $\chi_{vi,l}(G) \leq k_1$. Let $c_i : V(H_i) \to [k_i]$ be a $vi$-simultaneous $(k_i, l)$-coloring $(1 \leq i \leq 2)$ and $e = \{u, v\}$. Since $V(H_1) \cap V(H_2) = \{u, (u,v), (v,u), v\}$ and these four vertices induce a clique, so by suitable permutation on the colors of the coloring $c_1$, we reach the new coloring $c'_1$ such that $c'_1(x) = c_2(x)$ for any $x \in \{u, (u,v), (v,u), v\}$. Now we can easily prove that the following coloring is a $vi$-simultaneous $(k_1, l)$-coloring:

$$c(x) = \begin{cases} c'_1(x), & x \in V(H_1); \\ c_2(x), & x \in V(H_2). \end{cases}$$

□

**Lemma 2.3** *Let $G_1$ and $G_2$ be two graphs, $V(G_1) \cap V(G_2) = \{v\}$ and $G = G_1 \cup G_2$. Then*

$$\chi_{vi,1}(G) = \max\{\chi_{vi,1}(G_1), \chi_{vi,1}(G_2), d_G(v)+2\}.$$

**Proof:** Suppose that $k = \max\{\chi_{vi,1}(G_1), \chi_{vi,1}(G_2), d_G(v)+2\}$. Obviously $\chi_{vi,1}(G_1) \leq \chi_{vi,1}(G)$, $\chi_{vi,1}(G_2) \leq \chi_{vi,1}(G)$, and $d_G(v)+2 \leq \Delta(G)+2 \leq \chi_{vi}(G) \leq \chi_{vi,1}(G)$. So $k \leq \chi_{vi,1}(G)$. Now suppose that $c_1$ and $c_2$ are $vi$-simultaneous $(k,1)$-coloring of $G_1$ and $G_2$, respectively. Note that $I_1^{G_1}[v]$, $I_1^{G_2}[v]$, and $I_1^{G}[v]$ are cliques and $I_2^{G_1}(v)$, $I_2^{G_2}(v)$, and $I_2^{G}(v)$ are independent sets in $G_1$, $G_2$, and $G$, respectively. Also $c_i(I_1^{G_i}[v]) \cap c_i(I_2^{G_i}(v)) = \varnothing$ and $|c_i(I_2^{G_i}(v))| = 1$ for each $i \in [2]$. So by suitable permutations on the colors of $c_2$ in three steps, we reach the new coloring $c_3$:

(1) If $c_1(v) = a \neq b = c_2(v)$ then we just replace colors $a$ and $b$ together in $c_2$ and otherwise we do nothing. We denote the new coloring by $c'_2$.

(2) Let $c_1(x) = c$ and $c'_2(y) = d$ for each $x \in I_2^{G_1}(v)$ and $y \in I_2^{G_2}(v)$. If $c \neq d$ then we just replace colors $c$ and $d$ together in $c'_2$. Otherwise, we do nothing. We denote the new coloring by $c''_2$. Obviously, $c \neq a \neq d$ and so $c''_2(v) = a$.

(3) If $c''_2(I_1^{G_2}(v)) \cap c_1(I_1^{G_1}(v)) = \varnothing$ we do nothing. Otherwise, suppose that $c''_2(I_1^{G_2}(v)) \cap c_1(I_1^{G_1}(v)) = \{a_1, \ldots, a_s\}$. Since $k \geq d_G(v) + 2$ and $|c''_2(I_{G_2}[v]) \cup c_1(I_{G_1}[v])| = d_G(v) + 2 - s$, there are $s$ colors $b_1, \ldots, b_s$ which have not appeared in $c''_2(I_{G_2}[v]) \cup c_1(I_{G_1}[v])$. Now we replace $a_i$ and $b_i$ together for each $i \in \{1, \ldots, s\}$. We denote the new coloring by $c_3$.

Now we can easily show that the following function is a $vi$-simultaneous proper $(k, 1)$-coloring for $G$:

$$c(x) = \begin{cases} c_1(x), & x \in V(G_1) \cup I(G_1); \\ c_3(x), & x \in V(G_2) \cup I(G_2). \end{cases}$$

□

**Theorem 2.4** *Let $k \in \mathbb{N}$ and let $G$ be a graph with blocks $B_1, \ldots, B_k$. Then*

$$\chi_{vi,1}(G) = \max\{\chi_{vi,1}(B_1), \ldots, \chi_{vi,1}(B_k), \Delta(G) + 2\}.$$

*In particular, $\chi_{vi,1}(G) = \max\{\chi_{vi,1}(B_1), \ldots, \chi_{vi,1}(B_k)\}$ when $G$ has at least one $\Delta(G)$-vertex which is not a cut vertex.*

**Proof:** The proof is obtained by induction on the number $k$ and applying Lemma 2.3. □

We can determine an upper bound on the $vi$-simultaneous chromatic number $\chi_{vi,s}(G)$ in terms of $\Delta(G)$ and the list chromatic number of $G$.

**Definition 2.5** [3] Let $G$ be a graph and $L$ be a function that assigns to each vertex $v$ of $G$ a set $L(v) \subset \mathbb{N}$, called the list of $v$. A coloring $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V(G)$ is called a list coloring of $G$ with respect to $L$, or an $L$-coloring, and we say that $G$ is $L$-colorable. A graph $G$ is $k$-list-colorable if it has a list coloring whenever all the lists have length $k$. The smallest value of $k$ for which $G$ is $k$-list-colorable is called the list chromatic number of $G$, denoted $\chi_l(G)$.

**Theorem 2.6** *Let $G$ be a nonempty graph and let $s \in \mathbb{N}$. Then*

*(i)* $\chi_{vi,s}(G) \leq \max\{\chi_{i,s}(G), \chi_l(G) + \Delta(G) + s\}$,

*(ii) If* $\chi_{i,s}(G) \geq \chi_l(G) + \Delta(G) + s$, *then* $\chi_{vi,s}(G) = \chi_{i,s}(G)$.

**Proof:** *(i)* Suppose that $\max\{\chi_{i,s}(G), \chi_l(G) + \Delta(G) + s\} = k$. So there exists an incidence $(k, s)$-coloring $c_i : I(G) \to [k]$ of $G$ and hence $|c_i(I_2(u))| \leq s$ for any vertex $u \in V(G)$. So $|c_i(I_G(u))| \leq \Delta(G) + s$. Now we extend $c_i$ to a $vi$-simultaneous $(k, s)$-coloring $c$ of $G$. The set of available colors for the vetex $u$ is $L(u) = [k] \setminus c_i(I_G(u))$

which has at least $k-\Delta(G)-s \geq \chi_l(G)$ colors. Since $|L(u)| \geq \chi_l(G)$ for any vertex $u \in V(G)$, there exists a proper vertex coloring $c_v$ of $G$ such that $c_v(u) \in L(u)$. Now one can easily show that the following coloring is a $vi$-simultaneous $(k,s)$-coloring of $G$:

$$c(x) = \begin{cases} c_i(x), & x \in I(G); \\ c_v(x), & x \in V(G). \end{cases}$$

*(ii)* If $\chi_{i,s}(G) \geq \chi_l(G) + \Delta(G) + s$, then $\chi_{vi,s}(G) \leq \chi_{i,s}(G)$. In addition, any $vi$-simultaneous $(k,s)$-coloring of $G$ induces an incidence $(k,s)$-coloring of $G$ and so $\chi_{i,s}(G) \leq \chi_{vi,s}(G)$. Therefore, $\chi_{vi,s}(G) = \chi_{i,s}(G)$. □

**Corollary 2.7** *For any nonempty graph $G$, $\chi_{vi,1}(G) \leq \max\{\chi(G^2), \chi_l(G) + \Delta(G) + 1\}$. In particular, if $\chi(G^2) \geq \chi_l(G) + \Delta(G) + 1$, then $\chi_{vi,1}(G) = \chi(G^2)$.*

**Corollary 2.8** *Let $G$ be a graph of order $n$ with* $\operatorname{diam}(G) = 2$. *Then $\chi_{vi,1}(G) \leq \max\{n, \chi_l(G) + \Delta(G) + 1\}$. In particular, if $\Delta(G) \leq \frac{n}{2} - 1$, then $\chi_{vi,1}(G) = n$.*

**Remark 2.9** It was proved in [7] that the square of any cubic graph $G$, other than the Petersen graph, is 8-list-colorable and so $\chi(G^2) \leq 8$. In addition, the diameter of the Petersen graph $P$ is 2. Therefore, by Corollaries 2.7 and 2.8, $\chi_{vi,1}(P) = 10$ for the Petersen graph and $\chi_{vi,1}(G) \leq 8$ for any graph $G$ with $\Delta(G) = 3$, other than the Petersen graph.

## 3 $k$-degenerate graphs

A graph $G$ is said to be *$k$-degenerate* if any subgraph of $G$ contains a vertex of degree at most $k$. For example, a graph $G$ is 1-degenerate if and only if $G$ is a forest. We can give an upper bound for the $vi$-simultaneous chromatic number of a $k$-degenerate graph in terms of $k$ and its maximum degree.

Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a finite family of $n$ subsets of a finite set $X$. A *system of distinct representatives* (SDR) for the family $\mathcal{F}$ is a set $\{a_1, \ldots, a_n\}$ of distinct elements of $X$ such that $a_i \in A_i$, for all $i \in [n]$.

**Theorem 3.1** *Let $k \in \mathbb{N}$ and $G$ be a $k$-degenerate graph with $\Delta(G) \geq 2$. Then $\chi_{vi,k}(G) \leq \Delta(G) + 2k$.*

**Proof:** If $k = \Delta(G)$, then $\chi_{vi,k}(G) = \chi_{vi}(G) \leq 3\Delta(G) = \Delta(G) + 2k$. So we suppose that $1 \leq k \leq \Delta(G) - 1$. Assume the contrary, and let the theorem be false and $G$ be a minimal counter-example. Let $u$ be a vertex in $G$ with degree $r \leq k$ and $N_G(u) = \{u_1, \ldots, u_r\}$ and let $G' = G - u$. According to the minimality of $G$, $\chi_{vi,k}(G') \leq \Delta(G) + 2k$ and there exists a $vi$-simultaneous $(\Delta(G) + 2k, k)$-coloring $c'$ of $G'$. We extend $c'$ to a $vi$-simultaneous $(\Delta(G) + 2k, k)$-coloring $c$ of $G$, which is a contradiction.

First, we color the vertices of $I_1(u)$. For each $(u, u_i) \in I_1(u)$ there are at least $k$ available colors if $|c'(I_2(u_i))| = k$ and there are at least $2k$ available colors if

$|c'(I_2(u_i))| < k$. Let $A_i$ be the set of available colors for $(u, u_i) \in I_1(u)$. Since we must select distinct colors for the vertices of $I_1(u)$, we prove that the family $\mathcal{F} = \{A_1, \ldots, A_r\}$ has a system of distinct representatives. Because $| \cup_{j \in J} A_j| \geq k \geq |J|$ for any subset $J \subseteq [r]$, using Hall's Theorem (see Theorem 16.4 in [3]), we conclude that $\mathcal{F}$ has an SDR $\{a_1, \ldots, a_r\}$ such that $|\{a_j\} \cup c'(I_2(u_j))| \leq k$ for any $j \in [r]$. We color the vertex $(u, u_j)$ by $a_j$ for any $j \in [r]$. Now we color the vertices of $I_2(u)$. Since $|c'(I_{G'}[u_j] \cup c(I_1^G(u))| < \Delta(G) + 2k$ for each $j \in [r]$, there exists at least one available color for the vertex $(u_j, u)$. Finally, we select the color of the vertex $u$. Since $|I_G(u) \cup N_G(u)| = 3r < \Delta(G) + 2k$, we can color the vertex $u$ and complete the coloring $c$. □

**Corollary 3.2** *Let $F$ be a forest. Then*

$$\chi_{vi,1}(F) = \begin{cases} 1, & \Delta(F) = 0; \\ 4, & \Delta(F) = 1; \\ \Delta(F) + 2, & \Delta(F) \geq 2. \end{cases}$$

**Proof:** The proof is trivial for $\Delta(F) \leq 1$. So we suppose that $\Delta(F) \geq 2$. Each forest is a 1-degenerate graph. So by Theorem 3.1 we have $\chi_{vi,1}(F) \leq \Delta(F)+2$. In addition, $\chi_{vi,1}(F) \geq \chi_{vi}(F) = \chi(F^{\frac{3}{3}}) \geq \omega(F^{\frac{3}{3}}) = \Delta(F) + 2$. Hence $\chi_{vi,1}(F) = \Delta(F) + 2$. □

**Corollary 3.3** *For any $n \in \mathbb{N} \setminus \{1\}$, $\chi_{vi,1}(P_n) = 4$.*

**Remark 3.4** Using the following simple algorithm, we have a proper $(\Delta+2)$-coloring for the $\frac{3}{3}$-power of any tree $T$ with $\Delta(T) = \Delta$:

Suppose that $V(T) = \{v_1, \ldots, v_n\}$ and the vertex $v_1$ of degree $\Delta$ is the root of $T$. To achieve a $(\Delta + 2)$-coloring of $T^{\frac{3}{3}}$, assign color 1 to the vertex $v_1$ and color all $i$-vertices in $I_1(v_1)$ with distinct colors in $\{2, \ldots, \Delta + 1\}$. Note that, since these $i$-vertices in $I_1(v_1)$ are pairwise adjacent, they must have different colors. Also, color all $i$-vertices in $I_2(v_1)$ with color $\Delta + 2$.

Now, to color the other $t$-vertices and $i$-vertices of $T^{\frac{3}{3}}$, for the $t$-vertex $v_i$ with colored parent $p_{v_i}$, $2 \leq i \leq n$, color all the uncolored $i$-vertices in $I_2(v_i)$ the same as $(p_{v_i} v_i)_1$. Then color $v_i$ with a color from $[\Delta + 2] \setminus \{c(p_{v_i}), c((p_{v_i} v_i)_1), c((p_{v_i} v_i)_2)\}$. Now color all the uncolored $i$-vertices in $I_1(v_i)$ with $\Delta - 1$ distinct colors from $[\Delta + 2] \setminus \{c((p_{v_i} v_i)_1), c((p_{v_i} v_i)_2), c(v_i)\}$.

As each outerplanar graph is a 2-degenerate graph and each planar graph is a 5-degenerate graph [8], we can obtain the following corollary from Theorem 3.1.

**Corollary 3.5** *Let $G$ be a graph with maximum degree $\Delta$.*

*(i) If $G$ is an outerplanar graph, then $\chi_{vi,2}(G) \leq \Delta + 4$.*
*(ii) If $G$ is a planar graph, then $\chi_{vi,5}(G) \leq \Delta + 10$.*

We decrease the upper bound of Theorem 3.1 to $\Delta(G)+5$ for 3-degenerate graphs with maximum degree at least five.

**Theorem 3.6** *Every* 3*-degenerate graph* $G$ *with* $\Delta(G) \geq 5$ *admits a* $vi$*-simultaneous* $(\Delta(G)+5, 3)$*-coloring. Therefore,* $\chi_{vi,3}(G) \leq \Delta(G)+5$.

**Proof:** Assume the contrary, and let the theorem be false and $G$ be a minimal counter-example. Let $u$ be a vertex in $G$ with degree $r \leq 3$, $N_G(u) = \{u_1, \ldots, u_r\}$, and let $G' = G - u$. If $\Delta(G') = 4$, then by Theorem 3.1 we have $\chi_{vi,3}(G') \leq 4 + 6 = 10 = \Delta(G) + 5$ and if $\Delta(G') \geq 5$, according to the minimality of $G$, $\chi_{vi,3}(G') \leq \Delta(G) + 5$. So there exists a $vi$-simultaneous $(\Delta(G) + 5, 3)$-coloring $c'$ of $G'$. We extend $c'$ to a $vi$-simultaneous $(\Delta(G) + 5, 3)$-coloring $c$ of $G$, which is a contradiction.

First, we color the vertices of $I_1(u)$. For each $(u, u_i) \in I_1(u)$ there are at least three available colors if $|c'(I_2(u_i))| = 3$ and there are at least five available colors if $|c'(I_2(u_i))| \leq 2$. Let $A_i$ be the set of available colors for $(u, u_i) \in I_1(u)$ and $C_i = c'(I_2(u_i))$. Since we must select distinct colors for the vertices of $I_1(u)$, we prove that the family $\mathcal{F} = \{A_1, \ldots, A_r\}$ has an SDR. According to the degree of $u$ and the sizes of $C_1$, $C_2$, and $C_3$, we consider five cases:

(1) $r \leq 2$. Since $|A_i| \geq 3$, one can easily show that $\mathcal{F}$ has an SDR $\{a_j \mid j \in [r]\}$ such that $|\{a_j\} \cup c'(I_2(u_j))| \leq 3$ for any $j \in [r]$. We color the vertex $(u, u_j)$ by $a_j$ for any $j \in [r]$. Now we color the vertices of $I_2(u)$. Since $|c'(I_{G'}[u_j]) \cup c(I_1^G(u))| < \Delta(G) + 2 + r \leq \Delta(G) + 4$ for each $j \in [r]$, there exists at least one available color for the vertex $(u_j, u)$. Finally, we select the color of the vertex $u$. Since $|I_G(u) \cup N_G(u)| = 3r \leq 6 < \Delta(G) + 5$, we can color the vertex $u$ and complete the coloring $c$.

(2) $r = 3$ and $|C_j| \leq 2$ for any $j \in [3]$. Since $|\cup_{j \in J} A_j| \geq 5 \geq |J|$ for any subset $J \subseteq [r]$, using Hall's Theorem (see Theorem 16.4 in [3]), we conclude that $\mathcal{F}$ has an SDR $\{a_1, \ldots, a_r\}$ such that $|\{a_j\} \cup c'(I_2(u_j))| \leq 3$ for any $j \in [r]$. We color the vertex $(u, u_j)$ with $a_j$ for any $j \in [r]$. Now we color the vertices of $I_2(u)$. Since $|c'(I_{G'}[u_j]) \cup c(I_1^G(u))| < \Delta(G) + 2 + r - 1 \leq \Delta(G) + 4$ for each $j \in [r]$, there exists at least one available color for the vertex $(u_j, u)$. Finally, we select the color of the vertex $u$. Since $|I_G(u) \cup N_G(u)| = 9 < \Delta(G) + 5$, we can color the vertex $u$ and complete the coloring $c$.

(3) $r = 3$ and $|C_j| \leq 2$ for two sets of $C_j$s. Without loss of generality, let $|C_1| = |C_2| = 2$ and $|C_3| = 3$. If $C_j \cap c'(I_{G'}[u_3])$ is nonempty for some $j \in \{1, 2\}$ and $a \in C_j \cap c'(I_{G'}[u_3])$, then we color the vertex $(u, u_j)$ with $a$, the vertex $(u, u_i)$ ($j \neq i \in [2]$) with color $b$ from $C_i \setminus \{a\}$ ($b \in A_i \setminus \{a\}$ if $C_i = \{a\}$) and the vertex $(u, u_3)$ with color $d$ from $C_3 \setminus \{a, b\}$.
Because $|c'(I_{G'}[u_3])| = \Delta(G) + 3$, if $C_1 \cap c'(I_{G'}[u_3]) = \varnothing = C_2 \cap c'(I_{G'}[u_3])$ then $C_1 = C_2$. Suppose that $C_1 = C_2 = \{a, b\}$ and $d \in A_1 \setminus \{a, b\}$ (note that $|A_1| = 5$). So $d \in c'(I_{G'}[u_3])$. We color the vertex $(u, u_1)$ with $d$, the vertex $(u, u_2)$ with color $a$ and the vertex $(u, u_3)$ with color $f$ from $C_3 \setminus \{a, d\}$. Now we color the vertices of $I_2(u)$. Since $|c'(I_{G'}[u_j]) \cup c(I_1^G(u))| \leq \Delta(G) + 4$, for each $j \in [r]$, there exists at least one available color for the vertex $(u_j, u)$. Finally, we select the color of the vertex $u$. Since $|I_G(u) \cup N_G(u)| = 9 < \Delta(G) + 5$, we can color the vertex $u$ and complete the coloring $c$.

(4) $r = 3$ and $|C_j| \leq 2$ for only one set of $C_j$s. Without loss of generality, let $|C_1| = 2$ and $|C_2| = |C_3| = 3$. If $C_1 \cap c'(I_{G'}[u_j])$ is nonempty for some $j \in \{2,3\}$ and $a \in C_1 \cap c'(I_{G'}[u_j])$, then we color the vertex $(u, u_1)$ with $a$. Suppose that $j \neq i \in \{2,3\}$. Since $|C_i| + |c'(I_{G'}[u_j])| = \Delta(G) + 6$, $C_i \cap c'(I_{G'}[u_j]) \neq \varnothing$. Let $b \in C_i \cap c'(I_{G'}[u_j])$ and color the vertex $(u, u_i)$ with color $b$ and the vertex $(u, u_j)$ with color $d$ from $C_j \setminus \{a, b\}$.

Because $|c'(I_{G'}[u_2])| = |c'(I_{G'}[u_3])| = \Delta(G) + 3$, if $C_1 \cap c'(I_{G'}[u_2]) = \varnothing = C_1 \cap c'(I_{G'}[u_3])$ then $c'(I_{G'}[u_2]) = c'(I_{G'}[u_3])$. Since $|C_i| + |c'(I_{G'}[u_j])| = \Delta(G) + 6$, $C_i \cap c'(I_{G'}[u_j]) \neq \varnothing$ when $\{i, j\} = \{2, 3\}$. Therefore, there exist $b \in C_2 \cap c'(I_{G'}[u_3])$ and $d \in C_3 \cap c'(I_{G'}[u_2])$ such that $b \neq d$. Now we color the vertex $(u, u_1)$ with $a \in C_1$, the vertex $(u, u_2)$ with color $b$, and the vertex $(u, u_3)$ with color $d$. Now we color the vertices of $I_2(u)$. Since $|c'(I_{G'}[u_j]) \cup c(I_1^G(u))| \leq \Delta(G) + 4$ for each $j \in [r]$, there exists at least one available color for the vertex $(u_j, u)$. Finally, we select the color of the vertex $u$. Since $|I_G(u) \cup N_G(u)| = 9 < \Delta(G) + 5$, we can color the vertex $u$ and complete the coloring $c$.

(5) $r = 3$ and $|C_j| = 3$ for any $j \in [3]$. For any $i, j \in [3]$, since $|C_i| + |c'(I_{G'}[u_j])| = \Delta(G) + 6$, $C_i \cap c'(I_{G'}[u_j]) \neq \varnothing$. So there exist $a_1 \in C_1 \cap c'(I_{G'}[u_2])$, $a_2 \in C_2 \cap c'(I_{G'}[u_3])$, and $a_3 \in C_3 \cap c'(I_{G'}[u_1])$. If $|\{a_1, a_2, a_3\}| = 3$, then we color the vertex $(u, u_j)$ with color $a_j$ ($j \in [3]$), and similar to the previous cases, we can complete the coloring $c$. Now suppose that $|\{a_1, a_2, a_3\}| = 2$. Without loss of generality, suppose that $a_1 = a_2 \neq a_3$ and $b \in C_2 \setminus \{a\}$. Here, we color $(u, u_1)$ with $a_1$, the vertex $(u, u_2)$ with color $b$ and the vertex $(u, u_3)$ with color $a_3$.

Finally, suppose that $a_1 = a_2 = a_3$. If $(C_i \setminus \{a_1\}) \cap c'(I_{G'}[u_j]) \neq \varnothing$ for some $i, j \in [3]$ and $b \in (C_i \setminus \{a_1\}) \cap c'(I_{G'}[u_j])$, we color $(u, u_i)$ with $b$, the vertex $(u, u_2)$ with color $a_1$ and the vertex $(u, u_s)$ with color $d \in C_s \setminus \{a_1, b\}$ where $i \neq s \neq j$. Otherwise, we have $(C_1 \setminus \{a_1\}) \cap c'(I_{G'}[u_3]) = \varnothing = (C_2 \setminus \{a_1\}) \cap c'(I_{G'}[u_3])$ which concludes $C_1 = C_2$. Suppose that $C_1 = C_2 = \{a_1, b, d\}$. Now we color $(u, u_1)$ with $b$, the vertex $(u, u_2)$ with color $a_1$ and the vertex $(u, u_3)$ with color $f \in C_3 \setminus \{a_1, b\}$.

In these three subcases, we have $|c'(I_{G'}[u_j]) \cup c(I_1^G(u))| \leq \Delta(G) + 4$ for each $j \in [3]$ and similar to the previous cases, we can complete the coloring $c$.

□

**Problem 3.7** Let $G$ be a 3-degenerate graph with $\Delta(G) = 4$. We know that $\chi_{vi}(G) \leq 9$. What is the sharp upper bound for $\chi_{vi,1}(G)$, $\chi_{vi,2}(G)$, and $\chi_{vi,3}(G)$? By Theorem 3.1, $\chi_{vi,3}(G) \leq 10$. Is this upper bound sharp or, like Theorem 3.6, the upper bound is 9?

## 4 Cycles, Complete and Bipartite Graphs

The following theorem was proved in [13] about the chromatic number of a power of a cycle.

**Theorem 4.1** [13] *Let* $3 \le n \in \mathbb{N}$ *and* $k \in \mathbb{N}$. *Then*

$$\chi(C_n^k) = \begin{cases} n & k \ge \lfloor \frac{n}{2} \rfloor, \\ \lceil \frac{n}{\lfloor \frac{n}{k+1} \rfloor} \rceil & otherwise. \end{cases}$$

With a simple review, we can prove that $\chi(G^{\frac{3}{3}}) = \chi_{vi}(G) \le 5$ when $\Delta(G) = 2$, and $\chi(G^{\frac{3}{3}}) = \chi_{vi}(G) = 4$ if and only if any component of $G$ is a cycle of order divisible by 4 or a nontrivial path. In the first theorem, we show that any cycle of order at least 4 is $vi$-simultaneous $(5,1)$-colorable. To avoid drawing too many edges in the figures, we use $\frac{1}{3}$-powers of graphs instead of $\frac{3}{3}$-powers of graphs. Internal vertices are shown white and terminal vertices are shown black.

**Theorem 4.2** *Let* $3 \le n \in \mathbb{N}$. *Then*

$$\chi_{vi,1}(C_n) = \begin{cases} 6, & n = 3; \\ 4, & n \equiv 0 \ (mod\ 4); \\ 5, & otherwise. \end{cases}$$

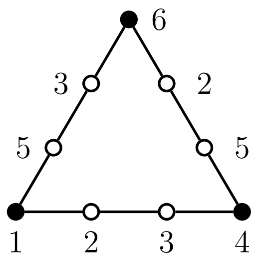


Figure 3: A $vi$-simultaneous proper $(6,1)$-coloring of $C_3$. Black vertices are corresponding to the vertices of $G$ and white vertices are corresponding to the incidences of $C_3$.

**Proof:** Suppose that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $c$ is a $vi$-simultaneous $(k,1)$-coloring of $C_3$. We have $c(v_i) \neq c((v_i, v_j)) = c((v_l, v_j))$ where $\{i, j, l\} = [3]$. So

$$|\{c(v_1), c(v_2), c(v_3), c((v_1, v_2)), c((v_2, v_1)), c((v_1, v_3))\}| = 6.$$

Therefore $k \ge 6$. Figure 3 shows a $vi$-simultaneous $(6,1)$-coloring of $C_3$ and so $\chi_{vi,1}(C_3) = 6$. In the second part, by Theorem 4.1 we have $\chi_{vi}(C_n) = \chi(C_n^{\frac{3}{3}}) = \chi(C_{3n}^3) = \lceil \frac{3n}{\lfloor \frac{3n}{4} \rfloor} \rceil = 4 = \Delta(C_n) + 2$ and hence Lemma 2.1 shows that any $vi$-simultaneous 4-coloring of $C_n$ is a $vi$-simultaneous $(4,1)$-coloring.

For the last part, we consider three cases:

(i) $n = 4q + 1$, $q \in \mathbb{N}$. Suppose that $c$ is a $vi$-simultaneous $(4,1)$-coloring of $C_{n-1}$ and

$$(c(v_1), c((v_1, v_{n-1})), c((v_{n-1}, v_1)), c(v_{n-1})) = (1, 4, 3, 2).$$

In this coloring, the colors of the other vertices are uniquely determined. To find a $vi$-simultaneous $(5,1)$-coloring of $C_n$, we replace the edge $\{v_1, v_{n-1}\}$ with the path $P = v_{n-1}v_nv_1$. Now we define the coloring $c'$ as follows (see Figure 4):

$$c'(x) = \begin{cases} 2, & x = v_n; \\ 3, & x \in \{v_{n-1}, (v_n, v_1)\}; \\ 4, & x = (v_n, v_{n-1}); \\ 5, & x \in \{v_{n-2}, (v_1, v_n), (v_{n-1}, v_n)\}; \\ c(x), & \text{otherwise.} \end{cases}$$

(ii) $n = 4q + 2$, $q \in \mathbb{N}$. Figure 5 shows a $vi$-simultaneous $(5,1)$-coloring of $C_6$.

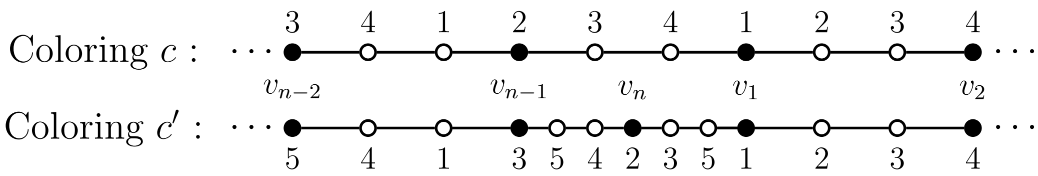


Figure 4: An extension of a $vi$-simultaneous $(4,1)$-coloring $c$ to a $vi$-simultaneous $(5,1)$-coloring $c'$.

Now suppose that $n \geq 10$. We can easily use the method of case (i) on two edges $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_4, v_5\}$ of $C_{n-2}$ to achieve a $vi$-simultaneous $(5,1)$-coloring of $C_n$.

(iii) $n = 4q + 3$, $q \in \mathbb{N}$. Figure 5 shows a $vi$-simultaneous $(5,1)$-coloring of $C_7$. Now suppose that $n \geq 11$. Again we use the method of case (i) on three edges $e_1 = \{v_1, v_2\}$ (changing the color of $v_3$ to 5 instead of vertex $v_{n-3}$), $e_2 = \{v_4, v_5\}$, and $e_3 = \{v_7, v_8\}$ of $C_{n-3}$ to achieve a $vi$-simultaneous $(5,1)$-coloring of $C_n$.

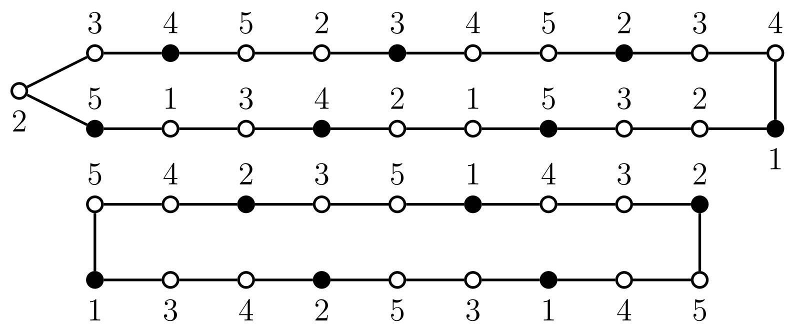


Figure 5: $vi$-simultaneous $(5,1)$-colorings of $C_6$ and $C_7$.

□

**Corollary 4.3** *Let $G$ be a nonempty graph with $\Delta(G) \leq 2$. Then $\chi_{vi,1}(G) = 4$ if and only if each component of $G$ is a cycle of order divisible by 4 or a nontrivial path.*

The following lemma is about the underlying digraph of any subgraph of the $\frac{3}{3}$-power of a graph induced by an independent set. We leave the proof to the reader.

**Lemma 4.4** *Let $G$ be a graph and $S$ be an independent set of $G^{\frac{3}{3}}$. Then each component of $D(G^{\frac{3}{3}}[S])$ is either trivial or a star whose arcs are directed towards the center. In addition, the vertices of trivial components form an independent set in $G$.*

**Theorem 4.5** *For each $n \in \mathbb{N} \setminus \{1\}$, $\chi_{vi}(K_n) = n + 2$.*

**Proof:** Let $G = K_n^{\frac{3}{3}}$, $c : V(G) \to [\chi(G)]$ be a proper coloring and $C_j = c^{-1}(j)$ $(1 \leq j \leq \chi(G))$. Lemma 4.4 concludes that each color class $C_j$ has at most $n - 1$ vertices. So

$$\chi(G) \geq \frac{|V(G)|}{n-1} = \frac{n^2}{n-1} = n + 1 + \frac{1}{n-1}.$$

Therefore $\chi(G) \geq n + 2$. Now we define a proper $(n + 2)$-coloring of $G$.

When $n = 2$, we have $\chi(G) = \chi(K_4) = 4$. Now we consider $n \geq 3$. Consider the hamiltonian cycle of $K_n$, namely $C = (v_1, v_2, \ldots, v_n)$. For $1 \leq j \leq n$, assign color $j$ to the $t$-vertex $v_j$ and all $i$-vertices $(v_k, v_{j+1})$, where $k \in [n] \setminus \{j, j + 1\}$ and $v_{n+1} = v_1$. It can be easily seen that all $t$-vertices of $G$ have a color in $[n]$ and the only uncolored vertices of $G$ are $(v_j, v_{j+1})$, for $1 \leq j \leq n$. Now, it suffices to color the aforementioned $i$-vertices. Suppose that $n$ is even. Assign color $n+1$ to the $i$-vertex $(v_j, v_{j+1})$ if $j$ is an odd number; otherwise color it with the color $n+2$. Now suppose that $n$ is an odd integer. Then for $1 \leq j \leq n - 1$, color the $i$-vertex $(v_j, v_{j+1})$ with color $n + 1$ if $j$ is odd and otherwise assign color $n + 2$ to it. Also, color the $i$-vertex $(v_n, v_1)$ with color $n$ and recolor the $t$-vertex $v_n$ with color $n + 1$. □

Suppose that $c$ is a $vi$-simultaneous $(n + 2)$-coloring of $K_n$. For any vertex $v$, $|c(I_1[v])| = n$ and so $|c(I_2(v))| = 2$. Therefore $\chi_{vi,2}(K_n) = \chi_{vi}(K_n) = n + 2$. In the following theorem, we determine $\chi_{vi,1}(K_n)$.

**Theorem 4.6** *Let $n \in \mathbb{N} \setminus \{1\}$ and let $G$ be a graph of order $n$. Then $\chi_{vi,1}(G) = 2n$ if and only if $G \cong K_n$.*

**Proof:** First, suppose that $G \cong K_n$. Since $\mathrm{diam}(G) = 1$, by Definition 1.9, any two vertices $(u, i)$ and $(v, j)$ of $\mathcal{T}_{vi,1}(G)$ are adjacent and so $\mathcal{T}_{vi,1}(G)$ is a complete graph of order $2n$. In addition, by Theorem 1.11 we have $\chi_{vi,1}(G) = \chi(\mathcal{T}_{vi,1}(G))$. So $\chi_{vi,1}(G) = \chi(\mathcal{T}_{vi,1}(G)) = \chi(K_{2n}) = 2n$. Conversely, suppose that $\chi_{vi,1}(G) = 2n$. Therefore, $\chi(\mathcal{T}_{vi,1}(G)) = 2n = |V(\mathcal{T}_{vi,1}(G))|$ which implies that $\mathcal{T}_{vi,1}(G)$ is a complete graph. Now for any two distinct vertices $u$ and $v$ of $G$, the vertices $(u, 1)$ and $(v, 2)$ of $\mathcal{T}_{vi,1}(G)$ are adjacent and so $d_G(u, v) = 1$. Thus, $G$ is a complete graph. □

A *dynamic coloring* of a graph $G$ is a proper coloring in which each vertex neighborhood of size at least 2 receives at least two distinct colors. The *dynamic chromatic number* $\chi_d(G)$ is the least number of colors in such a coloring of $G$ [10]. Akbari et al. proved the following theorem. We use it to give a proper coloring for the $\frac{3}{3}$-power of a regular bipartite graph.

**Theorem 4.7** [1] *Let $G$ be a $k$-regular bipartite graph, where $k \geq 4$. Then there is a 4-dynamic coloring of $G$, using two colors for each part.*

**Theorem 4.8** [3] *Every regular bipartite graph has a perfect matching.*

**Theorem 4.9** *If $G = G(A, B)$ is a $k$-regular bipartite graph with $k \geq 4$ and $|A| = |B| = n$, then $\chi_{vi}(G) \leq \min\{n + 3, 2k\}$.*

**Proof:** Suppose that $V(A) = \{v_1, \ldots, v_n\}$ and $V(B) = \{u_1, \ldots, u_n\}$. Since $G$ is a $k$-regular bipartite graph, by Theorem 4.8, $G$ has a perfect matching $M = \{v_1u_1, \ldots, v_nu_n\}$. First, we present an $(n+3)$-proper coloring for $G^{\frac{3}{3}}$. For $2 \leq i \leq n$, color two $t$-vertices $v_i$ and $u_i$ with colors 1 and $n + 1$, respectively. Also, for $u \in N(v_1)$ and $v \in N(u_1)$, color $i$-vertices $(u, v_1)$ and $(v, u_1)$ with colors 1 and $n+1$, respectively.

Now, for any $i \in \{2, \ldots, n\}$, any $u \in N(v_i) \setminus \{u_i\}$, and any $v \in N(u_i) \setminus \{v_i\}$, assign color $i$ to the incidence vertices $(u, v_i)$ and $(v, u_i)$. It can easily be seen that all the $t$-vertices of $G$ except $\{v_1, u_1\}$ and all the incidence vertices of $G$ except $\{(v_i, u_i), (u_i, v_i) \mid 2 \leq i \leq n\}$ have colors in $[n + 1]$. Now, assign colors $n + 2$ and $n + 3$ to $t$-vertices $v_1$ and $v_2$, respectively. Also, for $2 \leq i \leq n$, color the incidence vertices $(v_i, u_i)$ and $(u_i, v_i)$ with colors $n + 2$ and $n + 3$, respectively. With a simple review, it is easy to show that this coloring is a proper coloring for $G^{\frac{3}{3}}$ with $n + 3$ colors.

In the following, we present a $(2k)$-proper coloring for $G^{\frac{3}{3}}$. By Theorem 4.7, there is a 4-dynamic coloring of $G$, namely $c$, using two colors in each part. Without loss of generality, suppose that each $t$-vertex in $A$ has one of the colors 1 and 2 and each $t$-vertex in $B$ has one of the colors 3 or 4.

For $1 \leq i \leq n$, consider the $t$-vertex $u_i \in V(B)$ with set of neighbors $N(u_i)$. Note that $c$ is a 4-dynamic coloring, so $u_i$ has at least one neighbor of each color 1 and 2. Let $u$ and $u'$ be two $t$-vertices in $N(u_i)$, where $c(u) = 1$ and $c(u') = 2$. First, assign colors 1 and 2 to $i$-vertices $(u_i, u')$ and $(u_i, u)$, respectively. Then, for $w \in N(u_i) \setminus \{u, u'\}$, color all $i$-vertices $(u_i, w)$ with different colors in $\{5, \ldots, k + 2\}$.

Similarly, for a $t$-vertex $v_i \in V(A)$, suppose that $v$ and $v'$ are neighbors of $v$ with colors 3 and 4, respectively. Color the $i$-vertices $(v_i, v')$ and $(v_i, v)$ with colors 3 and 4, respectively. Then, for $w' \in N(v_i) \setminus \{v, v'\}$, color all $i$-vertices $(v_i, w')$ with different colors in $\{k + 3, \ldots, 2k\}$. It can be easily seen that the presented coloring is a proper $(2k)$-coloring for $G^{\frac{3}{3}}$. □

Since any bipartite graph with maximum degree $\Delta$ can be extended to a $\Delta$-regular bipartite graph, we have the following corollary.

**Corollary 4.10** *If $G$ is a bipartite graph with maximum degree $\Delta$, then $\chi_{vi}(G) \leq 2\Delta$.*

A *derangement* of a set $S$ is a bijection $\pi : S \to S$ such that no element $x \in S$ has $\pi(x) = x$.

**Theorem 4.11** *Let $n, m \in \mathbb{N}$ and $n \geq m$. Then* $\chi_{vi}(K_{n,m}) = \begin{cases} n+2, & m \leq 2; \\ n+3, & m \geq 3. \end{cases}$

**Proof:** Let $A = \{v_1, \ldots, v_n\}$ and $B = \{u_1, \ldots, u_m\}$ be two parts of $K_{n,m}$ and $G = K_{n,m}^{\frac{3}{3}}$. If $m = 1$, then $K_{n,1}$ is a tree and by Corollary 3.2, we have $\chi(G) = n+2$. Now suppose that $m = 2$. Since $\omega(G) = \Delta+2$, $\chi(G) \geq n+2$. It suffices to present a proper $(n+2)$-coloring for $G$ with colors in $[n+2]$. Suppose that $\pi$ is a derangement of the set $[n]$. Assign color $n+1$ to the vertices of $\{u_1\} \cup I_2(u_2)$ and color $n+2$ to the vertices of $u_2 \cup I_2(u_1)$. Also for $j \in [n]$, color $i$-vertices $(u_1, v_j)$ and $(u_2, v_j)$ with color $j$ and vertex $v_j$ with color $\pi(j)$. The given coloring is a proper $(n+2)$-coloring of $G$.

In the case $m \geq 3$, suppose that $c$ is a proper coloring of $G$ with colors $1, \ldots, n+2$. Since the vertices of $I_1[u_1]$ are pairwise adjacent in $G$, there are exactly $n+1$ colors in $c(I_1[u_1])$. Without loss of generality, suppose that $c(u_1) = 1$ and $c(I_1(u_1)) = [n+1] \setminus \{1\}$. By Theorem 2.1, all $i$-vertices of $I_2(u_1)$ have the same color $n+2$.

Now consider $t$-vertices $u_2$ and $u_3$. All $i$-vertices of $I_2(u_2)$ and all $i$-vertices of $I_2(u_3)$ have the same color and their colors are different from $\{2, \ldots, n+2\}$. Hence the only available color for these vertices is the color 1. But the subgraph of $G$ induced by $I_2(u_2) \cup I_2(u_3)$ is 1-regular and so for their coloring we need two colors, a contradiction.

To complete the proof, it suffices to show that $\chi((K_{n,n})^{\frac{3}{3}}) \leq n+3$. Since $n \geq 3$, $n+3 \leq 2n$ and by Theorem 4.9, we have $\chi(G) \leq \chi(K_{n,n}^{\frac{3}{3}}) \leq \min\{n+3, 2n\} = n+3$. Hence, $\chi(G) = n+3$. □

**Theorem 4.12** *Let $n, m \in \mathbb{N} \setminus \{1\}$. Then $\chi_{vi,1}(K_{n,m}) = n+m$.*

**Proof:** Since $(K_{n,m})^2 \cong K_{n+m}$, it follows that $K_{n+m}$ is a subgraph of $\mathcal{T}_{vi,1}(K_{n,m})$ and so $\chi_{vi.1}(K_{n,m}) = \chi(\mathcal{T}_{vi,1}(K_{n,m})) \geq n+m$. Now we show that $\chi(\mathcal{T}_{vi,1}(K_{n,m})) \leq n+m$. Let $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_m\}$ be two parts of $K_{n,m}$, $\pi$ be a derangement of $[n]$ and $\sigma$ be a derangement of $[m]$. One can easily show that the following vertex coloring of $\mathcal{T}_{vi,1}(K_{n,m})$ is proper.

$$c(x) = \begin{cases} i, & x = (v_i, 2); \\ n+j, & x = (u_j, 2); \\ \pi(i), & x = (v_i, 1); \\ n+\sigma(j), & x = (u_j, 1). \end{cases}$$

□

As the reader can see, there are some graphs such as trees and $K_{n,2}$ with maximum degree $\Delta$, whose $\frac{3}{3}$-power has chromatic number equal to $\Delta+2$. So it would be interesting to characterize all graphs with the desired property.

**Problem 4.13** Characterize all graphs $G$ with maximum degree at least 3 such that $\chi(G^{\frac{3}{3}}) = \omega(G^{\frac{3}{3}}) = \Delta(G)+2$.

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