AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **86(1)** (2023), Pages 222–232

# Cyclic sieving for a family of semistandard tableaux*

Joshua Basman Monterrubio
Graeme Henrickson Anna Stokke†

*University of Winnipeg*
*Department of Mathematics and Statistics*
*Winnipeg, Manitoba R3B 2E9*
*Canada*

**Abstract**

We give a new cyclic sieving phenomenon for semistandard Young tableaux $SSYT(\lambda, \mu)$ of shape $\lambda = (m, n^b)$ and content $\mu$, a $(b + 2)$-tuple. We prove that $(SSYT(\lambda, \mu), \langle \partial^{b+2} \rangle, f(q))$ exhibits the cyclic sieving phenomenon, where $\partial$ is the jeu de taquin promotion operator and $f(q)$ is a modified Kostka-Foulkes polynomial $\widetilde{K}_{\lambda,\mu}(q)$, up to a power of $q$.

## 1 Introduction

Given a finite set $X$, and $\langle g \rangle$ a cyclic group of order $n$ that acts on $X$, we can consider the cardinality of the fixed point set $X^{g^d}$, for a positive integer $d$. The triple $(X, \langle g \rangle, f(q))$, where $f(q) \in \mathbb{N}[q]$, is said to exhibit the cyclic sieving phenomenon (CSP) if $|X^{g^d}| = f(\omega^d)$ for all $d \geq 0$, where $\omega$ is a primitive $n$th root of unity. The cyclic sieving phenomenon was introduced by Reiner, Stanton and White in 2004 [15] and has been widely studied since then, in various settings (see [17] for details).

Several authors have produced CSPs for various sets of Young tableaux (see, for instance, [2, 3, 5, 7, 10, 11, 12, 14, 16]). Candidates for cyclic sieving polynomials are generally $q$-analogues of a natural counting formula (for example, the hook-length formula for standard tableaux) and a cyclic action on standard or semistandard tableaux is given by Schützenberger's jeu de taquin promotion operator $\partial$ ([18, 19]). One roadblock is that the order of promotion (the least positive integer that fixes all tableaux in the set under $\partial$) is unknown for most shapes. There are also situations where the order of promotion is known but the most natural cyclic sieving polynomial

* This research was supported by NSERC grant RGPIN-2018-05877.
† Email: a.stokke@uwinnipeg.ca

ISSN: 2202-3518 

does not yield a CSP. For example, the order of promotion for staircase tableaux was given in [13] but, so far, a CSP for staircase tableaux remains elusive.

For standard rectangular tableaux of shape $\lambda = (a^b)$, the order of promotion is $ab$ [8] and if $X = SSYT(a^b, k)$ is the set of semistandard rectangular tableaux with entries less than or equal to $k$, the order of promotion on $X$ is $k$. Rhoades proved CSPs for both standard and semistandard rectangular tableaux [16]. We give a summary of CSP results for semistandard tableaux thus far. For a list of CSPs in other settings, see [2, Table 1].

(1) Rhoades [16] proved that $(SSYT(\lambda, k), \langle\partial\rangle, q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{k-1}))$ is a CSP triple, where $\lambda$ is a rectangular partition, $s_\lambda(1, q, \ldots, q^{k-1})$ is a principal specialization of the Schur polynomial and $\kappa(\lambda) = \sum_i (i-1)\lambda_i$.

(2) In [5], the authors showed that $(SSYT(a^b, \gamma), \langle\partial^d\rangle, q^* K_{a^b,\gamma}(q))$ is a CSP triple, refining Rhoades's result. Here $SSYT(a^b, \gamma)$ is the set of rectangular tableaux with fixed content $\gamma$, with $\gamma$ invariant under the $d$th cyclic shift, where $d$ is the frequency of $\gamma$—the number of cyclic shifts to return $\gamma$ to itself—and $q^* K_{a^b,\gamma}(q)$ is a Kostka-Foulkes polynomial up to a power of $q$.

(3) A CSP for semistandard hook tableaux with content $\mu$ was given in [3] where it is shown that $(SSYT((n-m, 1^m), \mu), \langle\partial^d\rangle, f(q))$ is a CSP triple, with cyclic sieving polynomial $f(q) = \begin{bmatrix} nz(\mu) - 1 \\ m \end{bmatrix}_q$. Here $z(\mu)$ is the number of non-zero entries in $\mu$.

(4) Using the cyclic action $\mathsf{c}$ arising from the $U_q(\mathfrak{sl}_n)$ crystal structure for semistandard tableaux, Oh and Park [10] proved $(SSYT(\lambda), \langle\mathsf{c}\rangle, q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{k-1}))$ exhibits the CSP when the length of $\lambda$ is less than $k$ and $\gcd(k, |\lambda|) = 1$. The result was extended to skew shapes in [1].

(5) In [2], the authors gave a CSP for semistandard tableaux of stretched hook shape $\lambda = ((a+1)n, n^b)$ and rectangular content $\mu = (n^{a+b+1})$. They proved that $(SSYT((a+1)n, n^b), \mu), \langle\partial\rangle, f(q))$ exhibits the CSP, where

$$f(q) = \prod_{1 \le i \le a} \prod_{1 \le j \le b} \frac{[i+j+n-1]_q}{[i+j-1]_q} = q^{-n\binom{b+1}{2}} \widetilde{K}_{\lambda,\mu}(q).$$

Here $\widetilde{K}_{\lambda,\mu}(q)$ is a modified Kostka-Foulkes polynomial.

In this paper, we give a CSP for the set of semistandard tableaux $SSYT(\lambda, \mu)$ of shape $\lambda = (m, n^b)$ and content $\mu = (\mu_1, \ldots, \mu_{b+2})$, where $m, n, b$ are positive integers. The shape is a more general version of the stretched hook shape $\lambda = ((a+1)n, n^b)$ in (5) and our content is a $(b+2)$-tuple whereas the content in (5) is rectangular of the form $(n^{a+b+1})$. The CSP polynomial is a $q$-binomial coefficient, which is a modified Kostka-Foulkes polynomial. Our CSP coincides with (5) in the case where $a = 1$; that is when $\lambda = (2n, n^b)$ and $\mu = (n^{b+2})$.

After reviewing the necessary definitions and results in Sections 2 and 3 we prove our main result in Section 4, which is that $(SSYT(\lambda,\mu), \langle\partial^{b+2}\rangle, q^{-n\binom{b+1}{2}}\widetilde{K}_{\lambda,\mu}(q))$ is a CSP, for $\lambda = (m, n^b)$, $\mu = (\mu_1, \ldots, \mu_{b+2})$, and $\widetilde{K}_{\lambda,\mu}(q)$ a modified Kostka-Foulkes polynomial.

## 2 Semistandard tableaux and jeu de taquin promotion

A weakly decreasing $r$-tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of a positive integer $n$ if $\lambda_i \geq 0$ and $\sum_{i=1}^{r} \lambda_i = n$. The Young diagram of shape $\lambda$ consists of $n$ boxes in $r$ left-justified rows with the $i$th row containing $\lambda_i$ boxes. A $\lambda$-tableau $T$ is obtained by filling the Young diagram with positive integers. A $\lambda$-tableau is *semistandard* if the entries in its columns are strictly increasing from top to bottom and the entries in its rows are weakly increasing from left to right. If $T$ contains entries from the set $\{1, \ldots, k\}$, the *content* of $T$ is the $k$-tuple $\mu = (\mu_1, \ldots, \mu_k)$ where $\mu_i$ is equal to the number of entries equal to $i$ in $T$. We will denote the set of semistandard $\lambda$-tableaux with content $\mu$ by $SSYT(\lambda, \mu)$.

**Example 2.1.** The tableau $T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$ belongs to $SSYT(\lambda, \mu)$ where $\lambda = (5, 3, 1)$ and $\mu = (2, 2, 3, 1, 1)$.

*Jeu de taquin promotion* ([18, 19]) is a combinatorial algorithm that gives an action on semistandard tableaux. We will use the version defined in [3], which is the inverse of the operation used in [2]. For a semistandard tableau $T$ with entries in $\{1, \ldots, k\}$, first replace each entry equal to $k$ with a dot. If there is a dot in the figure that is not contained in a continuous strip of dots in the northwest corner, choose the westernmost dot and slide it north or west until it lands in a connected component of dots in the northwest corner according to the following rules:

$$\begin{array}{|c|c|}\hline a & b \\ \hline \bullet \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & b \\ \hline a \\ \cline{1-1}\end{array}\ ;\quad \begin{array}{|c|c|}\hline a & \bullet \\ \hline b \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & a \\ \hline b \\ \cline{1-1}\end{array}\ ;\quad \begin{array}{|c|c|}\hline a & b \\ \hline c & \bullet \\ \hline\end{array} \to \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline c & b \\ \hline\end{array} & \text{if } c \leq b \\[2ex] \begin{array}{|c|c|}\hline a & b \\ \hline \bullet & c \\ \hline\end{array} & \text{if } b < c. \end{cases}$$

Repeat for the remaining dots, then replace each dot with 1 and increase all other entries by one, giving $\partial(T)$, which is semistandard. If $T$ has content $\mu = (\mu_1, \ldots, \mu_k)$, then $\partial(T)$ has content $(\mu_k, \mu_1, \ldots, \mu_{k-1})$.

**Example 2.2.** Below is an illustration of jeu de taquin promotion.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & 5 \\ \hline 5 & 5 \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 4 & \bullet \\ \hline \bullet & \bullet \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet & 1 & 2 & 3 \\ \hline 1 & 3 & 4 & \bullet \\ \hline 2 & \bullet \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & 2 & 3 \\ \hline 1 & 1 & 4 & \bullet \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

$$\rightarrow \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & \bullet & 3 \\ \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 3 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} = \partial(T)$$

The *order of promotion* of a tableau $T$ is the least positive integer $r$ such that $\partial^r(T) = T$. If a set $X$ of semistandard tableaux is invariant under $\partial$, the least positive integer $r$ such that $\partial^r(T) = T$ for all $T \in X$ is the *order of promotion* on $X$.

## 3 Kostka-Foulkes polynomials

The *Kostka-Foulkes polynomials*, denoted $K_{\lambda,\mu}(q)$, relate Hall-Littlewood polynomials to Schur polynomials (see [4] for a comprehensive overview). They generalize the Kostka coefficients $K_{\lambda\mu}$, since $K_{\lambda,\mu}(1) = K_{\lambda\mu}$, which is the number of semistandard tableaux of shape $\lambda$ and content $\mu$. It was shown by Lascoux and Schützenberger [9] that the Kostka-Foulkes polynomials can be found using a statistic, called *charge*, which had previously been conjectured by Foulkes [6]:

$$K_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda,\mu)} q^{charge(T)}.$$

We will work with *modified Kostka-Foulkes polynomials* $\widetilde{K}_{\lambda,\mu}(q)$, which are related to the Kostka-Foulkes polynomials by the relation $\widetilde{K}_{\lambda,\mu}(q) = q^{\kappa(\mu)} K_{\lambda,\mu}(q^{-1})$, where $\kappa(\mu) = \sum_i (i-1)\mu_i$. These can be obtained via a statistic on tableaux called *cocharge*, denoted $cc(T)$, which we will define shortly:

$$\widetilde{K}_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda,\mu)} q^{cc(T)}.$$

Given a permutation $w = w_1 \dots w_n \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on $n$ letters, define the *cocharge* of $j$ in $w$ recursively as follows:

$$cc(w,j) := \begin{cases} 0 & \text{if } j = 1 \\ cc(w,j-1) + 1 & \text{if } j \text{ precedes } j-1 \text{ in } w \\ cc(w,j-1) & \text{otherwise.} \end{cases}$$

The cocharge of the word $w$ is $cc(w) = \sum_{j=1}^{n} cc(w,j)$ and $charge(w) = \binom{n}{2} - cc(w)$. The content of a word $w$ is $\mu = (\mu_1, \dots, \mu_n)$, where $\mu_i$ records the number of entries $i$ in $w$. We can define cocharge for a word $w$ whenever its content $\mu$ is a partition. To do so, obtain $\mu_1$ standard subwords from $w$ in the following way: start by selecting the rightmost 1, then move left to find the rightmost 2 that precedes the chosen 1 and if there is not a 2 preceding the 1, loop around to the beginning of the word to choose the rightmost 2. Continue for $3, 4$, etc., until the largest entry in the word has been selected. The selected entries, listed in the order they appear in $w$, form the first standard subword $w^{(1)}$. Delete the entries in $w^{(1)}$ from $w$ and repeat the process with the word consisting of the remaining entries to obtain $w^{(2)}$. Continue until no entries in the word remain, forming $\mu_1$ subwords. Each of the subwords $w^{(i)}$ is a permutation, and $w^{(1)}, \ldots, w^{(k)}$ are the parts of the conjugate partition $\mu^t$. Define the cocharge of $w$ as $cc(w) = \sum_{i=1}^{\mu_1} cc(w^{(i)})$.

To get the cocharge of a tableau, we work with its reading word $\mathrm{rw}(T)$, which is obtained by listing the entries of $T$, left to right, across the rows, starting with the bottom row of $T$. Define $cc(T) = cc(\mathrm{rw}(T))$. For the cocharge of $T$ to be well-defined, it is necessary for the content of $T$ to be a partition. However, $\widetilde{K}_{\lambda,\mu} = \widetilde{K}_{\lambda,\sigma\mu}$, where $\sigma$ is a permutation and $\sigma(\mu_1, \ldots, \mu_n) = (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)})$ so this does not impede the use of cocharge to find the modified Kostka-Foulkes polynomial.

**Example 3.1.** Let $T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 4 & 4 \\ \cline{1-3} \end{array}$ with $\mathrm{rw}(T) = 34422311112234$.

The content of $w$, which is the content of $T$, is $\mu = (4, 4, 3, 3)$. The four standard subwords obtained from $w$ are: $w^{(1)} = 3214$, $w^{(2)} = 4213$, $w^{(3)} = 4312$, $w^{(4)} = 12$. Then $cc(w^{(1)}) = cc(w^{(1)}, 1) + cc(w^{(1)}, 2) + cc(w^{(1)}, 3) + cc(w^{(1)}, 4) = 0 + 1 + 2 + 2 = 5$, $cc(w^{(2)}) = 0 + 1 + 1 + 2 = 4, cc(w^{(3)}) = 0 + 0 + 1 + 2 = 3, cc(w^{(4)}) = 0 + 0 = 0$ so $cc(w) = 12$.

## 4 Main result

Our aim in this section is to prove a CSP for semistandard tableaux of shape $\lambda = (m, n^b)$ and content $\mu = (\mu_1, \ldots, \mu_{b+2})$. For $\lambda$ and $\mu$ so defined, let

$$\beta(\lambda, \mu) = m - n - \sum_{i=1}^{b+2} \gamma_i, \text{ where } \gamma_i = \begin{cases} \mu_i - n & \text{if } \mu_i > n \\ 0 & \text{otherwise.} \end{cases}$$

If $\mu_i > n$, at least $\gamma_i = \mu_i - n$ entries equal to $i$ are forced into the last $m - n$ columns of $T \in SSYT(\lambda, \mu)$ and we will refer to these as *forced entries*. Thus $\sum_{i=1}^{b+2} \gamma_i$ is the number of entries in the last $m - n$ columns that are fixed and there are $m - n - \sum_{i=1}^{b+2} \gamma_i$ boxes in the last $m - n$ columns for which the entries can vary. The entries remaining in the last $m - n$ columns of $T$ after deleting $\gamma_i$ entries equal to $i$, for each $1 \leq i \leq b + 2$, will be called the *free entries* in $T$. Each tableau $T \in SSYT(\lambda, \mu)$ has $\beta(\lambda, \mu)$ free entries, which belong to the set $\{2, \ldots, b + 2\}$. Furthermore, since the sum $\sum_{i=1}^{b+2} \gamma_i$ is the same for any permutation $\sigma\mu$ of the content, any tableau in $SSYT(\lambda, \sigma\mu)$ also has $\beta(\lambda, \mu)$ free entries.

The free entries in $T \in SSYT(\lambda, \mu)$ can also be determined by considering a multiset of elements from $\{2, \ldots, b + 2\}$ that are missing from the first $n$ columns. Each of the first $n$ columns of $T$ is necessarily missing one element from $\{1, \ldots, b+2\}$ and the collection of these elements forms a multiset. If $\mu_i < n$, then $i$ is missing from at least $n - \mu_i$ of the first $n$ columns in any tableau, so for each $i$ in the multiset with $\mu_i < n$, remove $n - \mu_i$ entries equal to $i$ to get a multiset $A_T$ of elements from $\{2, \dots, b+2\}$. The set $A_T$ consists precisely of the free elements in $T$ so $\beta(\lambda, \mu) = |A_T| = n - \sum_{\mu_i<n}(n - \mu_i)$. Let $\mathcal{A}$ denote the set of $\beta(\lambda, \mu)$-element multisets of $\{2, \dots, b+2\}$ and define a map

$$\phi : SSYT(\lambda, \mu) \to \mathcal{A} \text{ where } \phi(T) = A_T.$$

Since $|A_T| \leq n$, for $T \in SSYT(\lambda, \mu)$, the following lemma is immediate.

**Lemma 4.1.** *Suppose that* $\lambda = (m, n^b)$ *and* $\mu = (\mu_1, \dots, \mu_{b+2})$. *Then* $\beta(\lambda, \mu) \leq n$.

**Example 4.2.** Let $T =$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | **1** | 2 | 4 | 4 | **4** | **4** | **6** |
| 2 | 2 | 2 | 3 | 3 | | | | | | | |
| 3 | 3 | 4 | 4 | 4 | | | | | | | |
| 5 | 5 | 5 | 5 | 5 | | | | | | | |
| 6 | 6 | 6 | 6 | 6 | | | | | | | |

,

where $\lambda = (12, 5^4)$ and $\mu = (6, 4, 4, 7, 5, 6)$. Then $\gamma_1 = \gamma_6 = 1, \gamma_4 = 2, \gamma_2 = \gamma_3 = \gamma_5 = 0$ and $\beta(\lambda, \mu) = 3$. Entries corresponding to $\gamma_1, \gamma_4, \gamma_6$, which are forced into the top row, are boldfaced in the tableau. The missing elements from the first five columns of $T$ are $\{4, 4, 3, 2, 2\}$. Since $\mu_2, \mu_3 < 5$ and $n - \mu_2 = n - \mu_3 = 1$, we remove both a 3 and a 2 to get $\phi(T) = A_T = \{4, 4, 2\}$. These are the free entries in the arm of the first row, which are not boldfaced.

We will use the following straightforward fact in the proofs that follow.

**Lemma 4.3.** *Suppose that $T$ is a semistandard tableau of shape* $\lambda = (m, n^b)$ *and content* $\mu = (\mu_1, \dots, \mu_{b+2})$. *Then any row $i$ of $T$, where $i \geq 2$, can contain only the entries $i$ or $i+1$.*

**Lemma 4.4.** *Suppose that* $\lambda = (m, n^b)$, $\mu = (\mu_1, \dots, \mu_{b+2})$ *and that* $T \in SSYT(\lambda, \mu)$. *Let* $\sigma = (2, 3 \dots, b+2) \in \mathfrak{S}_{b+2}$. *Then* $\phi(\partial(T)) = \sigma\phi(T)$.

*Proof.* Since jeu de taquin promotion permutes the content of $T$, $|\phi(\partial(T))| = |\phi(T)| = \beta(\lambda, \mu)$. Let $f_i^T$ denote the number of $i$'s in the multiset $\phi(T) = A_T$ and $c_i^T$ the number of $i$'s in the first $n$ columns of $T$. If $\mu_i > n$ then $f_i^T = n - c_i^T$, and if $\mu_i < n$ then $f_i^T = n - c_i^T - (n - \mu_i) = \mu_i - c_i^T$. Any entry $i$ in $T$ with $3 \leq i \leq b+1$ either belongs to the first $n$ columns below the first row or in the last $m - n$ columns and after jeu de taquin promotion becomes an $i+1$ that belongs to the first $n$ columns below the first row of $\partial(T)$ or in the last $m-n$ columns of $\partial(T)$, respectively, so for $3 \leq i \leq b+1$, $c_i^T = c_{i+1}^{\partial(T)}$, which yields $f_i^T = f_{i+1}^{\partial(T)}$.

Entries equal to 2 belong to either the first or second row of the tableau. Those below the first row or in the last $m - n$ columns move to boxes below the first row or in the last $m - n$ columns and become 3's under jeu de taquin promotion. Any of the first $n$ columns that contains a 2 in the first row contains a $b+2$ in the last row. If there are also $(b+2)$'s in row $b+1$ with 1's above them in the first row, these are moved first by jeu de taquin promotion. Since the top entry in the column in this case is equal to 1, then for some $i$ the $i$th row contains the entry $i$ while the row beneath it contains the entry $i+2$. Jeu de taquin promotion then commences in the following way, beginning with the leftmost column that contains a $b+2$ in row $b+1$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | → | 1 | 1 | → | 1 | 1 |
| 2 | 2 | | 2 | 2 | | 2 | 2 |
| ⋮ | ⋮ | | ⋮ | ⋮ | | ⋮ | ⋮ |
| $i$ | $i$ | | $i$ | $i$ | | $i$ | $i$ |
| $i{+}1$ | $i{+}2$ | | $i{+}1$ | • | | • | $i{+}1$ |
| ⋮ | ⋮ | | ⋮ | ⋮ | | ⋮ | ⋮ |
| $b{+}1$ | • | | $b{+}1$ | $b{+}1$ | | $b{+}1$ | $b{+}1$ |

The jeu de taquin promotion path then moves left across row $i+1$ to the first column without a dot and north to the first row, replacing a 1 with a dot. Promotion behaves in the same way for all remaining columns that contain a $b+2$ in the last row and a 1 in the top row. For columns that contain $(b+2)$'s in the last row and entries equal to 2 in the first row, it is now the case that for the leftmost such column, any entry $i$ in the column has an $i-1$ immediately to its left. Thus, jeu de taquin promotion slides the box in the last row directly to row one, which moves the 2 into the second row. Each remaining $b+2$ in row $b+1$ behaves in the same way, sliding each remaining 2 into the second row. It follows that $c_2^T = c_3^{\partial(T)}$ so $f_i^T = f_{i+1}^{\partial(T)}$ for $2 \le i \le b+1$.

Since $\beta(\lambda,\mu) = \sum_{i=2}^{b+2} f_i^{\partial(T)} = f_2^{\partial(T)} + \sum_{i=2}^{b+1} f_i^T$ and $\beta(\lambda,\mu) = \sum_{i=2}^{b+2} f_i^T = f_{b+2}^T + \sum_{i=2}^{b+1} f_i^T$, we have $f_2^{\partial(T)} = f_{b+2}^T$ and the result follows. □

The following lemma shows that if $\lambda = (m, n^b)$ and $T \in SSYT(\lambda,\mu)$ has fixed content $\mu = (\mu_1,\ldots,\mu_{b+2})$, then $T$ is uniquely determined by its free entries. It follows that the map $\phi : SSYT(\lambda,\mu) \to \mathcal{A}$ is a bijection.

**Lemma 4.5.** *Let $\lambda = (m, n^b)$, $\mu = (\mu_1,\ldots,\mu_{b+2})$ and let $T \in SSYT(\lambda,\mu)$. Then $T$ is uniquely determined by the multiset $A_T$.*

*Proof.* For each $i$ with $\mu_i < n$, add $n-\mu_i$ elements $i$ to $A_T$ to produce an $n$-element multiset $X$. Since $T$ is semistandard, listing the elements of $X$ in weakly decreasing order completely determines the entries in $\{1,\ldots,b+2\}$ that are missing from each of the first $n$ columns of $T$. The complement of the $k$th element in the multiset gives the $k$th column of $T$. The remaining entries in $T$, determined from $\mu$, appear in weakly increasing order in the first row of $T$. □

Since jeu de taquin promotion permutes the content of a tableau, the content of $T$ and $\partial^{b+2}(T)$ are equal so $\partial^{b+2} : SSYT(\lambda,\mu) \to SSYT(\lambda,\mu)$ for $\lambda = (m,n^b)$ and $\mu = (\mu_1,\ldots,\mu_{b+2})$. By Lemma 4.4, if $T \in SSYT(\lambda,\mu)$ and $\sigma = (2,\ldots,b+2)$

then $\phi(\partial^{(b+2)j}(T)) = \sigma^j\phi(T)$. Thus $\phi(\partial^{(b+2)(b+1)}(T)) = \sigma^{b+1}\phi(T) = \phi(T)$ so $\partial^{(b+2)(b+1)}(T) = T$. If $1 \leq j < b+1$ satisfies $\partial^{(b+2)j}(T) = T$ for all $T$, then $\sigma^j\phi(T) = \phi(T)$ for all $T$, which is not possible, so we have the following lemma.

**Lemma 4.6.** *Let $\lambda = (m, n^b)$ and $\mu = (\mu_1, \ldots, \mu_{b+2})$. The* order of promotion *on $SSYT(\lambda, \mu)$ under the cyclic action of $\partial^{b+2}$ is equal to $b+1$.*

For a positive integer $n$, let $[n]_q = \dfrac{q^n - 1}{q - 1}$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. The *q-binomial coefficients* are defined by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!}$. To prove our CSP, we will use the bijection $\phi$ and invoke the following theorem due to Reiner, Stanton and White.

**Theorem 4.7.** *(Reiner, Stanton and White [15]) Let $X$ be the set of $k$-element multisets of $\{1, 2, \ldots, n\}$, let $C = \mathbb{Z}/n\mathbb{Z}$ act on $X$ via the permutation $\theta = (1, 2, \ldots, n)$ and let $f(q) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$. Then $(X, C, f(q))$ exhibits the cyclic sieving phenomenon.*

**Theorem 4.8.** *Let $\lambda = (m, n^b)$, $\mu = (\mu_1, \ldots, \mu_{b+2})$, let $C = \mathbb{Z}/(b+1)\mathbb{Z}$ act on $SSYT(\lambda, \mu)$ via $\partial^{b+2}$ and let $f(q) = \begin{bmatrix} b + \beta(\lambda, \mu) \\ \beta(\lambda, \mu) \end{bmatrix}_q$. Then $(SSYT(\lambda, \mu), C, f(q))$ exhibits the cyclic sieving phenomenon.*

*Proof.* Adjust the map $\phi$ by decrementing each of the entries in $\phi(T)$ to get $\psi : SSYT(\lambda, \mu) \to \mathcal{B}$, where $\mathcal{B}$ is the set of $\beta(\lambda, \mu)$-element multisets of $\{1, \ldots, b+1\}$. Let $\theta = (1, 2, \ldots, n)$. By Lemma 4.4, $\psi(\partial^{b+2}(T)) = \theta(\psi(T))$ and $\psi(\partial^{(b+2)j}(T)) = \theta^j(\psi(T))$.

We have $\psi^{(b+2)j}(T) = T$ if and only if $\psi(\partial^{(b+2)j}(T)) = \psi(T)$, which, by the above, yields $\theta^j(\psi(T)) = \psi(T)$. But, by Theorem 4.7, $|\mathcal{B}^{\theta^j}| = f(\omega^j)$, where $\omega$ is a primitive $(b+1)$-th root of unity. The result now follows. □

We will now examine the relationship between the cyclic sieving polynomial in Theorem 4.8 and the modified Kostka-Foulkes polynomial $\widetilde{K}_{\lambda,\mu}$. To do so, we work with plane partitions to get a nice formula for cocharge in the case where $\lambda = (m, n^b)$ and $\mu = (\mu_1, \ldots, \mu_{b+2})$.

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that $\pi$ has finitely many nonzero entries and is weakly decreasing in rows and columns. If $\sum \pi_{ij} = n$, we write $|\pi| = n$ and say that $\pi$ is a plane partition of $n$. We can adjust the bijection $\phi$ between $SSYT(\lambda, \mu)$ and the set of $\beta(\lambda, \mu)$-element multisets of $\{2, \ldots, b+2\}$ by subtracting two from each of the entries in $\phi(T)$ and reversing the order to get a bijection between $SSYT(\lambda, \mu)$ and the set of one-row plane partitions $\pi = (\pi_1, \ldots, \pi_{\beta(\lambda,\mu)})$ with $\pi_1 \leq b$ and $\beta(\lambda, \mu)$ columns; we will denote the image of $T$ under this bijection by $\pi_T$.

**Theorem 4.9.** *Let $\lambda = (m, n^b)$, let $\mu = (\mu_1, \ldots, \mu_{b+2})$ be a partition of $m + nb$ and let $T \in SSYT(\lambda, \mu)$. Then $cc(rw(T)) = |\pi_T| + n\binom{b+1}{2}$, where $|\pi_T|$ is the sum of the entries in $\pi_T$.*

*Proof.* We will consider the contribution of a given entry $i$ in the tableau to the cocharge. An entry $i$ copies the cocharge contribution of $i-1$ in its subword if $i-1$ precedes $i$ in its subword (an $(i-1, i)$ pairing) and it increases cocharge otherwise (an $(i, i-1)$ pairing).

If an entry $i$ belongs to row $i$, each of the entries $1, \ldots, i-1$ appear above it in the same column so $i$ pairs with an $i-1$ in a row above it, giving associated subword $w = \cdots i(i-1)\cdots 321$. Thus, every entry $i$ in row $i$ contributes $i-1$ to the cocharge.

Any entry $i$ in row one, where $i \geq 2$, pairs as $(i-1, i)$ in the associated subword so copies the cocharge of $i-1$. Any subword containing a forced entry consists entirely of forced entries. Since there are $\mu_1 - n$ forced 1's, there are $\mu_1 - n$ subwords consisting of all forced entries, and these are of the form $w = 12\cdots i$, so each forced entry contributes zero to the cocharge.

If $i-1$ belongs to one of the first $n$ columns, it either belongs to row $i-1$ or to row $i-2$. If $i$ is a free entry in row one, it cannot pair with a forced entry $i-1$ and, if it pairs with an entry $i-1$ in row $i-2$ or with a free entry $i-1$ in row one, this forces an entry $i$ in row $i-1$. It would then follow that $i-1$ pairs with an entry $i$ in the row beneath it, instead of the $i$ in the first row. Thus each free entry $i$ in row one pairs with an $i-1$ in row $i-1$, creating an $(i-1, i)$ pairing in the subword so copies the contribution of $i-1$ to the cocharge. Thus each free entry $i$ in row one contributes $i-2$ to the cocharge.

Finally, we will show that entries $i+1$ in row $i$ contribute $i-1$ to the cocharge. If $i+1$ pairs with an $i$ in the same row, this yields an $(i, i+1)$ pairing in the subword so $i+1$ contributes the same value to cocharge as $i$, which is $i-1$. If $i+1$ pairs with a free entry $i$ in row one this yields an $(i+1, i)$ pairing in the subword so that $i+1$ increases the contribution of the free entry, giving a contribution of $i-1$. The last case is when $i+1$ pairs with an $i$ in row $i-1$ creating an $(i, i-1)$ pairing in the subword so increasing the contribution of $i-1$ by one. By induction, $i-1$ contributes $i-2$ to the cocharge so $i+1$ contributes $i-1$ to cocharge. It follows that all entries in row $i$ contribute $i-1$ to the cocharge so $cc(\mathrm{rw}(T)) = |\pi_T| + n\sum_{i=1}^{b+1}(i-1) = |\pi_T| + n\binom{b+1}{2}$. □

**Corollary 4.10.** *Let $\lambda = (m, n^b)$, $\mu = (\mu_1, \ldots, \mu_{b+2})$ and $f(q) = \begin{bmatrix} b+\beta(\lambda,\mu) \\ \beta(\lambda,\mu) \end{bmatrix}_q$. Then $f(q) = q^{-n\binom{b+1}{2}}\widetilde{K}_{\lambda,\mu}(q)$.*

*Proof.* Denoting $SSYT(\lambda, \mu)$ by $SSYT$, the result follows from Lemma 4.9 since

$$\widetilde{K}_{\lambda,\mu}(q) = \sum_{T \in SSYT} q^{cc(rw(T))} \quad = \quad q^{n\binom{b+1}{2}} \sum_{T \in SSYT} q^{|\pi_T|}$$

$$= q^{n\binom{b+1}{2}} \sum_{\substack{\pi=(\pi_1,\ldots,\pi_{\beta(\lambda,\mu)}) \\ \pi_1 \leq b}} q^{|\pi|} = q^{n\binom{b+1}{2}} \begin{bmatrix} b+\beta(\lambda,\mu) \\ \beta(\lambda,\mu) \end{bmatrix}_q,$$

by [20, I.3.19] (see also [21, §7.21]). □

## Acknowledgements

The authors wish to thank two anonymous referees who provided useful suggestions that improved the paper, including an equivalent variation on our map between tableaux and multisets.

## References

[1] P. Alexandersson, Free action and cyclic sieving on skew semi-standard Young tableaux, *Bull. Iran. Math. Soc.* 49(6) (2023).

[2] P. Alexandersson, E. Kantarci Oğuz and S. Linusson, Promotion and cyclic sieving on families of SSYT, *Ark. Mat.* 59 (2021), 247–274.

[3] M. Bennett, B. Madill and A. Stokke, Jeu-de-taquin promotion and a cyclic sieving phenomenon for semistandard hook tableaux, *Discrete Math.* 319 (2014), 62–67.

[4] L. M. Butler, *Subgroup lattices and symmetric functions*, Amer. Math. Soc., Providence, 1994.

[5] B. Fontaine and J. Kamnitzer, Cyclic sieving, rotation, and geometric representation theory, *Sel. Math. New Ser.* 20 (2013), 609–625.

[6] H. O. Foulkes, A survey of some combinatorial aspects of symmetric functions, In: *Permutations*, Gauthier-Villars, Paris, (1974), 79–92.

[7] C. Gaetz, O. Pechenik, J. Striker and J. P. Swanson, Curious cyclic sieving on increasing tableaux, *Enumer. Comb. Appl.* 2(3) (2022), Art. S2R18.

[8] M. D. Haiman, Dual equivalence with applications, including a conjecture of Proctor, *Discrete Math.* 99 (1992), 79–113.

[9] A. Lascoux and M.-P. Schützenberger, Sur une conjecture de H.O. Foulkes, *C.R. Acad. Sc. Paris* 286A (1978), 323–324.

[10] Y.-T. Oh and E. Park, Crystals, semistandard tableaux and cyclic sieving phenomenon, *Electron. J. Combin.* 26(4) (2019), #P4.39.

[11] Y.-T. Oh and E. Park, $q$-dimensions of highest weight crystals and cyclic sieving phenomenon, *European J. Combin.* 97 (2021), Paper No. 103372.

[12] O. Pechenik, Cyclic sieving of increasing tableaux and small Schröder paths, *J. Combin. Theory Ser. A* 125 (2014), 357–378.

[13] S. Pon and Q. Wang, Promotion and evacuation on standard Young tableaux of rectangle and staircase shape, *Electron. J. Combin.* 18(1) (2011), #P18.

[14] T. Pressey, A. Stokke and T. Visentin, Increasing tableaux, Narayana numbers and an instance of the cyclic sieving phenomenon, *Ann. Comb.* 20 (2016), 609–621.

[15] V. Reiner, D. Stanton and D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* 108(1) (2004), 17–50.

[16] B. Rhoades, Cyclic sieving, promotion, and representation theory, *J. Combin. Theory Ser. A*, 117(1) (2010), 38–76.

[17] B. Sagan, The cyclic sieving phenomenon: a survey, In: *Surveys in Combinatorics 2011*, Cambridge Univ. Press, Cambridge, 2011, 183–234.

[18] M.-P. Schützenberger, Quelques remarques sur une construction de Schensted, *Canad. J. Math.* 13 (1961), 117–128.

[19] M.-P. Schützenberger, Promotion des morphismes d'ensembles ordonnés, *Discrete Math.* 2 (1972), 93–74.

[20] R. Stanley, *Enumerative Combinatorics Vol. 1*, Cambridge Univ. Press, Cambridge, 1997.

[21] R. Stanley, *Enumerative Combinatorics Vol. 2*, Cambridge Univ. Press, Cambridge, 1999.

(Received 13 Oct 2022; revised 11 Feb 2023)