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Optimal point sets determining few distinct angles*

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Abstract

Let $P(k)$ denote the largest size of a non-collinear point set in the plane admitting at most k distinct angles. We prove $P(2) = P(3) = 5$, and we characterize the optimal sets. We also leverage results from Fleischmann et al. [*Disc. Comput. Geom.* (2023)] to provide the general bounds $k+2 \leq P(k) \leq 6k$, although the upper bound may be improved pending progress toward the Strong Dirac Conjecture. We conjecture that the lower bound is tight, providing infinite families of configurations meeting the bound and ruling out several classes of potential counterexamples.

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1 Introduction

1.1 Background

In 1946, Erdős [3] introduced the problem of finding asymptotic bounds on the minimum number of distinct distances among sets of n points in the plane. The Erdős distance problem, as it has become known, proved infamously difficult and was only finally (essentially) resolved by Guth and Katz [8] in 2015.

The Erdős distance problem has also spawned a wide variety of related questions, including the problem of finding maximal point sets with at most k distinct distances. Erdős and Fishburn [4] determine maximal planar sets with at most k distinct distances. Recent results by Szöllősi and Östergård [14] and Xianglin [15] classify the maximal 3-distance sets in $\mathbb{R}^4$, 4-distance sets in $\mathbb{R}^3$, and 6-distance sets in $\mathbb{R}^2$. Epstein et al. [12] and Brenner et al. [1, 2] investigate Euclidean point sets with a low number of distinct triangles. In even more recent work, Fleischmann et al. [6, 7] consider a number of angle analogues of distinct distance problems. Newfound connections to frame theory and engineering have renewed interest in few-distance sets [14].

Characterizing the largest possible point sets satisfying a given property is a classic problem in discrete geometry. As another example, Erdős and Kelly [5] introduced the problem of finding maximal point sets of all isosceles triangles in 1947. Ionin [10] completely answers this question in Euclidean space of dimension at most 7.

We study one variation of a related problem of Erdős and Purdy [11]. They ask about $A(n)$, the minimum number of distinct angles formed by n not-all-collinear points in the plane. In general, the best-known bounds are $n/6 \leq A(n) \leq n-2$ [6, 7]. We consider the related problem of maximum size planar point sets admitting at most k distinct angles in $(0, \pi)$. Throughout, we ignore angles of 0 and π to adhere to the convention in related research (see [13], for example), although we provide results including the 0 angle as corollaries. We completely answer this question for $k = 2$ and $k = 3$ and note that the results of Fleischmann et al. [6] immediately imply asymptotically tight linear bounds for $k > 3$. We conjecture that the computed explicit lower bounds are tight, providing infinite families of tight examples and ruling out several classes of potential counterexamples. In resolving this question for $k = 2$ and $k = 3$, we systematically consider all possible triangles in such configurations and then reduce to adding points in a finite number of positions by geometric casework. We classify all optimal configurations.

1.2 Definitions and Results

By convention, we only count angles of magnitude strictly between 0 and π. Our computations still answer the related optimal point configuration questions including 0 angles (see Corollaries 3.6, 4.4). We begin by introducing convenient notation.

Definition 1.1. Let $\mathcal{P} \subset \mathbb{R}^2$. Then

$$A(\mathcal{P}) := \#\{|\angle abc| \in (0, \pi) \, : \, a, b, c \text{ distinct}, \, a, b, c \in \mathcal{P}\}.$$

Now we define the quantity we are interested in studying.

Definition 1.2. For all $k \geq 1$, define

$$P(k) := \max\{\#\mathcal{P} : \mathcal{P} \subseteq \mathbb{R}^2, \text{ not all points in } \mathcal{P} \text{ are collinear}, A(\mathcal{P}) \leq k\}.$$

We first provide general linear lower and upper bounds for $P(k)$. In particular, we have the following theorem.

Theorem 1.3. *For all $k \geq 1$,*

$$\begin{aligned} 2k+3 &\leq P(2k) &&\leq 12k; \\ 2k+3 &\leq P(2k+1) &&\leq 12k+6. \end{aligned}$$

In the distance setting, the best known upper bound on the analogous parameter is the quadratic $(2+k)(1+k)$, and no lower bound is well-understood [14]. It is therefore interesting and surprising that we find $P(k) = \Theta(k)$ in the angle setting. We prove Theorem 1.3 in Section 2.

In fact, we conjecture that the lower bounds stated in Theorem 1.3 are sharp. In Section 2, we describe several infinite families of point configurations meeting the lower bound and rule out many classes of potential counterexamples (see Theorems 2.4, 2.7, 2.8).

Furthermore, we explicitly compute $P(1), P(2)$, and $P(3)$ and exhaustively identify all extremal point configurations for each.

Proposition 1.4. *We have $P(1) = 3$, and the equilateral triangle is the unique extremal configuration.*

In order to have only a single angle, every triangle of three points in the configuration must be equilateral. As this is impossible for point configurations that are not the vertices of an equilateral triangle, $P(1) = 3$. $P(2)$ and $P(3)$ are considerably less trivial quantities. We calculate $P(2)$ and $P(3)$ via exhaustive casework, simultaneously characterizing all of the unique optimal point configurations up to rigid motion transformations and dilation about the center of the configuration. We proceed by first considering sets of three points and then characterize the additional points that may be added without determining too many angles. We prove Theorem 1.5 in Section 3 and Theorem 1.6 in Section 4.

Theorem 1.5. *We have $P(2) = 5$. Moreover, the unique optimal point configuration is four vertices in a square with a fifth point at its center (see* **I** *in Figure 1).*

Theorem 1.6. *We have $P(3) = 5$. There are 5 unique optimal configurations, shown in Figure 1.*

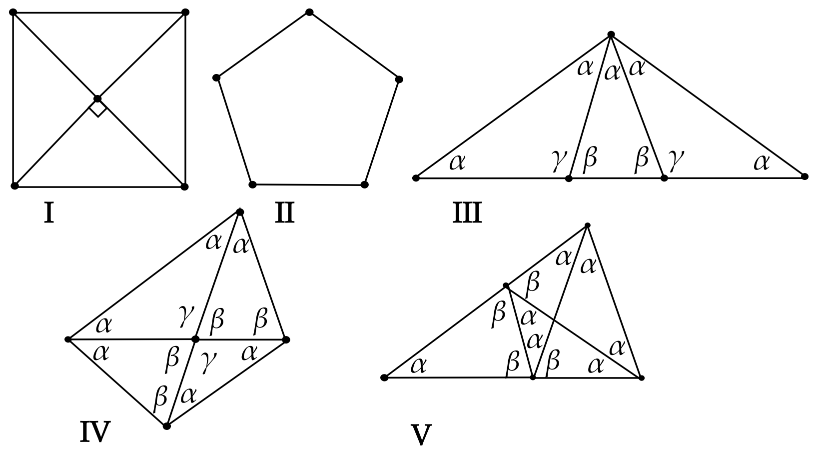


Figure 1: Optimal two and three angle configurations with $\alpha = \frac{\pi}{5}, \beta = \frac{2\pi}{5}, \gamma = \frac{3\pi}{5}$. Configuration **I** is the unique configuration on five points admitting two distinct angles.

2 General Bounds

Although one may in principle calculate $P(k)$ for small constant k by extensive casework (as we later calculate $P(2), P(3)$), it quickly becomes overwhelming. As such, we instead provide general bounds on $P(k)$. We conjecture that our lower bound is tight, providing several infinite families of point configurations meeting the bound. We also prove several results limiting the set of candidate counterexample configurations.

In [6] the authors study the quantity $A(n)$, the minimum number of angles admitted by a non-collinear point set of n points in the plane. They show in Lemma 2.2 and Theorem 2.5 that $n/6 \leq A(n) \leq n-2$. As mentioned in [6], the lower bound may be improved pending progress toward the Strong Dirac Conjecture.

Conjecture 2.1 (Strong Dirac Conjecture)**.** *For some $c_0 > 0$, every set $\mathcal{P}$ of non-collinear points in the plane contains a point incident to at least $\left\lfloor \frac{n}{2} \right\rfloor - c_0$ of the lines formed by the points in $\mathcal{P}$.*

Let $\ell(n)$ be the best known lower bound on this quantity. Then, from Lemma 2.2 of [6], $\frac{\ell(n)-1}{2} \leq A(n)$. By a result of Han [9], we know $\ell(n) \geq \left\lceil \frac{n}{3} \right\rceil + 1$. Indeed, this is where the lower bound $n/6 \leq A(n)$ arises from. Now, since $A(n) \leq n-2$, we have $n \geq A(n) + 2$, and so we deduce that $P(k) \geq k+2$. Similarly, we have $P(k) \leq 6k$. Resolution of Conjecture 2.1 would improve the upper bound in Proposition 2.2 to $4k + 2c_0 + 3$. Combining these bounds gives the following result.

Proposition 2.2. *$k + 2 \leq P(k) \leq 6k$.*

This implies all bounds in Theorem 1.3 except for the bound $2k+3 \leq P(2k)$. As stated in Remark 2.3 of [6], the point configuration of a regular $2n$-gon with a point added in the center admits precisely $(n-2)$ distinct angles. The added central point does not increase the number of distinct angles beyond that of the $2n$-gon. This implies that $2k+3 \leq P(2k)$. Configuration **I** in Figure 1 gives an example of this configuration starting from a regular 4-gon.

In fact, we conjecture that our lower bound constructions have the maximum number of possible points.

Conjecture 2.3. *The lower bound on $P(k)$ in Theorem 1.3 is tight. Namely, $P(2k) = 2k+3$ and $P(2k+1) = 2k+3$ for all $k \geq 1$.*

Via the correspondence between $A(n)$ and $P(k)$, a resolution of Conjecture 2.3 would amount to finding the optimal bounds of $A(n)$, thereby resolving a longstanding conjecture of Erdős and Purdy [11] and an open problem from [6]. Nonetheless, we provide some supporting evidence for Conjecture 2.3.

2.1 Tight Examples

We describe several infinite families of point configurations that meet the lower bound of Conjecture 2.3. For $P(2k)$, the vertices of a regular $(2k+2)$-gon with a point added in the center admit exactly $2k$ distinct angles. Another example is the stereographic projection of a regular $(2k+2)$-gon onto a line (see Theorem 2.8 and Figure 3), with the reflection of the point off the projected line added as a $(2k+3)^{\text{rd}}$ point.

For $P(2k+1)$, there are several more examples. Of course, the two configurations above admit $2k \leq 2k+1$ distinct angles and have $2k+3$ points. These both correspond to **I** in Figure 1. In addition, the regular $(2k+3)$-gon and the projection of the regular $(2k+3)$-gon onto a line (see Theorem 2.8 and Figure 3) admit exactly $2k+1$ distinct angles. These correspond to **II** and **III** of Figure 1, respectively.

2.2 Restrictions on Counterexamples

Now we prove that several large classes of point configurations cannot serve as counterexamples to Conjecture 2.3. We begin with a simple theorem.

Theorem 2.4. *Let $\mathcal{P} \subset \mathbb{R}^2$ with $|\mathcal{P}| = n$. If there exists $p \in \mathcal{P}$ such that p is on the convex hull of $\mathcal{P}$ and shares no line with two other points in $\mathcal{P}$, then $A(\mathcal{P}) \geq n-2$.*

Proof. Consider the angles formed with p as the apex. Let q be the next point in $\mathcal{P}$ on the convex hull, moving in a counterclockwise fashion. Now, consider all of the $n-2$ angles formed with p as the apex and q one of the legs. Since p is on the convex hull of $\mathcal{P}$ and shares no line with two other points in $\mathcal{P}$, these $n-2$ angles can be ordered such that each one is strictly contained in the next. See Figure 2. Namely, they are all distinct and $A(\mathcal{P}) \geq n-2$, as desired. □

As a consequence, any counterexample point configuration to Conjecture 2.3 must not satisfy the conditions of Theorem 2.4. As a corollary, this resolves the problem

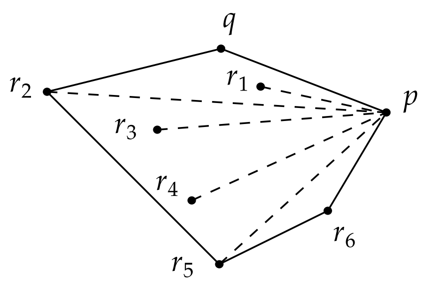


Figure 2: Choice of p and q inducing $n-2$ distinct angles. The third legs of the angles are marked r_1, r_2, etc. according to the ordering of the angles they induce.

completely for point configurations with no three points on a line, answering an open problem from [6].

Corollary 2.5. *We have $A_{no3l}(n) = n-2$, where $A_{no3l}(n)$ is the minimum number of distinct angles formed by $\mathcal{P} \subseteq \mathbb{R}^2$ with $|\mathcal{P}| = n$ and no three points in $\mathcal{P}$ sharing a line.*

To further restrict the classes of possible counterexamples to Conjecture 2.3, we begin with a constraining lemma.

Lemma 2.6. *Given a sequence of n positive integers, $x_1, x_2, \ldots, x_n$, there are at least n distinct sums of consecutive subsequences. Moreover, there are exactly n distinct sums of consecutive subsequences if and only if $x_i = x_j$ for all $i \neq j$.*

Proof. First, the number of distinct sums of consecutive subsequences is evidently at least n since x_1, $x_1 + x_2$, …, $x_1 + x_2 + \cdots + x_n$ are all distinct. Now suppose that not all x_i's are equal. Namely, suppose that $x_j \neq x_1$ and is the first element of the sequence to differ from x_1. Then, observe that $x_2 + x_3 + \cdots + x_j < x_1 + x_2 \cdots + x_j$ and $x_2 + x_3 + \cdots + x_j > x_1 + x_2 \cdots + x_{j-2}$ since $x_1, x_j > 0$. Then, since $x_j \neq x_1$, $x_2 + x_3 + \cdots + x_j \neq x_1 + x_2 + \cdots + x_{j-1}$. Hence, there are at least $n+1$ distinct sums of consecutive subsequences, yielding the forward implication. The reverse implication is clear. □

As a first application of Lemma 2.6, we show that the regular n-gon is a particularly special low angle point configuration.

Theorem 2.7. *Let $\mathcal{P} \subset \mathbb{R}^2$ with $|\mathcal{P}| = n$, each point of $\mathcal{P}$ on the convex hull of $\mathcal{P}$, and no three points in $\mathcal{P}$ on a line. Then, $A(\mathcal{P}) \geq n-2$. Moreover, $A(\mathcal{P}) = n-2$ if and only if the points in $\mathcal{P}$ form the vertices of a regular n-gon.*

Proof. Fix some point p. Label the remaining points x_1, …, x_{n-1}, in clockwise order around the convex hull starting at p. Let $\alpha_i = \angle x_i p x_{i+1}$. The angles formed with p as the apex are precisely the distinct sums of consecutive subsequences of $\alpha_1, \ldots, \alpha_{n-2}$. By Lemma 2.6, $A(\mathcal{P}) \geq n-2$. Suppose $A(\mathcal{P}) = n-2$. Then, again by Lemma 2.6,

$\alpha_i = \alpha_j$ for all $i \neq j$. Moreover, the value of the angles formed cannot depend on the choice of p. Then, consider $\triangle x_1 p x_{n-1}$. We know that $\angle x_1 p x_{n-1} = (n-2)\alpha$, $\angle p x_1 x_{n-1} = \alpha$, and $\angle p x_{n-1} x_1 = \alpha$. Hence, $\alpha = \pi/n$ and the points in $\mathcal{P}$ form the vertices of a regular n-gon, as desired. $\square$

We now show that the only n-point configurations with $n-1$ points on a line and (at most) $n-2$ distinct angles are the stereographic projections of regular polygons onto a line, as in **III** of Figure 1. See Figure 3 for examples for small n.

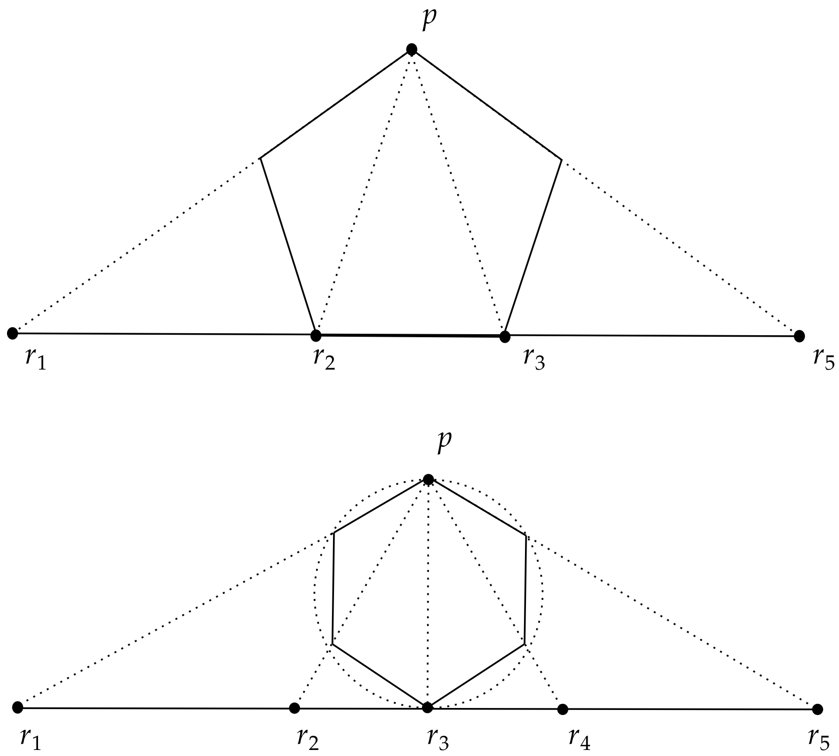


Figure 3: Point configurations for Theorem 2.8 with $n = 5$ and $n = 6$.

Theorem 2.8. *Let $\mathcal{P} \subset \mathbb{R}^2$ with all but one point on a single line. If $|\mathcal{P}| = n$, then $A(\mathcal{P}) \geq n-2$. Moreover, $A(\mathcal{P}) = n-2$ if and only if $\mathcal{P}$ is the stereographic projection of a regular polygon onto a line, with the line being tangent to the circumcircle at a vertex of the polygon for n even or coincident to an edge of the polygon for n odd.*

Proof. Let $p \in \mathcal{P}$ be the point off of the line. Without loss of generality, assume that the x-axis is the line. Then, let x_1 be the point in $\mathcal{P}$ on the x-axis with the most negative x-coordinate. Label the remaining points in $\mathcal{P}$ from $x_2, \ldots, x_{n-1}$ in increasing order of their x-coordinates. Now, observe that the distinct angles formed with p as the apex are exactly given by the distinct sums of consecutive subsequences of $\angle x_1 p x_2$, $\angle x_2 p x_3$, $\ldots$, $\angle x_{n-2} p x_{n-1}$. Hence, by Lemma 2.6, $A(\mathcal{P}) \geq n-2$.

Now, suppose that $A(\mathcal{P}) = n-2$. Then, again, by Lemma 2.6, we have that each of these angles must be equal to some α, and the $n-2$ angles formed with p as the apex are $\alpha, 2\alpha, \dots (n-2)\alpha$. Next, assume without loss of generality that $\lceil \frac{n-1}{2} \rceil$ points on the lines have x-coordinate at most the x-coordinate of p. Then, observe that the sequence $\angle px_1x_2, \angle px_2x_3, \dots \angle px_{\lceil \frac{n-1}{2} \rceil}x_{\lceil \frac{n-1}{2} \rceil+1}$ is strictly increasing. Moreover, the sequence $\angle px_2x_1, \angle px_3x_2, \dots \angle px_{\lceil \frac{n-1}{2} \rceil}x_{\lceil \frac{n-1}{2} \rceil-1}$ of supplementary angles to the first sequence is strictly decreasing. The angles in the former sequence are all at most $\pi/2$ and in the latter are all at least $\pi/2$ since the x coordinates of these points are at most that of p.

If n is even, then $\lceil \frac{n-1}{2} \rceil = n/2$. In this case, there are at least $n/2 + n/2 - 1$ angles. To get $n-2$ distinct angles, the angle $\pi/2$ must be repeated. As such, there are $n/2 - 1$ points with x-coordinate at most that of p, one point sharing the same x-coordinate as p, and the remaining $n/2 - 1$ points with x-coordinate at least that of p. If n is odd, there cannot be an angle of $\pi/2$ or else, without loss of generality, there are $(n+1)/2$ points with x-coordinate at most that of p. Then, the same sequences induce at least $n-1$ distinct angles.

Namely, if n is even, we then have that the sequence $\angle px_1x_2$, $\angle px_2x_3, \dots$ $\angle px_{\lceil \frac{n-1}{2} \rceil}x_{\lceil \frac{n-1}{2} \rceil+1}$ is precisely $\alpha, 2\alpha, \dots, n\alpha/2 = \pi$. The sequence of supplementary angles is then $(n-2)\alpha, (n-3)\alpha, \dots, \pi$. If n is odd, the sequences are $\alpha, 2\alpha, \dots$ $(n-1)\alpha/2$ and $(n-2)\alpha, (n-3)\alpha, \dots, (n+1)\alpha/2$, respectively.

To show that these configurations are exactly the stereographic projection of a regular polygon onto a line, it remains to show that $\alpha = \pi/n$. But consider the triangle $\triangle px_1x_{n-1}$. First, $\angle x_1px_{n-1} = (n-2)\alpha$. By the above and by symmetry, $\angle px_1x_{n-1} = \angle px_{n-1}x_1 = \alpha$. Hence $\alpha = \pi/n$, yielding the desired result. □

3 Proof of Theorem 1.5

Proof. In any valid point configuration with at least three points there are triangles. For any point configuration with at most two angles, all triangles must be isosceles. We divide into two cases based on whether or not there is an equilateral triangle. Unless otherwise specified, when considering points belonging to some region, we consider the interior of that region. Oftentimes the boundaries must be treated separately.

3.1 There is an equilateral triangle

We begin by considering point configurations with three of the points forming an equilateral triangle.

Claim 3.1. *In any configuration of four not all collinear points with three forming an equilateral triangle, there are at least three distinct angles.*

Proof. We consider adding a fourth point in cases (Figure 4).

Case 1: $p \in A$. Then $\angle acp < \pi/3$ and $\angle cap > \pi/3$, leading to more than two angles.

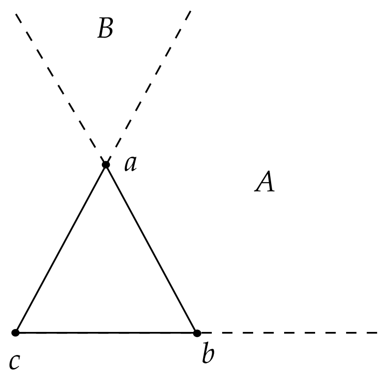


Figure 4: The regions for an equilateral triangle.

Case 2: $p \in \overline{ab}$. Then $\angle bcp < \pi/3$ and one of $\angle cpb$ and $\angle apc \geq \pi/2$, leading to more than two angles.

Case 3: $p \in \overleftrightarrow{ac}$ to the upper-right of a. Then $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, again leading to more than two angles.

Case 4: $p \in B$. In this case, $\angle cbp > \pi/3$ and $\angle cpb < \pi/3$, leading to more than two angles.

Case 5: p is in the interior of $\triangle abc$. In this case, one of $\angle apb, \angle bpc, \angle cpa \geq 2\pi/3$ and $\angle acp < \pi/3$, leading to more than two angles.

Up to symmetry, these cases are exhaustive. Thus if there is an equilateral triangle in the configuration, there can only be at most three points. $\square$

3.2 There is no equilateral triangle

Now, let a, b, and c be the vertices of an isosceles triangle with base angle β and α the apex vertex. We reduce the number of possibilities for additional points by partitioning the plane into regions A_i (Figure 5). Note that we may without loss of generality assume that no fourth point is added within the interior of $\triangle abc$ as we could then choose one of the resultant interior triangles as our initial triangle. Also note that A_1 and A_1', A_3 and A_3', and $\overleftrightarrow{ac}$ and $\overleftrightarrow{ab}$ are equivalent up to symmetry.

Claim 3.2. *No additional points may be added in $A_1, A_1', A_2, A_2', A_3,$ or A_3' without inducing a third distinct angle.*

Proof. We treat each case separately, appealing to symmetry for A_1', A_2', and A_3'.

Case 1: $p \in A_1$. In this case, $\angle pab > \alpha$ and $\angle pcb > \beta$. So, regardless of whether α or β is greater, adding p introduces an additional angle. So, no additional points can be in A_1.

Case 2: $p \in A_2$. In this case, $\angle pcb$ and $\angle pbc$ are greater than β, so both must be α to not add additional angles. But then $\angle cpb = \pi - 2\alpha \neq \beta$. Then, in order to not add additional angles, we must have $3\alpha = \pi$. But, this implies $\triangle pcb$ is an equilateral triangle. Thus no points may be added in A_2.

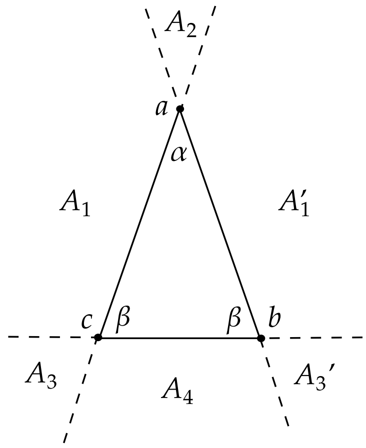


Figure 5: The regions for an isosceles triangle.

Case 3: $p \in A_3$. In this case, $\angle bap > \alpha$ and $\angle abp > \beta$, so there is an additional angle added regardless and no additional points are possible.

This completes the proof of the claim. □

A point may be added in the remaining regions, but the placement of that point heavily restricts the point configuration.

Claim 3.3. *If $p \in A_4$, then acpb is a square.*

Proof. The conditions of Theorem 2.7 apply, so, in order for the point configuration to have two distinct angles, *acpb* must be a square. □

Claim 3.4. *If $p \in \overleftrightarrow{bc}$, then $\beta = \pi/4$, $\alpha = \pi/2$, and p is the midpoint of edge bc.*

Proof. This is immediate by Theorem 2.8 since the projection of a onto $\overleftrightarrow{bc}$ must be the midpoint of the edge bc since $\triangle abc$ is isosceles. □

Claim 3.5. *If $p \in \overleftrightarrow{ac}$, then $\beta = \pi/4$, $\alpha = \pi/2$, and $\triangle cbp$ is an isosceles right triangle with b the apex vertex, p on $\overleftrightarrow{ac}$ to the upper right of a, and a at the center of side $\overline{pc}$.*

Proof. This will again follow from Theorem 2.8. Since $\angle acb < \pi/2$, the projection of b onto $\overleftrightarrow{ac}$ cannot be at c. By Theorem 2.8, the projection must then either be a or p. If the projection is at a, the desired result holds. If the projection were at p, then p would have to be the midpoint of $\overline{ac}$ and $\beta = \pi/2$, contradicting the assumption that $\triangle abc$ is isosceles. □

As such, in order to add additional points to an isosceles triangle point configuration without adding additional angles, we must have $\alpha = \pi/2$ and $\beta = \pi/4$. The four additional possible points are marked in Figure 6.

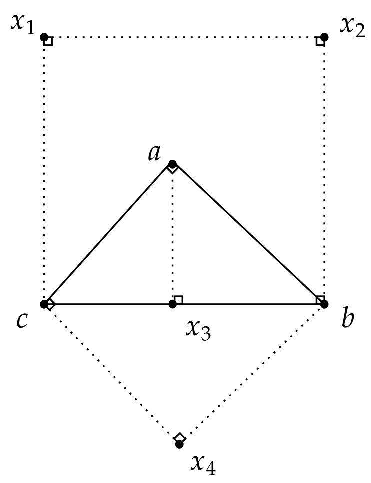


Figure 6: Compatible points with the right triangle.

Note that $\angle x_4 a x_1, \angle x_4 a x_2 > \pi/2$. So, x_4 cannot be in the same point configuration as x_1 or x_2. The same follows for x_3. However, we may have both x_1 and x_2 or both x_3 and x_4, either of which give the unique extremal configuration **I** in Figure 1. □

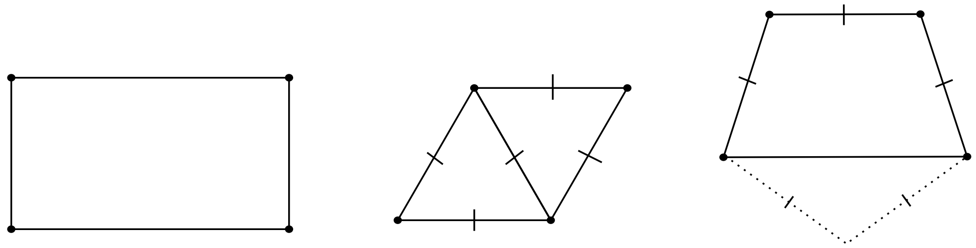
Corollary 3.6. *If the trivial* 0*-angle is included in the count, then* $P(2) = 4$ *and the unique configuration is the square.*

Proof. The only 5-point configuration no longer holds when we count the 0-angle. Figure 6 displays all valid four point configurations which define only 2 angles excluding 0, as detailed in the proof of $P(2)$. All the shown points but x_4 define a 0-angle, so the only valid 4-point configuration is the square. □

4 Proof of Theorem 1.6

Lemma 4.1. *Let abcd be a convex quadrilateral defining three angles or fewer. Then, they form one of the three configurations of Figure 7, where* 5.1 *is a rectangle,* 5.2 *is two attached equilateral triangles, and* 5.3 *is four of the five vertices of a regular pentagon.*

Proof. If all the angles of the quadrilateral are $\pi/2$, we are in the case of Figure 5.1. Then we may assume that there is at least one obtuse angle, γ, and one acute angle, β. Any angle α formed by splitting β is less than β and thus must be exactly $\beta/2$ so as not to create two additional angles for a total of four. These three angles $\alpha = \beta/2$, β, γ are then exactly the three angles in the configuration. Now we consider each of the four cases of placing β and γ about the quadrilateral, with the first listed angle corresponding to vertex a, the second to b, and so on, and with a, b, c, and d in clockwise cyclic order.



5.1 5.2 5.3

Figure 7: Configurations of points in a convex quadrilateral defining at most three distinct angles.

Case $\gamma\beta\gamma\beta$: Equal opposite angles implies that the quadrilateral is a parallelogram. The fact that bd bisects the two β angles implies that $abcd$ is in fact a rhombus. Thus, ac also bisects the γ angles, implying that $\gamma/2 = \beta$. So, $6\beta = 2\pi$ and $\alpha = \pi/6, \beta = \pi/3$, and $\gamma = 2\pi/3$ in this case. Given that $abcd$ is a rhombus, the configuration in this case is similar to Figure 5.2.

Case $\gamma\gamma\beta\beta$: Note that we have $\gamma + \beta = \pi$ from the angle sum of the quadrilateral. This implies that ab and cd are parallel. So, by analyzing the alternate interior angles given by the transversal ac, we have $\gamma = \alpha + (\gamma - \alpha)$, where $\alpha = \angle cab$ and $\gamma - \alpha = \angle cad$. Thus, $\gamma - \alpha = \beta$ and $3\beta/2 = \gamma$, so $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. Then by considering isosceles triangles dab and abc, we see that segments $\overline{da}, \overline{ab}$, and $\overline{bc}$ are all of equal length. Thus, the configuration is similar to Figure 5.3 in this case.

Case $\gamma\gamma\gamma\beta$: Diagonal bd bisects the angle β. Then, since the sum of the angles of $\triangle bcd$ and $\triangle abd$ are both π and $\beta \neq \gamma$, we must have $\angle abd = \angle dbc = \beta/2$. The diagonal ac must then also bisect angles $\angle dab$ and $\angle dcb$ or else yield more than three distinct angles. But then, $3\beta = \pi = 4\beta$ from the angle sums of $\triangle acd$ and $\triangle abc$, a contradiction.

Case $\beta\beta\beta\gamma$: By an argument analogous to the previous case, we must have that diagonal bd bisects the angle γ at d and $\angle abd = \angle dbc = \beta$. But then, $4\alpha = \pi = 6\alpha$ by looking at the angle sums of $\triangle abd$ and $\triangle bcd$, a contradiction. □

To handle the configurations without convex quadrilaterals, we will make use of the following proposition.

Proposition 4.2. *Let a, b, c, d be points such that d is contained in the interior of $\triangle abc$ and the configuration induces at most three distinct angles. Then, $\triangle abc$ must be equilateral and d must be in the center of $\triangle abc$.*

Proof. Note that $\angle adb > \angle acb > \angle acd$. This is similarly true of $\angle bdc, \angle bac, \angle bad$ and of $\angle adc, \angle abc, \angle abd$. Symmetry and the maximum of three distinct angles then allows the completion of all angles in the configuration, finishing the proof. See Figure 8. □

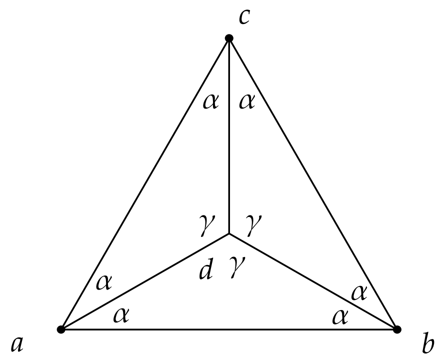


Figure 8: Triangular configuration resultant from Proposition 4.2.

Lemma 4.3. *Let a, b, c, d, and E be five points such that their convex hull is $\triangle abc$, no four of them form a convex quadrilateral, and the configuration induces at most three distinct angles. Then there is only one possible configuration (namely, the stereographic projection of the points of a regular pentagon onto a line,* **III** *of Figure 1).*

Proof. We proceed by casework on the number of points in the interior of $\triangle abc$.

Case 1: No points in the interior of $\triangle abc$. If neither d nor e are in the interior of $\triangle abc$ then, since the convex hull of the five points is $\triangle abc$, d and e must both be on the edges of $\triangle abc$. If they are not on the same side of the triangle, then the quadrilateral formed by d, e, and the ends of the edge which neither d nor e lie on is convex, yielding a contradiction.

Now, suppose without loss of generality that d and e lie on $\overline{ab}$ with the order of the points being a, d, e, and then b. Three distinct angles are immediately induced in this case. Namely, $\angle acd = \alpha < \angle ace = \beta < \angle acb = \gamma$. Since the difference between each pair of angles is also induced by this configuration, we have that $\beta = 2\alpha$ and $\gamma = 3\alpha$. Since $\angle adc > \angle aec > \angle abc$, we have $\angle adc = \gamma, \angle aec = \beta$, and $\angle abc = \alpha$. This is similarly true of $\angle ceb, \angle cdb$, and $\angle cab$ by symmetry. Thus, the angle sum of $\triangle acb$ implies $5\alpha = \pi$ and thus $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. So, in this case the points are configuration of **III** of Figure 1.

Case 2: One point in the interior of $\triangle abc$. Suppose without loss of generality that d is the point along an edge of $\triangle abc$, say $\overline{ab}$. Then, e is in the interior of $\triangle abc$. Now e must be on $\overline{cd}$ or else one of $adec$ or $bced$ is a convex quadrilateral.

Now, from Proposition 4.2, $\triangle abc$ must be equilateral and e must be the center of the triangle. This induces angles of $\pi/6, \pi/3, 2\pi/3$. However, d and e form a right angle, yielding more than three distinct angles. Hence, there are no valid configurations in this case.

Case 3: Both points in the interior of $\triangle abc$. From Proposition 4.2, $\triangle abc$ must be equilateral and both d and e must be the center of the triangle, a contradiction. □

Now we exhaustively check the points that may be added to the configurations given by Lemmas 4.1 and 4.3. All valid configurations of five points inducing at most three distinct angles arise from either adding a point to a configuration from Lemma 4.1 or the configuration given in 4.3. This is because the convex hull of the configuration must have at least three vertices (by definition of $P(k)$) and, if the convex hull has five vertices, any four of the vertices forms a convex quadrilateral.

Figure 5.1: Consider adding a point to the configuration shown in Figure 5.1, with the angles formed by the vertices of the rectangle being $\alpha < \beta < \gamma$ with $\alpha + \beta = \gamma = \pi/2$. Label the vertices of the rectangle shown in 5.1 of Figure 7 as a, b, c, and d starting from the top left as a and proceeding clockwise. Then, if a point e is added in the exterior of $abcd$, it will form an obtuse angle with one edge of the angle being a side of the rectangle. For example, if e is added below $\overline{cd}$, then $\angle bce$ is obtuse. If e is added to edge ab, then $\angle deb$ is obtuse. It will similarly induce an obtuse angle if it is added to any other edge. Finally, if e is added to the interior of $abcd$, then the only way e may be added without inducing an obtuse angle is if all the segments from e to the vertices of the rectangle form angles of $\pi/2$ with each other at e. However, this would imply that the diagonals of $abcd$ intersect at e at a right angle, implying that $abcd$ is a square.

So, the only valid configurations require that $abcd$ form a square. Moreover, if $abcd$ form a square, we can still not induce any obtuse angles. This is because the other two angles in any triangle with an obtuse angle could not both be $\pi/4$ (and cannot be $\pi/2$), yielding more than three distinct angles. Thus, the only extremal configuration in this case is adding a fifth point e as the centerpoint of a square, **I** of Figure 1.

Figure 5.2: In Figure 5.2, the angles are all determined: $\alpha = \pi/6, \beta = \pi/3$, and $\gamma = 2\pi/3$. Let the points a, b, c, and d be in clockwise order around the configuration such that $\overline{ac}$ is the segment dividing the two equilateral triangles. In order to not contradict Lemma 4.1, any added point must be in the interior of the rhombus (no point may be added to decrease the number of vertices in the convex hull since $abcd$ is a parallelogram). In order for e to not yield any angles smaller than α, e must be in the center of $\triangle abc$ or $\triangle cda$. However, in either case, this yields a new angle of $\pi/2$. So, no points may be added in this case.

Figure 5.3: As in Figure 5.2, the angles in Figure 5.3 are all determined: $\alpha = \pi/5, \beta = 2\pi/5$, and $\gamma = 3\pi/5$. Label the points $abcd$ clockwise starting from the top left as in the diagram of 5.3 in Figure 7. In order to not violate Lemma 4.1, any added point must be in the interior of $abcd$, must result in a triangular convex hull, or must be outside of $abcd$ and have every convex quadrilateral in the configuration an instance of Figure 5.3. In the former case, in order to not add an angle smaller than α, e must be added at the intersection of $\overline{ac}$ and $\overline{bd}$. In the second case, e must be added at the intersection of $\overleftrightarrow{ad}$ and $\overleftrightarrow{bc}$. In the last case, the configuration with the added point cannot have a convex hull of a quadrilateral, as that quadrilateral could not be an instance of Figure 5.3. Thus, it must be a pentagon. In order to guarantee that every convex quadrilateral in the configuration is a copy of Figure 5.3, it must be regular. All three configurations (**II**, **IV**, and **V** of Figure 1) are valid, but are

not mutually compatible as adding multiple of these points would form an angle of magnitude less than α.

III of Figure 1: As in Lemma 4.3, suppose that the convex hull of the configuration is $\triangle abc$ with d and e on $\overline{ab}$ such that the points are in the order a, d, e, and then b.

If another point were added to this configuration, either the convex hull would remain a triangle or there would be four points which form a convex quadrilateral. In the former case, no point could be in the interior of a triangle, as that would force the angles to be as in Proposition 4.2, which they are not. Thus, an additional added point would have to be placed on an existing edge. It could not be placed on $\overline{ab}$, as it would split an angle of α. If it were placed on $\overline{ac}$ or $\overline{bc}$ it would form a convex quadrilateral with c, d, and e. Given the induced values of the angles in this case, that quadrilateral would have to be similar to the configuration in Figure 5.3. However, from the prior casework, no configuration containing a similar copy of Figure 5.3 may have more than five points. Hence, the only extremal configuration in this case is **III** of Figure 1.

Therefore, $P(3) = P(2) = 5$, with five optimal configurations as in Figure 1.

Corollary 4.4. *If the trivial* 0*-angle is included in the count, then* $P(3) = 5$*, and the square with the center-point and the regular pentagon are the only valid configurations.*

Proof. Since this is a more restricted setting, the set of valid five-point configurations be a subset of the configurations identified above. By direct inspection, the square with the center-point and the pentagon are the only of the five in Figure 1 which define only three angles. All the others define three angles greater than zero and also the 0-angle by collinearity. □

5 Future Work

While it seems possible to compute $P(k)$ by exhaustive casework for higher values of k, the casework quickly becomes overwhelming. Nonetheless, as is visible from the applications of Theorems 2.7 and 2.8 in Section 3, stronger general structural restrictions could facilitate such a pursuit.

An important future direction is the resolution of Conjecture 2.3. Given the supporting evidence provided in Section 2, we believe that future work should improve the upper bound of $P(n) \leq 6n$, either via progress towards the Strong Dirac Conjecture (which would still fall short of our conjecture) or by some other means. There is significant room for ruling out broad swaths of potential counterexamples.

Alternatively, future research may find a more efficient method of constructing viable point sets without the need for the exhaustive search we perform. It is also an open problem to investigate $P(k)$ with point sets in more than two dimensions. Low angle configurations using variations of Lenz's construction, as in [6], may yield insight into optimal structures in higher dimensions.

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