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On Kleinewillinghöfer types of finite Laguerre planes with respect to translations

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Abstract

Kleinewillinghöfer classified Laguerre planes with respect to various linearly transitive groups of central automorphisms and obtained a multitude of types. In this paper we investigate the Kleinewillinghöfer types of finite Laguerre planes with respect to Laguerre translations and the full automorphism groups of these planes. This yields, among other results, a characterization of finite elation Laguerre planes of odd order by their Kleinewillinghöfer types.

1 Introduction

A finite *Laguerre plane* $\mathcal{L}$ *of order* n is a transversal (or group divisible) 3-design with block size $n+1$ and n points in each 'group', or equivalently, an orthogonal array of strength 3 on n symbols (levels), $n+1$ constraints and index 1, cf. [1]. Since we have a more geometric point of view we rather use the term Laguerre plane instead of orthogonal array or transversal design. Explicitly, a finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order n is an incidence structure consisting of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators (where circles and generators are both subsets of P) such that $\mathcal{G}$ partitions P and each generator contains n points, such that each circle intersects each generator in precisely one point, and such that three points no two of which are on the same generator can be joined by a unique circle; see [38] or [33, §2] for a more geometric definition of (general) Laguerre planes.

Models of finite Laguerre planes can be obtained as follows. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, q)$, q a prime power. Embed $\mathcal{P}_2$ into 3-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, q)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Generators are the traces of lines of $\mathcal{P}_3$ through v that are

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contained in the cone. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* q. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order* q. All known finite Laguerre planes of odd order are Miquelian and all known finite Laguerre planes of even order are ovoidal. In fact, it is a long standing open problem whether or not these are the only finite Laguerre planes. (There are many non-ovoidal infinite Laguerre planes though, see, for example, [19] or [24, §2].)

Similar to the Lenz–Barlotti classification of projective planes, Kleinewillinghöfer classified Laguerre planes with respect to linearly transitive subgroups of *central automorphisms*, that is, automorphisms that fix at least one point and a central collineation is induced in the derived projective plane at this fixed point. Furthermore, the induced groups of central collineations are transitive on each central line except for the obvious fixed points, the centre and the point of intersection with the axis. In this paper we are mainly concerned with two of the four kinds of central automorphisms in Laguerre planes that Kleinewillinghöfer used, namely Laguerre translations, that is, G-translations and $B(q, C)$-translations; see Section 3 for definitions.

The (Kleinewillinghöfer) type of a Laguerre plane $\mathcal{L}$ is the type of the (full) automorphism group of $\mathcal{L}$ in Kleinewillinghöfer's classification. We investigate the Kleinewillinghöfer types of finite Laguerre planes with respect to Laguerre translations, labelled A to K, making use of the special combinatorial situation in finite planes. As so often is the case in finite geometry Laguerre planes of even order behave quite differently from those of odd order. This is reflected in the results we obtain for the Kleinewillinghöfer types of finite Laguerre planes. In particular, for one of the types, type D, one has the following.

Main Theorem *A finite Laguerre plane* $\mathcal{L}$ *of Kleinewillinghöfer type D has* $\mathcal{E} = \mathcal{G}$. *Furthermore, if the order of* $\mathcal{L}$ *is odd, then* $\mathcal{L}$ *is an elation Laguerre plane.*

In Section 2 we give a brief summary of basic properties of and results about finite Laguerre planes. The following section deals with Laguerre translations and the types of finite Laguerre planes with respect to these central automorphisms are investigated. In particular, some feasible types are excluded in even/odd order and finite elation Laguerre planes of odd order are characterized by their Kleinewillinghöfer types. In the last section we combine the results of Section 3 with respect to Laguerre translations with those from [33] about Kleinewillinghöfer types of finite Laguerre planes with respect to Laguerre homotheties and provide examples of finite Laguerre planes of four of the combined types (with respect to both Laguerre translations and homotheties). We further exclude four combined types from the list of feasible combined types as Kleinewillinghöfer types of finite Laguerre planes.

2 Laguerre planes

Since the generators partition the point set P, being on the same generator defines an equivalence relation on P, and points on the same generator are often called *parallel*. We denote the generator that contains the point p by $[p]$.

It readily follows that for each point p of a Laguerre plane $\mathcal{L}$ the residual incidence structure $\mathcal{A}_p = (A_p, \mathcal{L}_p)$ whose point set A_p consists of all points of $\mathcal{L}$ that are not parallel to p and whose line set $\mathcal{L}_p$ consists of all traces in A_p of circles of $\mathcal{L}$ passing through p and of all generators not passing through p is an affine plane, called the *derived affine plane at* p. This affine plane extends to a projective plane $\mathcal{P}_p$, which we call the *derived projective plane at* p.

When forming the derived projective plane $\mathcal{P}_p$ of a Laguerre plane at a point p circles C not passing through the distinguished point p induce ovals in $\mathcal{P}_p$ by removing from C the point on $[p]$ and adding in $\mathcal{P}_p$ the ideal point ω of the lines that come from generators of $\mathcal{L}$. The ideal line of $\mathcal{P}_p$ (relative to the derived affine plane $\mathcal{A}_p$) is a tangent to this oval.

In case of an ovoidal Laguerre plane each derived affine plane is Desarguesian. Given a parabolic function f (that is, $\{(x, f(x)) \mid x \in \mathbb{F}\} \cup \{\omega\}$ is an oval in the Desarguesian projective plane over the field $\mathbb{F}$ where ω is the ideal point of the y-axis in the projective plane), the circles of the corresponding ovoidal Laguerre plane $\mathcal{L}(f)$ are the sets

$$\{(x, y) \in \mathbb{F}^2 \mid y = af(x) + bx + c\} \cup \{(\infty, a)\},$$

where $a, b, c \in \mathbb{F}$. For example, $f(x) = x^2$ describes the Miquelian Laguerre planes. Moreover, every ovoidal Laguerre plane is isomorphic to a plane $\mathcal{L}(f)$ for a suitable parabolic function f.

Note that every oval in a finite Desarguesian projective plane $\mathrm{PG}(2, q)$ of odd order q is a conic by [28]. As a consequence, a finite Laguerre plane of odd order that admits a Desarguesian derivation is Miquelian, see [3] and [22, VII.2].

There may be many different ovals in a finite Desarguesian projective plane of even order 2^h. One particularly nice class of ovals are *translation ovals*. These are ovals $\mathcal{O}$ that have a point c such that the translations of the projective plane with axis the tangent to $\mathcal{O}$ at c that fix $\mathcal{O}$ are transitive on $\mathcal{O} \setminus \{c\}$. They can be described by $f(x) = x^{2^k}$ where k is co-prime to h, see [20] or [11].

If $\mathcal{O}$ is an oval in a projective plane of even order, then the tangents to $\mathcal{O}$ all pass through a common point, the *nucleus* or *knot* of $\mathcal{O}$, see [25] or [7, 3.2.23]. This fact implies the following property of Laguerre planes of even order as noted in [15, p. 27] where we say that two circles C and C' *touch* at a point p if $C \cap C' = \{p\}$ or $C = C'$.

Lemma 2.1 *In a finite Laguerre plane of even order the relation of "touching" between circles is an equivalence relation, that is, if circles C_1 and C_2 touch each other at a point p and C_2 and C_3 touch each other at a point q (not necessarily the same as p), then C_1 and C_3 also touch each other at a point.*

An *automorphism* of a Laguerre plane $\mathcal{L}$ is a permutation of the point set that maps circles to circles and generators to generators. The collection of all automorphisms of $\mathcal{L}$ forms a group with respect to composition, the automorphism group $\Gamma = \mathrm{Aut}(\mathcal{L})$ of $\mathcal{L}$. The collection of all automorphisms of $\mathcal{L}$ that fix each generator is a normal subgroup of Γ, called the *kernel* of $\mathcal{L}$. Finally, the collection of all automorphisms that fix each generator globally but fix no circle, together with the identity forms a normal subgroup Ξ, see [31, Corollary 1.5], called the *elation group* of $\mathcal{L}$.

In [31] finite elation Laguerre planes were introduced and their basic structure investigated. They generalize finite ovoidal Laguerre planes and were characterized by Knarr [17] as weakly Miquelian Laguerre planes, that is, those Laguerre planes in which a certain variation M2 of Miquel's configuration, which characterizes the Miquelian Laguerre planes, is satisfied. More precisely, a finite *elation Laguerre plane* is a finite Laguerre plane that admits a group of automorphisms that acts trivially on the set of generators and regularly on the set of circles. The group in question is the elation group defined above. Clearly one has

$$\text{Miquelian} \implies \text{ovoidal} \implies \text{elation}.$$

The latter implication follows from the spatial model of ovoidal Laguerre planes by considering central collineations of $\mathrm{PG}(3, q)$ with centre the vertex of the cone and 'axis' a plane through the vertex of the cone.

Finite ovoidal Laguerre planes are characterized by the kernel having maximum order, see [31, Proposition 1.9]. Similarly, finite elation Laguerre planes are characterized by the kernel being transitive on the circle set $\mathcal{C}$ of $\mathcal{L}$, see [31, Theorem 2].

Theorem 2.2 *A finite Laguerre plane $\mathcal{L}$ of order n is ovoidal if and only if the kernel of $\mathcal{L}$ has order $(n-1)n^3$.*

A finite Laguerre plane $\mathcal{L}$ of order n is an elation Laguerre plane if and only if the kernel of $\mathcal{L}$ has order divisible by n^3.

Each derived projective plane of a finite elation Laguerre plane $\mathcal{L}$ is a dual translation plane with translation centre the ideal point ω. In particular, $\mathcal{L}$ has order a prime power. This allows us to use the rich theory of translation planes. Furthermore, the automorphism group of an elation Laguerre plane is linearly represented on the elation group (considered as a vector space over the corresponding prime field). Hence representation theory of finite groups is also available. Finite elation Laguerre planes seem to give the best candidates so far to find finite non-ovoidal Laguerre planes.

A finite elation Laguerre plane of order q is equivalent to a generalized oval (or pseudo-oval) with $q + 1$ points and thus to a translation generalized quadrangle of order q. This connection has been used in [27] to characterize those finite elation Laguerre planes of even order that are ovoidal. In case of odd order one also obtains a translation generalized quadrangle with an anti-regular point via the associated Lie geometry; compare [2], [14] and [37].

3 Laguerre Translations

Following Kleinewillinghöfer [15] or [16] a *Laguerre translation* of a Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points of one generator and induces a translation in the derived affine plane at one of its fixed points. Depending on the direction of this translation in the derived affine plane there are two kinds of Laguerre translations. A *G-translation* where G is a generator of $\mathcal{L}$ is an automorphism in the kernel of $\mathcal{L}$ that is either the identity or fixes precisely the points of G. A group of G-translations of $\mathcal{L}$ is called *G-transitive*, if it acts transitively on each generator $H \neq G$. We say that the automorphism group Γ of $\mathcal{L}$ is G-transitive if Γ contains a G-transitive subgroup of G-translations.

The second kind of Laguerre translations is determined by a circle C passing through $p \in G$. Let $B(p, C)$ denote the *tangent pencil with carrier* p, that is, $B(p, C)$ consists of all circles that touch the circle C at the point p (i.e., circles D such that $D \cap C = \{p\}$ or $D = C$). In the derived affine plane at p the tangent pencil represents a parallel class of lines and one considers translations in the direction of this parallel class of lines. Then a *$B(p, C)$-translation* of $\mathcal{L}$ is an automorphism of $\mathcal{L}$ that fixes $[p]$ pointwise and each circle in $B(p, C)$ globally. (Note that $B(p, C) = B(p, D)$ for any circle D in the pencil $B(p, C)$.) A group of $B(p, C)$-translations of $\mathcal{L}$ is called *$B(p, C)$-transitive*, if it acts transitively on $C \setminus \{p\}$. We say that the automorphism group Γ of $\mathcal{L}$ is $B(p, C)$-transitive if Γ contains a $B(p, C)$-transitive subgroup of $B(p, C)$-translations.

With respect to Laguerre translations Kleinewillinghöfer obtained 11 types of Laguerre planes, labelled A to K; see [15, Satz 3.3] or [16, Satz 2]. If $\mathcal{E} \subseteq \mathcal{G}$ denotes the set of all generators G for which the automorphism group Γ of the Laguerre plane is G-transitive, and $\mathcal{B}$ denotes the set of all tangent pencils $B(p, C)$ with carrier p for which Γ is $B(p, C)$-transitive, then exactly one of the following statements is valid:

A. $\mathcal{E} = \emptyset$; $\mathcal{B} = \emptyset$.

B. $|\mathcal{E}| = 1$; $\mathcal{B} = \emptyset$.

C. $|\mathcal{E}| = 2$; $\mathcal{B} = \emptyset$.

D. $|\mathcal{E}| \geq 3$; $\mathcal{B} = \emptyset$.

E. $\mathcal{E} = \emptyset$; $|\mathcal{B}| = 1$.

F. $\mathcal{E} = \emptyset$; there are a generator G, a subset $U \subseteq G$, $|U| \geq 2$, and an injective map $\phi : U \to \mathcal{C}$ such that $q \in \phi(q)$ and $\mathcal{B} = \{B(q, \phi(q)) \mid q \in U\}$.

G. $\mathcal{E} = \emptyset$; there is a circle C such that $\mathcal{B} = \{B(q, C) \mid q \in C\}$.

H. There is a point p such that $\mathcal{E} = \{[p]\}$ and $\mathcal{B} = \{B(p, C) \mid p \in C \in \mathcal{C}\}$.

I. There is a generator G such that $\mathcal{E} = \{G\}$ and $\mathcal{B} = \{B(q, C) \mid C \in \mathcal{C}, q \in C \cap G\}$.

J. $\mathcal{E} = \mathcal{G}$; there is a generator G such that $\mathcal{B} = \{B(q, C) \mid C \in \mathcal{C}, q \in C \cap G\}$.

K. $\mathcal{E} = \mathcal{G}$; $\mathcal{B} = \{B(q, C) \mid q \in C \in \mathcal{C}\}$.

Remark 3.1 Since the elation group of a finite elation Laguerre plane acts regularly on the circle set and transitively on each generator, one readily sees that in a finite elation Laguerre plane one has $\mathcal{E} = \mathcal{G}$ so that the Laguerre plane is of type D, J or K with respect to translations. The examples in 4.1 show that each of these types occurs as a type of a finite ovoidal Laguerre plane and thus as the type of a finite elation Laguerre plane. We now have

$$\text{Miquelian} \implies \text{ovoidal} \implies \text{elation} \implies \mathcal{E} = \mathcal{G} \implies \text{type D, J, K.}$$

Clearly, each Miquelian Laguerre plane is of type K. Conversely, finite Miquelian Laguerre planes are described by this type, see [10, Sätze 1 and 3].

Proposition 3.2 *A finite Laguerre plane is of Kleinewillinghöfer type K if and only if it is Miquelian.*

In case of finite Laguerre planes of odd order, Hartmann [10, Satz 3] shows that a configuration as in type J with respect to Laguerre translations already implies Miquelian. In [10, Satz 2] he deals with the even-order case.

Proposition 3.3 *A finite Laguerre plane of Kleinewillinghöfer type J has even order.*

A finite non-Miquelian Laguerre plane of even order is ovoidal over a translation oval if and only if $\mathcal{L}$ is of type J with distinguished generator G and there is a point p on G such that the derived affine plane at p is Desarguesian.

Kleinewillinghöfer [15, Lemma 3.4] shows that $\mathcal{E} = \emptyset$, $|\mathcal{E}| = 1$ or $\mathcal{E} = \mathcal{G}$ in a finite Laguerre plane of even order. This follows from Lemma 2.1. Furthermore, [15, Lemma 3.7] or [16, Lemma 6] implies that in a finite Laguerre plane $\mathcal{L}$ of even order the $B(p, C)$-transitivity of $\mathcal{L}$ for all circles C through a given point p results in the $B(q, D)$-transitivity of $\mathcal{L}$ for all points q on $[p]$ and all circles D through q. (Briefly, the derived affine plane $\mathcal{A}_p$ at p is a translation plane, and circles not passing through p induce translation ovals.) Hence, one has the following.

Proposition 3.4 *A finite Laguerre plane of Kleinewillinghöfer type C or H has odd order.*

As in the even-order case we show that $\mathcal{E} = \mathcal{G}$ if $\mathcal{E}$ contains sufficiently many generators. To do this we need another kind of central Laguerre automorphism, Laguerre shears. They are defined in [15, 16] and [10] but have not been investigated systematically further. Let G and H be two distinct generators of a Laguerre plane $\mathcal{L}$. A $\{G, H\}$*-shear* is an automorphism of $\mathcal{L}$ that is either the identity or fixes precisely the points on the two generators G and H. By [19, Lemma 3.1] a Laguerre shear is in the kernel or an involution.

Proof of Main Theorem. As mentioned above $|\mathcal{E}| \geq 2$ implies $\mathcal{E} = \mathcal{G}$ in Laguerre planes of even order by [15, Lemma 3.4]. Therefore we now assume that $\mathcal{L}$ is a Laguerre plane of odd order n. Let G_1, G_2, G_3 be three distinct generators in $\mathcal{E}$. The G_i-translations form a normal subgroup Δ_i in the kernel T of $\mathcal{L}$. Since by assumption Δ_i is G_i-transitive, Δ_i has order n. Furthermore, $\Delta_i \cap \Delta_j = \{\text{id}\}$ for $i \neq j$, so that Δ_i and Δ_j generate a normal subgroup $\Delta_{ij} \simeq \Delta_i \times \Delta_j$ of T of order n^2.

Assume that there are $\gamma_i \in \Delta_i$, $i = 1, 2, 3$, and a circle C of $\mathcal{L}$ such that $\gamma_1\gamma_2\gamma_3$ fixes C. Consider the circles $C_1 = \gamma_1^{-1}(C)$ and $C_3 = \gamma_3(C)$. Then C_i touches C in $p_i = C \cap G_i$ for $i = 1, 3$ and, because $C_1 = \gamma_2(C_3)$, the circle C_1 also touches C_3 in $p_2 = C_3 \cap G_2$. We consider the Lie geometry $\mathcal{Q}$ associated with $\mathcal{L}$. This geometry has points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point ∞; the lines of $\mathcal{Q}$ are the extended generators $G \cup \{\infty\}$, $G \in \mathcal{G}$, and the extended tangent pencils $B(p, C) \cup \{p\}$, $p \in C \in \mathcal{C}$, with incidence being the natural one; compare [22]. The circles C, C_1, C_3 give rise to three points in $\mathcal{Q}$, any two of which are collinear. But $\mathcal{Q}$ is a generalized quadrangle by [22, VII.1], so that no proper triangle exists in $\mathcal{Q}$. Therefore two of the points (i.e., circles in $\mathcal{L}$) must be the same, which then implies $\gamma_1 = \gamma_2 = \gamma_3 = \text{id}$.

This shows that $\Delta_3 \cap \Delta_{12} = \{\text{id}\}$. Hence Δ_1, Δ_2, Δ_3 generate a normal subgroup $\Delta_{123} \simeq \Delta_1 \times \Delta_2 \times \Delta_3$ of T of order n^3. Furthermore, as seen above, its stabilizer of a circle consists of the identity only. Hence Δ_{123} acts regularly on the set of circles of $\mathcal{L}$.

Given two distinct circles C and D that intersect in two points p and q, there is an automorphism $\delta \in \Delta_{123}$ such that $\delta(C) = D$. Then δ fixes p and q. In the derived projective plane $\mathcal{P}_p$ at p, the automorphism δ induces a central collineation $\widehat{\delta}$ with centre ω. Since q is a fixed point of $\widehat{\delta}$, the axis A of $\widehat{\delta}$ passes through q. But δ is in the kernel of $\mathcal{L}$ so that A must come from a generator of $\mathcal{L}$. This shows that δ fixes every point of $[q]$. By symmetry, δ also fixes every point of $[p]$, and δ is a $\{[p], [q]\}$-shear. Moreover, since D can be chosen as an arbitrary circle through p and q, we see that the group of all $\{[p], [q]\}$-shears is transitive on this set of circles and thus transitive one each generator different from $[p]$ and $[q]$.

Finally, let C and C' two circles that touch each other at a point p. Choose a circle D through p that intersects each of C and C' in another point q and q', respectively. From the above we know that there is a $\{[p], [q]\}$-shear that takes C to D and a $([p], [q'])$-shear that takes D to C'. The composition δ of these two Laguerre shears fixes every point of $[p]$ and maps C to C'. In the derived projective plane at p, we see a central collineation with centre ω, and similar to the former case the axis must be the ideal line. This shows that δ is a $[p]$-translation. Since C' was arbitrary, we see that $\mathcal{L}$ is $[p]$-transitive. Hence $\mathcal{E} = \mathcal{G}$.

Theorem 2.2 shows that $\mathcal{L}$ is an elation Laguerre plane. □

An immediate consequence of Proposition 3.3 and the Main Theorem is the following result which characterizes finite elation Laguerre planes of odd order in terms of their Kleinewillinghöfer types.

Corollary 3.5 *A finite non-Miquelian Laguerre plane of odd order is an elation Laguerre plane if and only if it is of Kleinewillinghöfer type D.*

Remark 3.6

1. With the above results Remark 3.1 can be sharpened for finite Laguerre planes of odd order:

$$\text{Miquelian} \iff \text{ovoidal} \implies \text{elation} \iff \mathcal{E} = \mathcal{G} \iff \text{type D, K.}$$

(There is no type J in odd order.)

2. Under certain additional assumptions the implication "ovoidal $\implies$ elation" above can also be reversed. For example, in [34, Main Theorem 3.10], it was shown that if the automorphism group of an elation Laguerre plane $\mathcal{L}$ of odd order is 2-transitive on $\mathcal{G}$, then $\mathcal{L}$ is Miquelian. Similarly, if the automorphism group of an elation Laguerre plane $\mathcal{L}$ of odd order fixes a generator G_0 and acts 2-transitively on $\mathcal{G} \setminus \{G_0\}$, then $\mathcal{L}$ is Miquelian; see [35, Theorem 3.6].

   The first of the results mentioned above remains valid for finite elation Laguerre planes of even order, but in the second case there are non-Miquelian elation Laguerre planes of even order. If the order is 2^h where h is either a prime number or the square of a prime number, then the Laguerre plane is ovoidal over a translation oval, see [35, Corollary 5.7].

3. There are infinite (locally compact, 4-dimensional, topological) elation Laguerre planes that are not ovoidal, see for example [32].

4. Note that in case of even order although types D and J have $\mathcal{E} = \mathcal{G}$ the group generated by all G-translations for $G \in \mathcal{G}$ may not be transitive on $\mathcal{C}$. For example, in the ovoidal Laguerre plane $\mathcal{L}(f)$ of order 2^h over a translation oval described by $f(x) = x^{2^k}$ where k and h are co-prime, the collection

$$\Sigma = \{(x,y) \mapsto \begin{cases} (x, y + af(x) + c), & \text{if } x \in \mathbb{F}, \\ (\infty, y + a), & \text{if } x = \infty, \end{cases} \mid a, c \in \mathbb{F}\}$$

   of automorphisms is a group consisting entirely of Laguerre translations such that for each $G \in \mathcal{G}$ the subgroup of all G-translations in Σ is linearly transitive. However, Σ is clearly not transitive on $\mathcal{C}$.

5. Furthermore, in the even-order case type J is potentially possible as the type of a finite non-ovoidal elation Laguerre plane. By Proposition 3.3 though no derived affine plane at a point on the distinguished generator G as in type J can be Desarguesian.

A slight weakening of the conditions in Proposition 3.3 yields the following.

Proposition 3.7 *A finite Laguerre plane of Kleinewillinghöfer type J with distinguished generator G is an elation Laguerre plane if there is a point p on G such that* *the derived affine plane at p is a semifield plane. Moreover, each circle not passing through p induces a translation oval in $\mathcal{A}_p$.*

Conversely, a finite non-Miquelian elation Laguerre plane has Kleinewillinghöfer type J if it admits a generator G such that the derived affine plane at a point p of G is a translation plane and such that each circle not passing through p induces a translation oval in $\mathcal{A}_p$.

Proof. Let $\mathcal{L}$ be a finite Laguerre plane of Kleinewillinghöfer type J with distinguished generator G and let $p \in G$ be such that $\mathcal{A}_p$ is a semifield plane. Then conditions (1*), (2*), (3*) of [10, Section 3] are satisfied. Lemma 3.1 of that paper then shows that circles of $\mathcal{L}$ can be described as $C_{a,b,c} = \{(x, f_a(x) + b * x + c) \mid x \in K\} \cup \{(\infty, a)\}$ where $a, b, c \in K$ and K is the semifield with multiplication $*$ that coordinatises $\mathcal{A}_p$ and where the f_a are suitable functions. Furthermore, by [10, Lemma 3.2], one has that $f_a(x) + f_{a'}(x) = f_{a+a'}(x)$ for all $a, a', x \in K$. Hence $(x, y) \mapsto (x, y + f_r(x) + s * x + t)$ where $r, s, t \in K$ extends to an automorphism of $\mathcal{L}$. The collection of all these automorphisms is a group that acts regularly on the set of circles. This shows that $\mathcal{L}$ is an elation Laguerre plane.

By Proposition 3.3 a finite Laguerre plane of Kleinewillinghöfer type J has even order. In this case [10, Lemma 3.3] shows that all functions f_a as above are additive. Hence each $C_{a,b,c}$ where $a \neq 0$ induces a translation oval in the semifield plane.

For the converse direction note that $\mathcal{E} = \mathcal{G}$ in any elation Laguerre plane. In particular, every derived affine plane is a dual translation plane. The assumption that $\mathcal{A}_p$ is a translation plane thus implies that $\mathcal{A}_p$ is a semifield plane. Furthermore, because all circles not passing through p induce translation ovals in $\mathcal{A}_p$, one sees that each translation of $\mathcal{A}_p$ is an automorphism of the Laguerre plane. Hence the Laguerre plane is of Kleinewillinghöfer type J. □

Translation ovals in proper translation planes have been found in commutative semifield planes; cf. [4] and [13]. For ovals in Desarguesian planes of even order see [11], [12, Section 8.4] or [5].

The following lemma will be used to exclude Kleinewillinghöfer type F in finite Laguerre planes of odd order.

Lemma 3.8 *If there is a generator G and at least two points $p_1, p_2 \in G$ and circles $C_1, C_2 \in \mathcal{C}$, $p_i \in C_i$, for which the finite Laguerre plane $\mathcal{L}$ of odd order is $B(p_i, C_i)$-transitive, $i = 1, 2$, then $\mathcal{L}$ is $B(p_1, C)$-transitive for all circles C through p_1.*

Proof. Let Σ_i, $i = 1, 2$, be the collection of all $B(p_i, C_i)$-translations. Then Σ_2 is transitive on $C_2 \setminus \{p_2\}$. Since Σ_2 also fixes the point p_1, it induces a group Σ_2' of collineations of the derived projective plane $\mathcal{P}_1 = \mathcal{P}_{p_1}$ at p_1.

The circles in $B(p_1, C_1)$ give rise to affine lines in $\mathcal{P}_1$, all passing through the same point ω_{C_1} on the ideal line W. The $B(p_1, C_1)$-transitivity implies that $\mathcal{P}_1$ is (ω_{C_1}, W)-transitive, that is, the induced group of all translations with axis W and centre ω_{C_1} is linearly transitive.

The circle C_2 induces an oval $\mathcal{O}$ in $\mathcal{P}_1$ such that W is a tangent line to $\mathcal{O}$ at the distinguished ideal point ω. Furthermore, Σ_2' is transitive on $\mathcal{O} \setminus \{\omega\}$. Since in a finite projective plane of odd order there are either no or precisely two tangent lines to an oval through a point not on the oval (see for example [7, 3.2.23]), there is a unique tangent line $\neq W$ to $\mathcal{O}$ through every point of $W \setminus \{\omega\}$. We therefore see that Σ_2' is also transitive on $W \setminus \{\omega\}$. Hence $\mathcal{P}_1$ is (q, W)-transitive for all $q \in W$, $q \neq \omega$. Furthermore, all these collineations are induced by automorphisms of the Laguerre plane. But this implies the $B(p_1, C)$-transitivity of $\mathcal{L}$ for all circles C through p_1. □

Since type F contains a configuration of points and circles as in Lemma 3.8, we obtain the following.

Corollary 3.9 *A finite Laguerre plane of Kleinewillinghöfer type F has even order.*

Lemma 3.10 *Let $\mathcal{L}$ be a finite Laguerre plane of order n and let G be a generator in $\mathcal{L}$ such that the automorphism group of $\mathcal{L}$ is $B(p, C)$-transitive for all $p \in G$ and $C \in \mathcal{C}$, $p \in C$. Then $\mathcal{L}$ is Miquelian in case $\mathcal{L}$ has odd order. In case of even order each derived affine plane at a point of G is a translation plane and circles not passing through the point of derivation are induced by translation ovals.*

Proof. Clearly, each derived affine plane $\mathcal{A}_p$ at a point p of G is a translation plane so that n is a prime power. For each $p \in G$ let Δ_p be the group generated by all $B(p, C)$-translations for all circles $C \ni p$. This group induces the full translation group in $\mathcal{A}_p$. If Q_p is a coordinatising quasifield of $\mathcal{A}_p$, then $\Delta_p \simeq (Q_p)^2 \simeq \mathbb{F}_n^2$.

We fix two distinct points $p, q \in G$. We coordinatise $\mathcal{A}_p$ in such a way that Δ_p consists of the transformations $(x, y) \mapsto (x + u, y + v)$ where $u, v \in \mathbb{F}_n$. Since Δ_p also fixes q, this group induces a group Σ of collineations of the derived plane $\mathcal{A}_q$ at q. Furthermore, because $\Sigma \simeq \mathbb{F}_n^2$, this group is transitive on the points of $\mathcal{A}_q$, fixes the ideal line W and the ideal point ω. By [6, Satz 3] the group Σ is either the translation group of $\mathcal{A}_q$ or a shift group of $\mathcal{A}_q$ (so that Σ is transitive on the set of non-vertical lines of $\mathcal{A}_q$).

The latter case occurs if and only if n is odd, see [8, Lemma 9]. Since the translation plane $\mathcal{A}_p$ admits at least one shift group, it can be coordinatised by a commutative semifield; see [30] or [18, 9.12]. The shift plane $\mathcal{A}_q$ can be coordinatised in such a way that non-vertical lines have the form $\{(x, f(x-a)+b) \mid x \in \mathbb{F}_n\}$ where $a, b \in \mathbb{F}_n$ and where f is a planar function, that is, for each $a \in \mathbb{F}_n \setminus \{0\}$ the map $x \mapsto f(x + a) - f(x)$ is a permutation of $\mathbb{F}_n$; cf. [8]. Since this can be done for each $q \in [p]$, $q \neq p$, Theorem 2.2 in [36] yields that the derived plane $\mathcal{A}_p$ is Desarguesian. Hence, $\mathcal{L}$ is Miquelian.

In the former case n has to be even, and non-identity translations have order 2. Let C be a circle through q and let $r, s \in C \setminus \{q\}$, $r \neq s$. There is a translation τ that takes r to s. Since τ is an involution, it fixes q and $\{r, s\}$ and thus C. It follows that C induces a translation oval in $\mathcal{P}_p$. □

As a consequence of Lemma 3.10, types I and J cannot occur in finite Laguerre planes of odd order. We already know this in case of type J by Proposition 3.3.

Corollary 3.11 *A finite Laguerre plane of Kleinewillinghöfer type I or J has even order.*

Corollary 3.12 *A finite Laguerre plane of odd order as in Lemma 3.8 is Miquelian.*

Proof. Note that by symmetry, a finite Laguerre plane $\mathcal{L}$ as in Lemma 3.8 is also $B(p_2, C)$-transitive for all circles C through p_2. But then [16, Lemma 6] shows that $\mathcal{L}$ is $B(p, C)$-transitive for all circles C through $p \in [p_1]$. Hence, $\mathcal{L}$ is of type I, J or K. With Corollary 3.9 we obtain that $\mathcal{L}$ has type K and thus is Miquelian. □

In summary, we now obtain the following feasible Kleinewillinghöfer types for finite Laguerre planes with respect to Laguerre translations.

Theorem 3.13 *A finite Laguerre plane of odd order is of Kleinewillinghöfer type A, B, C, D, E, G, H or K.*

A finite Laguerre plane of even order is of Kleinewillinghöfer type A, B, D, E, F, G, I, J or K.

There are examples of finite Laguerre planes of Kleinewillinghöfer types D, J and K, see Example 4.1.

4 Combined Types and Examples

Kleinewillinghöfer further considered Laguerre homotheties and obtained 13 different types, labelled 1 to 13; see [15, Satz 3.2] or [16, Satz 1]. A (Laguerre) homothety of a Laguerre plane $\mathcal{L}$ is determined by two non-parallel points p and q of $\mathcal{L}$. An automorphism of $\mathcal{L}$ is a $\{p, q\}$-*homothety* if it the identity or fixes precisely p and q and induces a homothety with centre q in the derived affine plane $\mathcal{A}_p$ at p. The automorphism group Γ of $\mathcal{L}$ is said to be $\{p, q\}$-transitive if Γ contains a subgroup of $\{p, q\}$-homotheties that is transitive on each circle through p and q minus the two points p and q.

In [33] finite Laguerre planes and their Kleinewillinghöfer types with respect to homotheties were investigated. The known finite Laguerre planes are of one of the following types where $\mathcal{H}$ denotes the set of all unordered pairs of non-parallel points $\{p, q\}$ for which the automorphism group of the Laguerre plane $\mathcal{L}$ is $\{p, q\}$-transitive.

1. $\mathcal{H} = \emptyset$.
8. There are two distinct generators F, G such that $\mathcal{H} = \{\{p, q\} \mid p \in F, q \in G\}$.
12. There is a generator G such that $\mathcal{H} = \{\{p, q\} \mid p \in G, q \in P \setminus G\}$.
13. $\mathcal{H}$ consists of all pairs of non-parallel points.

The full list of possible Kleinewillinghöfer types with respect to homotheties can be found in [15, Satz 3.2], [16, Satz 1] or [33, §3].

It was shown in [33, Proposition 3.9] that a finite elation Laguerre plane is of type 1, 8, 12 or 13 with respect to homotheties and that a finite non-ovoidal elation Laguerre plane must be of type 1 or 8. Furthermore, by [33, Theorem 3.8], a finite Laguerre plane of type 8 is an elation Laguerre plane.

When combining both classifications with respect to Laguerre translations and Laguerre homotheties of course not every combination X.m with X $\in \{A, \ldots, K\}$ and $m \in \{1, \ldots, 13\}$ can occur as the Kleinewillinghöfer type of a Laguerre plane. In fact, there are at most a total of potentially 25 combined types A.1, ..., K.13, see [15, Satz 3.4] or [24, §6] where a third kind of central automorphisms, Laguerre homologies, are also used. Explicitly the feasible combined types with respect to Laguerre translations and Laguerre homotheties are

A.	1, 2, 5, 7, 9,	B.	1, 3, 10,	C.	1, 8,	D.	1, 8,
E.	1, 4,	F.	1,	G.	1, 6,	H.	1, 11,
I.	1, 11,	J.	1, 12,	K.	1, 13.		

Example 4.1 All known finite Laguerre planes are ovoidal and thus are represented as $\mathcal{L}(f)$ over a field $\mathbb{F}$ for some parabolic function $f : \mathbb{F} \to \mathbb{F}$. In a Laguerre plane $\mathcal{L}(f)$ the transformations

$$(x, y) \mapsto \begin{cases} (x, y + t(af(x) + bx + c)), & \text{if } x \in \mathbb{F} \\ (\infty, y + ta), & \text{if } x = \infty, \end{cases}$$

where $t \in \mathbb{F}$, form a transitive group of $[(x_0, 0)]$-translations whenever the circle

$$\{(x, af(x) + bx + c) \mid x \in \mathbb{F}\} \cup \{(\infty, a)\}$$

touches the circle C given by $y = 0$ in the point $(x_0, 0)$.

The known finite Laguerre planes have combined Kleinewillinghöfer types as follows. We just state the respective linearly transitive groups of translations (other than G-translations) and homotheties without explicitly verifying that no further linearly transitive groups of these central automorphisms exist. Of course, there may be more models of finite ovoidal Laguerre planes of a given type, we just specify some that are easiest to write down.

D.1 An ovoidal Laguerre plane over an oval that is neither a translation oval nor admits a group that fixes two points and is transitive on the remaining points of the oval. For example, if the finite Laguerre plane is represented as $\mathcal{L}(f)$ over the field $\mathbb{F}_{2^h}$ of order 2^h, where h odd and f is given by $f(x) = x^{1/6} + x^{3/6} + x^{5/6}$, exponents are modulo $2^h - 1$, see [21], then this plane does not admit a linearly transitive group of homotheties or $B(p, C)$-translations.

D.8 An ovoidal Laguerre plane over an oval $\mathcal{O}$ that is not a translation oval but admits a group that fixes two points of $\mathcal{O}$ and is transitive on the remaining points of $\mathcal{O}$. For example, if the finite Laguerre plane is represented as $\mathcal{L}(f)$ over the field $\mathbb{F}_{2^h}$, where h odd and f is given by $f(x) = x^6$, compare [29], then the transformations

$$(x,y) \mapsto \begin{cases} (sx, sy + (1+s)c + a(s+f(s))f(x)), & \text{if } x \in \mathbb{F}_{2^h} \\ \left(\infty, \frac{s}{f(s)}y + \left(1 + \frac{s}{f(s)}\right)a\right), & \text{if } x = \infty, \end{cases}$$

where $s \in \mathbb{F}_{2^h}$, $s \neq 0$, form a transitive group of $((\infty, a), (0, c))$-homotheties. There is no linearly transitive group of $B(p, C)$-translations.

J.12 A non-Miquelian ovoidal Laguerre plane over a translation oval. In this case, the oval is not a conic, the finite Laguerre plane has even order 2^h, and the plane can be represented in the form $\mathcal{L}(f)$ over the field $\mathbb{F}_{2^h}$ where $f(x) = x^{2^k}$ and k is co-prime to h. The distinguished generator is $\{\infty\} \times \mathbb{F}_{2^h}$. For each $a, b, c \in \mathbb{F}_{2^h}$ the transformations

$$(x,y) \mapsto \begin{cases} (sx + (1+s)b, sy + (1+s)c + a((s+f(s))f(x)+f(sb)), & \text{if } x \in \mathbb{F}_{2^h} \\ \left(\infty, \frac{s}{f(s)}y + \left(1 + \frac{s}{f(s)}\right)a\right), & \text{if } x = \infty, \end{cases}$$

where $s \in \mathbb{F}_{2^h}$, $s \neq 0$, form a transitive group of $((\infty, a), (b, c))$-homotheties. Furthermore, for each $a, b, c \in \mathbb{F}_{2^h}$ the transformations

$$(x,y) \mapsto \begin{cases} (x+t, y + af(t) + bt), & \text{if } x \in \mathbb{F}_{2^h} \\ (\infty, y), & \text{if } x = \infty, \end{cases}$$

where $t \in \mathbb{F}_{2^h}$, form a transitive group of $B(\infty, a), C)$-translations where C is the circle given by $y = af(x) + bx + c$.

K.13 A Miquelian Laguerre plane. Here all admissible subgroups of Laguerre translations and Laguerre homotheties are linearly transitive.

One sees that the above examples comprise all possible types of finite ovoidal Laguerre planes.

Theorem 4.2 *A finite Laguerre plane cannot have type C.8, H.11, I.11 or K.1.*

Proof. By [33, Theorem 3.8], a finite Laguerre plane of type 8 is an elation Laguerre plane. Hence $\mathcal{E} = \mathcal{G}$, and type C is impossible in combination with type 8.

In type 11 there is a point p such that $\mathcal{H} = \{\{p, q\} \mid q \in P \setminus [p]\}$. It has been noted in [33] after Theorem 3.4 that a finite Laguerre plane cannot have type 11 with respect to homotheties. Hence combined types H.11 and I.11 are excluded too. Alternately, by [33, Theorem 3.4], a finite Laguerre plane of type 11 is Miquelian or ovoidal over a translation oval. But ovoidal Laguerre planes have $\mathcal{E} = \mathcal{G}$ so that types H and I are impossible in combination with type 11.

By Proposition 3.2 a finite Laguerre plane of type K is Miquelian and thus of type 13 with respect to homotheties. □

Remark 4.3 1. As a direct consequence of [33, Proposition 3.5] one obtains that a finite Laguerre plane of type A.5 or A.9 must have odd order.

With Theorem 3.13 we further know that a finite Laguerre plane of type C.1 or H.1 must have odd order. Moreover, a finite Laguerre plane of type F.1, I.1, J.1 or J.12 must have even order.

2. In a finite elation Laguerre plane $\mathcal{L}$ of type D.8 each derived projective plane at a point of the two distinguished generators F and G as in type 8 is a dual nearfield plane by the dual of [23, 3.5.46]. In suitable coordinates $F = \{\infty\} \times N$, $G = \{0\} \times N$ where N is the coordinatising dual nearfield, and circles through $(\infty, 0)$ are

$$\{(x, x * b + c) \mid x \in N\} \cup \{(\infty, 0)\}$$

where $b, c \in N$ and $*$ is multiplication in N. The $\{(\infty, 0), (0, 0)\}$-homotheties are

$$(x, y) \mapsto \begin{cases} (r * x, r * y), & \text{if } x \in N \\ (\infty, r * y), & \text{if } x = \infty, \end{cases}$$

where $r \in N \setminus \{0\}$. Furthermore, there is a parabolic function $f : N \to N$ with $f(0) = 0$ and $f(x) = 0$ if and only if $x = 0$ such that the circles of $\mathcal{L}$ not passing through $(\infty, 0)$ are of the form

$$\{(x, a * f(a^{-1} * x) + x * b + c) \mid x \in N\} \cup \{(\infty, a)\}$$

where $a, b, c \in N$, $a \neq 0$.

Finite nearfields have been classified by Dickson [9] and Zassenhaus [39]. If N is a regular nearfield of dimension 2 over its centre, the projective plane coordinatised by it contains ovals, see [26]. However these ovals are 'hyperbolic', that is, they have two points on the translation axis of the plane. Dualising these ovals in case of odd order does not result in parabolic ovals as is required for Laguerre planes considered here.

3. Each derived projective plane of a finite elation Laguerre plane of type J.12 at a point of the distinguished generator G as in type J or 12 is a dual nearfield plane and a semifield plane. Thus, derived planes at these points are Desarguesian. Proposition 3.7 and Theorem 3.3 then show that the Laguerre plane is ovoidal. Hence, a non-ovoidal elation Laguerre plane must be of type D.1, D.8 or J.1.

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