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Large non-trivial t-intersecting families of signed sets

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Abstract

For positive integers n, r, k with $n \geqslant r$ and $k \geqslant 2$, a set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_r, y_r)\}$ is called a k-signed r-set on $[n]$ if $x_1, \ldots, x_r$ are distinct elements of $[n]$ and $y_1, \ldots, y_r \in [k]$. We say that a t-intersecting family consisting of k-signed r-sets on $[n]$ is trivial if each member of this family contains a fixed k-signed t-set. In this paper, we determine the structure of large maximal non-trivial t-intersecting families of k-signed r-sets. In particular, we characterize the non-trivial t-intersecting families with maximum size for $t \geqslant 2$, extending a Hilton-Milner-type result for signed sets given by Borg.

1 Introduction

Let n, r and t be positive integers with $n \geqslant r \geqslant t$. For an n-set X, let 2^X and $\binom{X}{r}$ denote the family of subsets and the set of r-subsets of X, respectively. A family $\mathcal{F} \subset 2^X$ is called *t-intersecting* if $|F \cap F'| \geqslant t$ for every $F, F' \in \mathcal{F}$. Moreover, we say $\mathcal{F}$ is *trivial* if the members of $\mathcal{F}$ contain a fixed t-subset of X.

The famous Erdős-Ko-Rado Theorem [13, 15, 24] states that the largest t-intersecting subfamilies of $\binom{X}{r}$ are trivial if $n > (t+1)(r-t+1)$. In [15], Frankl

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conjectured the structure of the maximum-sized t-intersecting subfamilies of $\binom{X}{r}$ for all n, r and t. Frankl's conjecture was partially settled by Frankl and Füredi [18], and was completely confirmed by Ahlswede and Khachatrian [2].

The maximum-sized non-trivial t-intersecting subfamilies of $\binom{X}{r}$ have been characterized. Hilton and Milner [21] gave the first result for the structure of such families when $t = 1$, which was also proved by Frankl and Füredi [17] via the shifting technique. In [16], Frankl proved the corresponding result for all t and sufficiently large n. The complete result was given by Ahlswede and Khachatrian [1]. Extending this further, Han and Kohayakawa [20] described the structure of the second largest maximal non-trivial 1-intersecting familes with $n > 2r \geqslant 6$. Kostochka and Mubayi [22] determined the structure of 1-intersecting families with sizes quite a bit smaller than $\binom{n-1}{r-1}$ for large n. Recently, Cao et al. [11] gave the structure of large maximal non-trivial t-intersecting families for all t and large n.

The t-intersection problem has been studied for some other mathematical objects, for example, signed sets. Write $[n] = \{1, 2, \ldots, n\}$. For $k \geqslant 2$, each element of

$$\mathcal{L}_{n,r,k} := \left\{ \{(x_1, y_1), \ldots, (x_r, y_r)\} : \{x_1, \ldots, x_r\} \in \binom{[n]}{r}, y_1, \ldots, y_r \in [k] \right\}$$

is called a k-*signed* r-*set on* $[n]$. When $r = n$ and $k = 2$, the family $\mathcal{L}_{n,n,2}$ is considered as $2^{[n]}$. Notice that the family $\binom{[n]}{r}$ can be viewed as the set of all "1-signed r-sets" on $[n]$. Signed sets generalize the classical sets and so the t-intersection problem for this setting has attracted much attention.

A t-intersecting subfamily of $\mathcal{L}_{n,r,k}$ is said to be *trivial* if all its members contain a fixed k-signed t-sets and *non-trivial* otherwise. There are a lot of Erdős-Ko-Rado results for $\mathcal{L}_{n,r,k}$, see [3, 4, 5, 19, 23] for $r = n$ and [5, 6, 7, 8, 12, 14] for $r < n$. In general, the Erdős-Ko-Rado theorem for $\mathcal{L}_{n,r,k}$ can be stated as follows.

Theorem 1.1. *Let n, r, k and t be positive integers with $n \geqslant r \geqslant t$ and $k \geqslant 2$. If n or k is sufficiently large, then each maximum-sized t-intersecting subfamily of $\mathcal{L}_{n,r,k}$ is trivial.*

We remark here that the t-intersection problem of signed sets does not focus solely on $\mathcal{L}_{n,r,k}$, and refer readers to [10] for an Erdős-Ko-Rado result about a family which is more general than $\mathcal{L}_{n,r,k}$.

In this paper, we study the structure of maximal non-trivial t-intersecting subfamilies of $\mathcal{L}_{n,r,k}$. To present our main results, we introduce two constructions of non-trivial t-intersecting subfamilies of $\mathcal{L}_{n,r,k}$. For each $d \in [n]$, write $M_d = \{(1, 1), (2, 1), \ldots, (d, 1)\}$.

Construction 1. *Suppose that n, r, k, ℓ and t are positive integers with $2 \leqslant k$, $t+1 \leqslant r \leqslant n$ and $t + 2 \leqslant \ell \leqslant \min\{r + 1, n\}$. Let $\mathcal{H}_1(n, r, k, \ell, t)$ be the set of all elements F of $\mathcal{L}_{n,r,k}$ such that*

- *$M_t \subset F$ and $|F \cap M_\ell| \geqslant t + 1$, or*

- $M_t \not\subset F$ *and* $|F \cap M_\ell| = \ell - 1$.

Construction 2. *Suppose that* n, r, k, c *and* t *are positive integers with* $2 \leqslant k$, $t+2 \leqslant r \leqslant n$ *and* $r+2 \leqslant c \leqslant \min\{2r-t, n\}$. *Let* $\mathcal{H}_2(n, r, k, c, t)$ *be the set of all elements* F *of* $\mathcal{L}_{n,r,k}$ *such that*

- $M_t \subset F$ *and* $|F \cap M_r| \geqslant t+1$, *or*
- $F \cap M_r = M_t$ *and* $M_c \setminus M_r \subset F$, *or*
- $M_t \not\subset F$, $|F \cap M_r| = r-1$ *and* $|F \cap (M_c \setminus M_r)| = 1$.

Indeed, the sizes of these families are difficult to compute and the formulas are quite messy, but in most cases we do not need exact values. For each $d \in [n]$, write

$$f(n, r, k, d, t) = (d-t)\binom{n-t-1}{r-t-1}k^{r-t-1} - \binom{d-t}{2}\binom{n-t-2}{r-t-2}k^{r-t-2}, \quad (1)$$

$$g(n, r, t) = \frac{(r-t+3)(r-t-1)}{n-t-1} \cdot \max\left\{\binom{t+2}{2}, \frac{r-t+1}{2}\right\}. \quad (2)$$

In the proofs of our main results, we will use $f(n, r, k, d, t)$ to give lower bounds of families defined above, and show some inequalities for sizes of non-trivial t-intersecting families based on the assumption that $k \geqslant g(n, r, t)$.

In the rest of this paper, for two subfamilies $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{L}_{n,r,k}$, if there exists a bijection σ from $[n] \times [k]$ to itself such that $\mathcal{G} = \{\sigma(F) : F \in \mathcal{F}\}$, then we say $\mathcal{F}$ is *isomorphic* to $\mathcal{G}$, and denote this by $\mathcal{F} \cong \mathcal{G}$. One of our main results is stated as follows, describing the structure of maximal non-trivial t-intersecting subfamilies of $\mathcal{L}_{n,r,k}$ with sizes no less than $f(n, r, k, r, t)$.

Theorem 1.2. *Let* n, r, k *and* t *be positive integers with* $n \geqslant t+2$, $n \geqslant r \geqslant t+1$ *and* $k \geqslant \max\{2, g(n, r, t)\}$. *Suppose that* $\mathcal{F}$ *is a maximal non-trivial* t*-intersecting subfamily of* $\mathcal{L}_{n,r,k}$. *Then* $|\mathcal{F}| \geqslant f(n, r, k, r, t)$ *if and only if one of the following holds.*

(i) $r \geqslant t+2$ *and* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, m, t)$ *for some* $m \in \{r, \min\{r+1, n\}\}$.

(ii) $n \geqslant r+2 \geqslant t+4$ *and* $\mathcal{F} \cong \mathcal{H}_2(n, r, k, c, t)$ *for some* $c \in \{r+2, \ldots, \min\{2r-t, n\}\}$.

(iii) $r \leqslant 2t+2$, $r \neq t+2$ *and* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, t+2, t)$.

The size of a largest non-trivial t-intersecting subfamily of $\mathcal{L}_{n,r,k}$ was determined in [5]. In [9], Borg determined the structure of the largest non-trivial 1-intersecting subfamilies of $\mathcal{L}_{n,r,k}$.

Theorem 1.3. ([9]) *Let* n, r, k *and* t *be positive integers with* $n \geqslant 3$, $n \geqslant r \geqslant 2$, $k \geqslant 2$ *and* $(r, k) \neq (n, 2)$. *If* $\mathcal{F}$ *is a maximum-sized non-trivial intersecting subfamily of* $\mathcal{L}_{n,r,k}$, *then one of the following holds.*

(i) $\mathcal{F} \cong \mathcal{H}_1(n, r, k, \min\{r+1, n\}, 1)$.

(ii) $\mathcal{F} \cong \mathcal{H}_1(n, r, k, 3, 1)$ *when* $r = 3$ *or* $r = n = 4$.

By comparing the sizes of the families given in Theorem 1.2, we can describe the structure of maximum-sized nontrivial t-intersecting subfamilies of $\mathcal{L}_{n,r,k}$ when k is sufficiently large. Notice that Theorem 1.3 is the result for the case $t = 1$. Our second main result focuses on the case $t \geqslant 2$.

Theorem 1.4. *Let n, r, k and t be positive integers with $n \geqslant t+2 \geqslant 4$, $n \geqslant r \geqslant t+1$ and $k \geqslant \max\{2, g(n, r, t)\}$. Suppose that $\mathcal{F}$ is a largest non-trivial t-intersecting subfamily of $\mathcal{L}_{n,r,k}$.*

(i) *If* $\min\{r+1, n\} \leqslant 2t+2$, *then* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, t+2, t)$.

(ii) *If* $\min\{r+1, n\} > 2t+2$, *then* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, \min\{r+1, n\}, t)$.

The rest of this paper is organized as follows. In Section 2, we will prove some properties for t-intersecting families with t-covering number $t+1$ in preparation for the proof of our main results. In Sections 3 and 4, we will prove Theorems 1.2 and 1.4, respectively.

2 t-intersecting families with t-covering number $t+1$

For a t-intersecting subfamily $\mathcal{F}$ of $\mathcal{L}_{n,r,k}$, a k-signed set T on $[n]$ is said to be a *t-cover* of $\mathcal{F}$ if $|T \cap F| \geqslant t$ for each $F \in \mathcal{F}$, and the minimum size $\tau_t(\mathcal{F})$ of a t-cover of $\mathcal{F}$ is called the *t-covering number* of $\mathcal{F}$. Observe that $t \leqslant \tau_t(\mathcal{F}) \leqslant r$, and $\mathcal{F}$ is trivial if and only if $\tau_t(\mathcal{F}) = t$. In this section, we determine some properties of t-intersecting subfamilies of $\mathcal{L}_{n,r,k}$ with t-covering number $t+1$.

For convenience, we write $\mathcal{F}_X := \{F \in \mathcal{F} : X \subset F\}$ where $\mathcal{F}$ is a subset of $\mathcal{L}_{n,r,k}$ and X a k-signed set on $[n]$. We make the following assumption when proving our lemmas in this section and will handle the remaining case, i.e. $\tau_t(\mathcal{F}) \geqslant t+2$, in the proof of Theorem 1.2.

Assumption 2.1. Let n, r, k and t be positive integers with $n \geqslant r \geqslant t+1$ and $k \geqslant 2$. Suppose $\mathcal{F} \subset \mathcal{L}_{n,r,k}$ is a maximal t-intersecting family with $\tau_t(\mathcal{F}) = t+1$. Let $\mathcal{T}$ denote the set of all t-covers of $\mathcal{F}$ with size $t+1$. Set $M = \bigcup_{T \in \mathcal{T}} T$ and $\ell = |M|$.

We first claim that $\mathcal{T}$ is a t-intersecting family with $t \leqslant \tau_t(\mathcal{T}) \leqslant t+1$. In fact, for $T \in \mathcal{T}$ and $F \in \mathcal{L}_{n,r,k}$ containing T, we have $F \in \mathcal{F}$ by the maximality of $\mathcal{F}$. Then for each $T' \in \mathcal{T}$, there exists $F' \in \mathcal{F}$ such that $T' \subset F'$ and $T' \cap T = F' \cap F$, which implies that $|T' \cap T| \geqslant t$, as desired. To describe the structure of some t-intersecting families, we need the following lemma, which shows a relationship between elements of $\mathcal{F}$ and the set M defined in Assumption 2.1.

Lemma 2.2. *Let $n, r, k, t, \ell, \mathcal{F}, \mathcal{T}$ and M be as in Assumption 2.1.*

(i) *If* $\tau_t(\mathcal{T}) = t+1$*, then* $M \in \mathcal{L}_{n,t+2,k}$ *and* $|F \cap M| \geqslant t+1$ *for each* $F \in \mathcal{F}$*.*

(ii) *If* $\tau_t(\mathcal{T}) = t$*, then* $M \in \mathcal{L}_{n,\ell,k}$ *with* $t+1 \leqslant \ell \leqslant \min\{r+1, n\}$*, and for any* t*-cover* S *of* $\mathcal{T}$ *with size* t*,* $|F \cap M| = \ell - 1$ *for each* $F \in \mathcal{F} \setminus \mathcal{F}_S$*.*

Proof. (i) Let T_1 and T_2 be distinct members of $\mathcal{T}$. We claim that $T_1 \Delta T_2 \in \mathcal{L}_{n,2,k}$. Indeed, since $|T_1 \cap T_2| = t$ and $\mathcal{F}$ is non-trivially t-intersecting, we have $|T_1 \Delta T_2| = 2$ and there exists a member of $\mathcal{F} \setminus \mathcal{F}_{T_1 \cap T_2}$ containing $T_1 \Delta T_2$, so $T_1 \Delta T_2 \in \mathcal{L}_{n,2,k}$.

Since $\tau_t(\mathcal{T}) = t+1$, there exists $T_3 \in \mathcal{T}$ such that $T_1 \cap T_2 \not\subset T_3$. From $|T_1 \cap T_3| \geqslant t$ and $|T_2 \cap T_3| \geqslant t$, we get $T_1 \Delta T_2 \subset T_3$ and $|T_3 \cap (T_1 \cap T_2)| = t-1$, which imply that $T_3 \subset T_1 \cup T_2$. For each $T_4 \in \mathcal{T} \setminus \{T_1\}$ containing $T_1 \cap T_2$, we have $T_1 \cap T_3 \not\subset T_4$. Similarly, we have $T_4 \subset T_1 \cup T_3 \subset T_1 \cup T_2$. Hence $M \subset T_1 \cup T_2 \subset M$. Together with $T_1 \Delta T_2 \in \mathcal{L}_{n,2,k}$, we get $M = T_1 \cup T_2 \in \mathcal{L}_{n,t+2,k}$. For each $F \in \mathcal{F}$, we have $|F \cap M| \geqslant t$. If $|F \cap M| = t$, then $F \cap M$ is contained in each member of $\mathcal{T}$, but this contradicts $\tau_t(\mathcal{T}) = t+1$. Therefore, $|F \cap M| \geqslant t+1$, as desired.

(ii) By the claim in (i), it is routine to check that $M \in \mathcal{L}_{n,\ell,k}$. Let S be a t-cover of $\mathcal{T}$. For each $F \in \mathcal{F} \setminus \mathcal{F}_S$ and $T \in \mathcal{T}$, we have $|F \cap T| = t$, from which we get $r+1 \leqslant |S \cup F| \leqslant |T \cup F| = r+1$. Then $S \cup F = T \cup F$, which implies that $|M \cup F| = |S \cup F| = r+1$. Hence $|F \cap M| = \ell - 1$ and $\ell \leqslant r+1$. Together with $M \in \mathcal{L}_{n,\ell,k}$ and $\mathcal{T} \neq \emptyset$, we obtain $t+1 \leqslant \ell \leqslant \min\{r+1, n\}$, as required. □

For a k-signed set $Q = \{(s_1, t_1), \ldots, (s_q, t_q)\}$ on $[n]$ with $s_1 \leqslant \ldots \leqslant s_q$, consider the permutation $\pi_0 = (q\ s_q)(q-1\ s_{q-1}) \cdots (1\ s_1)$, and for each $x \in [n]$, let π_x be a permutation on $[k]$ with $\pi_x = (1\ t_i)$ if $x = s_i$ for some $i \in [q]$, and $\pi_x = (1)$ otherwise. We get a bijection π from $[n] \times [k]$ to itself with $\pi(x,y) = (\pi_0(x), \pi_x(y))$ for each $(x,y) \in [n] \times [k]$. Observe that $\pi(Q) = M_q$, and $\pi(\mathcal{L}_{n,s,k}) = \mathcal{L}_{n,s,k}$ for each $s \in [n]$. It is routine to check that there exists a bijection σ from $[n] \times [k]$ to itself such that $\sigma(\mathcal{F})$ is a t-intersecting subfamily of $\mathcal{L}_{n,r,k}$ with t-covering number $t+1$, $M_\ell = \bigcup_{T \in \mathcal{T}'} T$, and M_t is a t-cover of $\mathcal{T}'$ if $\tau_t(\mathcal{T}) = t$, where $\mathcal{T}'$ is the set of all t-covers of $\sigma(\mathcal{F})$ with size $t+1$. Let $\mathcal{G}$ denote the family $\sigma(\mathcal{F})$. In the following two lemmas, based on Lemma 2.2, we characterize some special t-intersecting families.

Lemma 2.3. *Let* $n, r, k, t, \ell, \mathcal{F}, \mathcal{T}$ *and* M *be as in Assumption 2.1. Suppose that* $|F \cap M| \geqslant t+1$ *for each* $F \in \mathcal{F}$*.*

(i) *If* $\tau_t(\mathcal{T}) = t+1$*, then* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, t+2, t)$*.*

(ii) *If* $\tau_t(\mathcal{T}) = t$*, then* $\mathcal{F} \cong \mathcal{H}_1(n, r, k, \ell, t)$ *and* $\ell \in \{t+3, \ldots, \min\{r+1, n\}\}$*.*

Proof. (i) If $\tau_t(\mathcal{T}) = t+1$, then $M \in \mathcal{L}_{n,t+2,k}$ by Lemma 2.2 (i). By the assumption that $\mathcal{F} \cong \mathcal{G}$ and $|F \cap M| \geqslant t+1$ for each $F \in \mathcal{F}$, we have $|G \cap M_{t+2}| \geqslant t+1$ for each $G \in \mathcal{G}$. Then $\mathcal{G} \subset \mathcal{H}_1(n, r, k, t+2, t)$. Since $\mathcal{H}_1(n, r, k, t+2, t)$ is t-intersecting and $\mathcal{G}$ is maximal, we have $\mathcal{F} \cong \mathcal{G} = \mathcal{H}_1(n, r, k, t+2, t)$.

(ii) Since $\mathcal{F}$ is non-trivially t-intersecting, by Lemma 2.2 (ii), we have $t+2 \leqslant \ell \leqslant \min\{r+1, n\}$. Notice that each $(t+1)$-subset of M_ℓ containing M_t is a t-cover of $\mathcal{G}$. Then $\{G \in \mathcal{L}_{n,r,k} : M_t \subsetneq G \cap M_\ell\} \subset \mathcal{G}$. By Lemma 2.2 (ii), we have

$|G \cap M_\ell| = \ell - 1$ for each $G \in \mathcal{G} \setminus \mathcal{G}_{M_t}$. Hence $\mathcal{G} \subset \mathcal{H}_1(n, r, k, \ell, t)$. Since $\mathcal{G}$ is maximal and $\mathcal{H}_1(n, r, k, \ell, t)$ is t-intersecting, we have $\mathcal{F} \cong \mathcal{G} = \mathcal{H}_1(n, r, k, \ell, t)$. Notice that $\tau_t(\mathcal{T}) = t + 1$ if $\ell = t + 2$. Then $\ell \geqslant t + 3$, as desired. □

Lemma 2.4. *Let $n, r, k, t, \ell, \mathcal{F}, \mathcal{T}$ and M be as in Assumption 2.1. Suppose that there exists $F_0 \in \mathcal{F}$ such that $|F_0 \cap M| = t$. Then $t \leqslant r - 2$ and $\ell < \min\{r + 1, n\}$. Moreover, if $\ell = \min\{r + 1, n\} - 1$, then $r \leqslant n - 2$ and $\mathcal{F} \cong \mathcal{H}_2(n, r, k, c, t)$ for some $c \in \{r + 2, \ldots, \min\{2r - t, n\}\}$.*

Proof. By Lemma 2.2 (i), we have $\tau_t(\mathcal{T}) = t$. If $r = t+1$, then $\mathcal{T} = \mathcal{F}$, which implies that $\tau_t(\mathcal{T}) = t+1$, a contradiction. Hence $r \geqslant t+2$. Observe that $F_0 \cap M$ is a t-cover of $\mathcal{T}$. Let $F \in \mathcal{F} \setminus \mathcal{F}_{F_0 \cap M}$. If $\ell = \min\{r + 1, n\}$, then by Lemma 2.2 (ii), we have $|F \cap F_0| = |F \cap (F_0 \cap M)| < t$, which is impossible. Therefore, $\ell < \min\{r + 1, n\}$.

Now suppose that $\ell = \min\{r + 1, n\} - 1$. Since $\mathcal{F} \cong \mathcal{G}$, there exists $G_0 \in \mathcal{G}$ such that $G_0 \cap M_\ell = M_t$. Let $G \in \mathcal{G} \setminus \mathcal{G}_{M_t}$. If $r \geqslant n - 1$, then $\ell = n - 1$. By Lemma 2.2 (ii), we have $|G_0 \cap G \cap ([n - 1] \times [k])| = t - 1$, which implies that $(n, x_0) \in G_0 \cap G$ for some $x_0 \in [k]$. Then $M_t \cup \{(n, x_0)\}$ is a t-cover of $\mathcal{G}$, which is impossible since $\ell < n$ and each member of $\mathcal{T}'$ is contained in M_ℓ. Hence $r \leqslant n - 2$ and $\ell = r$.

By $|G_0 \cap G| \geqslant t$ and Lemma 2.2 (ii), we obtain $G \setminus ([r] \times [k]) \in \binom{G_0}{1}$. Let

$$E = \{(i, j) : i \geqslant r + 1,\ (i, j) \in G \text{ for some } G \in \mathcal{G} \setminus \mathcal{G}_{M_t}\}.$$

Observe that E is a non-empty subset of G_0 and $E \cap M_r = \emptyset$. We have $1 \leqslant |E| \leqslant \min\{r - t, n - r\}$. If $E = \{(e_1, e_2)\}$ for some $e_1 \geqslant r + 1$ and $e_2 \in [k]$, then (e_1, e_2) is contained in each member of $\mathcal{G} \setminus \mathcal{G}_{M_t}$, which implies that $M_t \cup \{(e_1, e_2)\} \in \mathcal{T}'$, a contradiction. Therefore $|E| \geqslant 2$. Since M_t is a t-cover of $\mathcal{T}'$, then each $(t + 1)$-subset of M_r containing M_t is a member of $\mathcal{T}'$, which implies that $\{H \in \mathcal{L}_{n,r,k} : M_t \subsetneq H \cap M_r\} \subset \mathcal{G}$. For each $G_0' \in \mathcal{G}_{M_t}$ with $|G_0' \cap M_r| = t$, observe that $G \setminus ([r] \times [k]) \subset G_0'$. Then we have $E \subset G_0'$. For each $G' \in \mathcal{G} \setminus \mathcal{G}_{M_t}$, we have $|G' \cap M_r| = r - 1$ and $G' \cap E \neq \emptyset$. Together with $2 \leqslant |E| \leqslant \min\{r - t, n - r\}$, it is routine to check that $\mathcal{G}$ is isomorphic to a subset of $\mathcal{H}_2(n, r, k, c, t)$ where $r + 2 \leqslant c \leqslant \min\{2r - t, n\}$. Since that $\mathcal{G}$ is maximal and $\mathcal{H}_2(n, r, k, c, t)$ is t-intersecting, we have $\mathcal{F} \cong \mathcal{G} \cong \mathcal{H}_2(n, r, k, c, t)$, as desired. □

Now we prove upper bounds for sizes of families under Assumption 2.1 with $\tau_t(\mathcal{T}) = t$. We begin with a frequently used lemma.

Lemma 2.5. *Let n, r, k, t and u be positive integers with $n \geqslant r \geqslant u + 1$. Suppose $\mathcal{F} \subset \mathcal{L}_{n,r,k}$ is a t-intersecting family and $U \in \mathcal{L}_{n,u,k}$. If $|U \cap F| = s < t$ for some $F \in \mathcal{F}$, then there exists $R \in \mathcal{L}_{n,u+t-s,k}$ such that $U \subseteq R$ and $|\mathcal{F}_U| \leqslant \binom{r-s}{t-s} |\mathcal{F}_R|$.*

Proof. W.l.o.g., assume that $\mathcal{F}_U \neq \emptyset$. Let $\mathcal{R}$ denote the set of $R \in \mathcal{L}_{n,u+t-s,k}$ such that $U \subset R \subset F \cup U$. For $G \in \mathcal{F}_U$, from $|G \cap F| \geqslant t$ and $|F \cap U| = s < t$, we obtain $|G \cap (F \cup U)| \geqslant u + t - s$, which implies that $\mathcal{R} \neq \emptyset$ and $\mathcal{F}_U = \bigcup_{R \in \mathcal{R}} \mathcal{F}_R$.

Since $|F \cup U| = u + r - s$, we have $|\mathcal{R}| \leqslant \binom{r-s}{t-s}$. Then the desired result holds by $|\mathcal{F}_U| \leqslant \sum_{R\in\mathcal{R}} |\mathcal{F}_R|$. □

Lemma 2.6. *Let $n, r, k, t, \ell, \mathcal{F}, \mathcal{T}$ and M be as in Assumption 2.1 with $|\mathcal{T}| = 1$. Then*

$$|\mathcal{F}| \leqslant \binom{n-t-1}{r-t-1} k^{r-t-1} + (t+1)(r-t)^2 \binom{n-t-2}{r-t-2} k^{r-t-2}.$$

Proof. Suppose that T_0 is the unique element of $\mathcal{T}$. We have

$$\mathcal{F} = \mathcal{F}_{T_0} \cup \left(\bigcup_{W \in \binom{T_0}{t}} \mathcal{F}_W \setminus \mathcal{F}_{T_0} \right). \tag{3}$$

For each $W \in \binom{T_0}{t}$, there exists $F_1 \in \mathcal{F} \setminus \mathcal{F}_{T_0}$ such that $|W \cap F_1| < t$. Since $|F_1 \cap T_0| = t$ and $|T_0| = t+1$, we have $|F_1 \cap W| = t-1$. Let $H_1 = F_1 \cup W$. It is routine to check that $|H_1| = r+1$ and $T_0 \subset H_1$. For each $F_1' \in \mathcal{F}_W \setminus \mathcal{F}_{T_0}$, we have $|F_1' \cap H_1| \geqslant t+1$ by $|F_1 \cap F_1'| \geqslant t$. Then

$$\mathcal{F}_W \setminus \mathcal{F}_{T_0} = \bigcup_{I \in \mathcal{L}_{n,t+1,k} \setminus \{T_0\},\ W \subset I \subset H_1} \mathcal{F}_I \setminus \mathcal{F}_{T_0}. \tag{4}$$

Suppose $I \in \mathcal{L}_{n,t+1,k} \setminus \{T_0\}$ with $W \subset I \subset H_1$. Since $I \notin \mathcal{T}$, there exists $F_1'' \in \mathcal{F}$ such that $t-1 \leqslant |F_1'' \cap W| \leqslant |F_1'' \cap I| \leqslant t-1$. Observe that $I \cup T_0 \in \mathcal{L}_{n,t+2,k}$. Since $\mathcal{F}$ is maximal and T_0 is a t-cover of $\mathcal{F}$, each element of $\mathcal{L}_{n,r,k}$ containing T_0 is a member of $\mathcal{F}$, which implies that $|\mathcal{F}_{I\cup T_0}| = \binom{n-t-2}{r-t-2} k^{r-t-2}$. By Lemma 2.5 and $|F_1' \cap I| = t-1$, we have $|\mathcal{F}_I| \leqslant (r-t+1)|\mathcal{F}_R|$ for some $R \in \mathcal{L}_{n,t+2,k}$. Together with $|\mathcal{F}_R| \leqslant \binom{n-t-2}{r-t-2} k^{r-t-2}$, this produces $|\mathcal{F}_I| \leqslant (r-t+1)\binom{n-t-2}{r-t-2} k^{r-t-2}$. Then

$$|\mathcal{F}_I \setminus \mathcal{F}_{T_0}| = |\mathcal{F}_I| - |\mathcal{F}_{I\cup T_0}| \leqslant (r-t)\binom{n-t-2}{r-t-2} k^{r-t-2}. \tag{5}$$

Notice that $|\mathcal{F}_{T_0}| = \binom{n-t-1}{r-t-1} k^{r-t-1}$ and the number of $I \in \mathcal{L}_{n,t+1,k} \setminus \{T_0\}$ with $W \subset I \subset H_1$ is at most $r-t$. Together with (3), (4) and (5), we get the desired bound of $|\mathcal{F}|$. □

Lemma 2.7. *Let $n, r, k, t, \ell, \mathcal{F}, \mathcal{T}$ and M be as in Assumption 2.1 with $|\mathcal{T}| \geqslant 2$ and $\tau_t(\mathcal{T}) = t$.*

(i) *If $\ell = t+2$, then*

$$|\mathcal{F}| \leqslant 2\binom{n-t-1}{r-t-1} k^{r-t-1} + (r-1)(r-t+1)\binom{n-t-2}{r-t-2} k^{r-t-2}.$$

(ii) *If $\ell \geqslant t+3$, then*

$$|\mathcal{F}| \leqslant (\ell-t)\binom{n-t-1}{r-t-1} k^{r-t-1} + ((r-\ell+1)(r-t+1)+t)\binom{n-t-2}{r-t-2} k^{r-t-2}.$$

Proof. Suppose that S is a t-cover of $\mathcal{T}$ with size t.

We first prove an upper bound for $|\mathcal{F}_S|$. Let $F_2 \in \mathcal{F} \setminus \mathcal{F}_S$ and $H_2 = S \cup F_2$. It follows from Lemma 2.2 (ii) that $M \subset H_2$ and $|H_2| = r+1$. For each $F_2' \in \mathcal{F}_S$, if $F_2 \cap M = S$, then from $|F_2 \cap F_2'| \geqslant t$ we get $|F_2' \cap H_2| \geqslant t+1$. Write

$$\mathcal{A} = \{A \in \mathcal{L}_{n,t+1,k} : S \subset A \subset H_2, A \not\subset M\}, \quad \mathcal{B} = \{B \in \mathcal{L}_{n,t+1,k} : S \subset B \subset M\}.$$

Observe that each member of $\mathcal{F}_S$ contains at least one element of $\mathcal{A} \cup \mathcal{B}$. For each $A \in \mathcal{A}$, since $A \notin \mathcal{T}$, there exists $F_2'' \in \mathcal{F}$ such that $t-1 \leqslant |F_2'' \cap S| \leqslant |F_2'' \cap A| \leqslant t-1$. Then by Lemma 2.5, we have $|\mathcal{F}_A| \leqslant (r-t+1)\binom{n-t-2}{r-t-2}k^{r-t-2}$. Notice that $|\mathcal{A}| \leqslant r-\ell+1$, $|\mathcal{B}| = \ell - t$ and $|\mathcal{F}_B| \leqslant \binom{n-t-1}{r-t-1}k^{r-t-1}$ for each $B \in \mathcal{B}$. Then we obtain

$$|\mathcal{F}_S| \leqslant (\ell - t)\binom{n-t-1}{r-t-1}k^{r-t-1} + (r-\ell+1)(r-t+1)\binom{n-t-2}{r-t-2}k^{r-t-2}. \quad (6)$$

Let $\mathcal{C} = \{C \in \mathcal{L}_{n,\ell-1,k} : S \not\subset C \subset M\}$. We have $|\mathcal{C}| = t$ and $\mathcal{F} \setminus \mathcal{F}_S \subset \bigcup_{C \in \mathcal{C}} \mathcal{F}_C$.

(i) Suppose $\ell = t+2$. For each $C \in \mathcal{C}$, since $C \notin \mathcal{T}$, there exists $F_3 \in \mathcal{F}$ such that $|F_3 \cap C| \leqslant t-1$. Together with $|F_3 \cap M| \geqslant t$, we have $|F_3 \cap C| = t-1$. By Lemma 2.2 (ii), Lemma 2.5 and $|\mathcal{C}| = t$, we have

$$|\mathcal{F} \setminus \mathcal{F}_S| \leqslant \sum_{C \in \mathcal{C}} |\mathcal{F}_C| \leqslant t(r-t+1)\binom{n-t-2}{r-t-2}k^{r-t-2}.$$

Together with (6), this produces the desired result.

(ii) Suppose $\ell \geqslant t+3$. Observe that $|\mathcal{F}_C| \leqslant \binom{n-\ell+1}{r-\ell+1}k^{r-\ell+1}$ for each $C \in \mathcal{C}$. By Lemma 2.2 (ii), $\ell \geqslant t+3$ and $|\mathcal{C}| = t$, we have

$$|\mathcal{F} \setminus \mathcal{F}_S| \leqslant \sum_{C \in \mathcal{C}} |\mathcal{F}_C| \leqslant t\binom{n-\ell+1}{r-\ell+1}k^{r-\ell+1} \leqslant t\binom{n-t-2}{r-t-2}k^{r-t-2}.$$

Together with (6), this produces the desired bound on $|\mathcal{F}|$. □

3 Proof of Theorem 1.2

Let n, r, k and t be positive integers with $n \geqslant t+2$, $n \geqslant r \geqslant t+1$ and $k \geqslant \max\{2, g(n,r,t)\}$. Suppose that $\mathcal{F}$ is a maximal non-trivial t-intersecting subfamily of $\mathcal{L}_{n,r,k}$. If $r = t+1$, then $\tau_t(\mathcal{F}) = t+1$ and $\mathcal{F}$ is the set of its t-covers with size $t+1$. It follows from Lemmas 2.2 (i) and 2.3 (i) that $\mathcal{F} \cong \mathcal{H}_1(n, t+1, k, t+2, t)$ and $|\mathcal{F}| = t+2 > 1 = f(n, t+1, k, t+1, t)$. In the following, we may assume that $r \geqslant t+2$. Write

$$\varphi(n,r,k,t) = \frac{f(n,r,k,r,t) - |\mathcal{F}|}{\binom{n-t-2}{r-t-2}k^{r-t-2}}.$$

It is sufficient to show that $\varphi(n,r,k,t) < 0$ if one of (i), (ii) and (iii) in Theorem 1.2 holds, and $\varphi(n,r,k,t) > 0$ otherwise.

Case 1. $\tau_t(\mathcal{F}) = t+1$.

In this case, let $\mathcal{T}$ be the set of all t-covers of $\mathcal{F}$ with size $t+1$ and $\ell = |\bigcup_{T\in\mathcal{T}} T|$. Recall from Section 2 that $t \leqslant \tau_t(\mathcal{T}) \leqslant t+1$, and $t+1 \leqslant \ell \leqslant \min\{r+1,n\}$ by Lemma 2.2.

Case 1.1. $\tau_t(\mathcal{T}) = t$.

In this case, (iii) does not hold since the corresponding $\mathcal{T}$ for $\mathcal{H}_1(n,r,k,t+2,t)$ has t-covering number $t+1$. Therefore, in this case, we need to show that $\varphi(n,r,k,t) < 0$ when (i) or (ii) holds and $\varphi(n,r,k,t) > 0$ when neither (i) nor (ii) holds.

Case 1.1.1. (i) or (ii) holds.

We may assume that $\mathcal{F} = \mathcal{H}_1(n,r,k,m,t)$ for some $m \in \{r, \min\{r+1,n\}\}$, or $n \geqslant r+2 \geqslant t+4$ and $\mathcal{F} = \mathcal{H}_2(n,r,k,c,t)$ for some $c \in \{r+2,\ldots,\min\{2r-t,n\}\}$. Note that $\ell \geqslant r$.

Let a be an integer with $a \geqslant t+1$. For each $b \in \{t+1,\ldots,a\}$, set

$$\mathcal{N}_b(M_a, M_t) = \{F \in \mathcal{L}_{n,r,k} : M_t \subset F, |F \cap M_a| = b\}.$$

We claim that

$$f(n,r,k,a,t) = \sum_{i=1}^{a-t} \frac{3i - i^2}{2} \cdot |\mathcal{N}_{t+i}(M_a, M_t)|. \tag{7}$$

For each $b \in \{t+1,\ldots,a\}$, let $\mathcal{M}_b(M_a, M_t)$ denote that set of all $(I,F) \in \mathcal{L}_{n,b,k} \times \mathcal{L}_{n,r,k}$ with $M_t \subset I \subset M_a$ and $I \subset F$. By double counting $|\mathcal{M}_{t+1}(M_a, M_t)|$ and $|\mathcal{M}_{t+2}(M_a, M_t)|$, we obtain

$$\sum_{i=1}^{a-t} i|\mathcal{N}_{t+i}(M_a, M_t)| = (a-t)\binom{n-t-1}{r-t-1}k^{r-t-1},$$

$$\sum_{i=2}^{a-t} \binom{i}{2}|\mathcal{N}_{t+i}(M_a, M_t)| = \binom{a-t}{2}\binom{n-t-2}{r-t-2}k^{r-t-2},$$

which imply that (7) holds. If $t+2 \leqslant a \leqslant \ell$, then we have

$$\begin{aligned} f(n,r,k,a,t) &\leqslant |\mathcal{N}_{t+1}(M_a, M_t)| + |\mathcal{N}_{t+2}(M_a, M_t)| \\ &\leqslant |\mathcal{N}_{t+1}(M_\ell, M_t)| + |\mathcal{N}_{t+2}(M_\ell, M_t)| < |\mathcal{F}| \end{aligned} \tag{8}$$

by (7). Then $\varphi(n,r,k,t) < 0$, as desired.

Case 1.1.2. Neither (i) nor (ii) holds.

In this case, we have $\ell < r$. Indeed, if $|F \cap \bigcup_{T\in\mathcal{T}} T| \geqslant t+1$ for each $F \in \mathcal{F}$, then by Lemma 2.3 (ii) and the assumption that (i) does not hold, we get $\ell < \min\{r+1,n\} \leqslant$

$r+1$ and $\ell \neq r$, which produce $\ell < r$. On the other hand, if $|F_0 \cap \bigcup_{T\in\mathcal{T}} T| = t$ for some $F_0 \in \mathcal{F}$, then by Lemma 2.4 and the assumption that (ii) does not hold, we have $\ell < \min\{r+1, n\} - 1 \leqslant r$.

If $\ell = t+1$, then from (1), Lemma 2.6 and $(n-t-1)k \geqslant \binom{t+2}{2}(r-t)^2$, we obtain

$$\varphi(n,r,k,t) \geqslant (n-t-1)k - \binom{r-t}{2} - (t+1)(r-t)^2 \geqslant \frac{(t^2+t-1)(r-t)^2}{2} > 0.$$

If $\ell = t+2$, then, since $\ell < r$, $r-t \geqslant 3$. From (1), (2), Lemma 2.7 (i) and $k \geqslant g(n,r,t)$, we obtain

$$\begin{aligned}
\varphi(n,r,k,t) &\geqslant \frac{(r-t-2)(n-t-1)k}{r-t-1} - \binom{r-t}{2} - (r-1)(r-t+1) \\
&\geqslant (r-t-2)(r-t+3)\left(\binom{t+2}{2} - \frac{3(r-t)^2+(2t-1)(r-t)+2(t-1)}{2(r-t-2)(r-t+3)}\right) \\
&\geqslant (r-t-2)(r-t+3)\left(\binom{t+2}{2} - \frac{4t+11}{6}\right) \\
&> 0.
\end{aligned}$$

If $\ell \geqslant t+3$, then, since $\ell < r$, $r-t \geqslant 4$. Notice that

$$\begin{aligned}
g(n,r,t) &\geqslant \left(\alpha\binom{t+2}{2} + (1-\alpha)\cdot\frac{r-t+1}{2}\right)\cdot\frac{(r-t+3)(r-t-1)}{n-t-1} \\
&\geqslant \left(t + \left(1-\frac{1}{3(r-t+3)}\right)\cdot\frac{(r-t+1)(r-t+3)}{2}\right)\cdot\frac{r-t-1}{n-t-1} \qquad (9)\\
&= \left(t + \frac{3(r-t)^2+11(r-t)+8}{6}\right)\cdot\frac{r-t-1}{n-t-1},
\end{aligned}$$

where α is a real number such that $\binom{t+2}{2}(r-t+3)\alpha = t$. Together with (1), (2), Lemma 2.7 (ii), $k \geqslant g(n,r,t)$ and $r-\ell \geqslant 1$, we get

$$\begin{aligned}
\varphi(n,r,k,t) &\geqslant \frac{(r-\ell)(n-t-1)k}{r-t-1} - \binom{r-t}{2} - (r-\ell+1)(r-t+1) - t \\
&\geqslant (r-\ell)\left(\frac{(n-t-1)k}{r-t-1} - \binom{r-t}{2} - 2(r-t+1) - t\right) \\
&\geqslant \frac{3(r-t)^2+11(r-t)+8}{6} - \binom{r-t}{2} - 2(r-t+1) \\
&> 0,
\end{aligned}$$

as desired.

Case 1.2. $\tau_t(\mathcal{T}) = t+1$.

In this case, by Lemmas 2.2 (i) and 2.3 (i), we have $\mathcal{F} \cong \mathcal{H}_1(n,r,k,t+2,t)$. Then (ii) does not hold. Next we show that $\varphi(n,r,k,t) < 0$ if either (i) holds with

$r \leqslant 2t+2$ or (iii) holds, and $\varphi(n,r,k,t) > 0$ otherwise. Observe that

$$|\mathcal{H}_1(n,r,k,t+2,t)| = (t+2)\binom{n-t-1}{r-t-1}k^{r-t-1} - (t+1)\binom{n-t-2}{r-t-2}k^{r-t-2}, \quad (10)$$

and it follows from (1) that

$$\varphi(n,r,k,t) = \frac{(r-2t-2)(n-t-1)k}{r-t-1} - \binom{r-t}{2} + (t+1). \quad (11)$$

Suppose that either (i) holds with $r \leqslant 2t+2$ or (iii) holds. Then $r \leqslant 2t+2$. If $r = 2t+2$, then by (11), we have

$$\varphi(n,r,k,t) = -\binom{t+2}{2} + (t+1) = -\binom{t+1}{2} < 0.$$

If $r < 2t+2$, then by (2), (11) and $k \geqslant g(n,r,t)$, we get

$$\varphi(n,r,k,t) \leqslant -\frac{(n-t-1)k}{r-t-1} - \binom{r-t}{2} + (t+1) \leqslant -\binom{t+2}{2}(r-t+3) + (t+1) < 0,$$

as desired.

Now suppose that we neither have (i) with $r \leqslant 2t+2$ nor have (iii). Then $r > 2t+2$. From (2), (11) and $k \geqslant g(n,r,t)$, we obtain

$$\varphi(n,r,k,t) \geqslant \frac{(n-t-1)k}{r-t-1} - \binom{r-t}{2} + (t+1) \geqslant \frac{(r-t+3)(r-t+1)}{2} - \binom{r-t}{2} > 0,$$

as required.

Case 2. $\tau_t(\mathcal{F}) \geqslant t+2$.

Observe that none of (i), (ii) and (iii) holds. To show $\varphi(n,r,k,t) > 0$, we first prove an upper bound on $|\mathcal{F}|$.

Claim 1. $|\mathcal{F}| \leqslant (r-t+1)^2\binom{t+2}{2}\binom{n-t-2}{r-t-2}k^{r-t-2}$.

Proof of Claim 1. Suppose $\tau_t(\mathcal{F}) = z$ and Z is a t-cover of $\mathcal{F}$ with size z. For $Y_0 \in \binom{Z}{t}$, without loss of generality, assume that $\mathcal{F}_{Y_0} \neq \emptyset$. Since Y_0 is not a t-cover of $\mathcal{F}$, there exists $X_0 \in \mathcal{F}$ such that $|X_0 \cap Y_0| < t$. By Lemma 2.5, there exists $Y_1 \in \mathcal{L}_{n,2t-|X_0\cap Y_0|,k}$ containing Y_0 such that

$$|\mathcal{F}_{Y_0}| \leqslant \binom{r-|X_0\cap Y_0|}{t-|X_0\cap Y_0|}|\mathcal{F}_{Y_1}| \leqslant (r-t+1)^{t-|X_0\cap Y_0|}|\mathcal{F}_{Y_1}|.$$

Note that $\mathcal{F}_{Y_1} \neq \emptyset$ by $|\mathcal{F}_{Y_0}| > 0$. Similarly, we deduce that there exist k-signed sets $Y_0, Y_1, \ldots, Y_w$ on $[n]$ such that $Y_0 \subset \cdots \subset Y_w$ with $|Y_{w-1}| < z$, $|Y_w| \geqslant z$ and

$$|\mathcal{F}_{Y_i}| \leqslant (r-t+1)^{|Y_{i+1}|-|Y_i|}|\mathcal{F}_{Y_{i+1}}|$$

for each $i = 0, \dots, w-1$. Therefore

$$|\mathcal{F}_{Y_0}| \leqslant (r-t+1)^{|Y_w|-t}|\mathcal{F}_{Y_w}| \leqslant (r-t+1)^{|Y_w|-t}\binom{n-|Y_w|}{r-|Y_w|}k^{r-|Y_w|}.$$

Together with $k \geqslant g(n,r,t)$, we obtain

$$\frac{|\mathcal{F}_{Y_0}|}{(r-t+1)^{z-t}\binom{n-z}{r-z}k^{r-z}} \leqslant \prod_{i=z}^{|Y_w|-1} \frac{(r-t+1)(r-i)}{(n-i)k} \leqslant \left(\frac{2}{r-t+3}\right)^{|Y_w|-z} \leqslant 1.$$

Notice that $\mathcal{F} = \bigcup_{Y \in \binom{Z}{t}} \mathcal{F}_Y$. Then

$$|\mathcal{F}| \leqslant (r-t+1)^{z-t}\binom{z}{t}\binom{n-z}{r-z}k^{r-z}.$$

For each $y \in \{t+2, \dots, r\}$, write

$$\psi(y) = (r-t+1)^{y-t}\binom{y}{t}\binom{n-y}{r-y}k^{r-y}.$$

If $y \leqslant r-1$, then by $y \geqslant t+2$, $k \geqslant g(n,r,t)$ and (2), we have

$$\begin{aligned}\frac{\psi(y+1)}{\psi(y)} &= \frac{y+1}{y+1-t} \cdot \frac{(r-t+1)(r-y)}{(n-y)k} \\ &\leqslant \frac{t+3}{3} \cdot \frac{r-t-1}{n-t-1} \cdot \frac{(r-t+1)(n-t-1)}{\binom{t+2}{2}(r-t+3)(r-t-1)} \leqslant 1.\end{aligned}$$

Then from $z \geqslant t+2$, we get $|\mathcal{F}| \leqslant \psi(t+2)$, as desired. □

Observe that

$$\begin{aligned}g(n,r,t) &\geqslant \left((1-\beta)\binom{t+2}{2} + \beta \cdot \frac{r-t+1}{2}\right) \cdot \frac{(r-t+3)(r-t-1)}{n-t-1} \\ &= \left(\frac{(r-t)^2+3(r-t)+4}{r-t+1}\binom{t+2}{2} + \frac{1}{r-t}\binom{r-t}{2}\right) \cdot \frac{r-t-1}{n-t-1},\end{aligned}$$

where β is a real number such that $(r-t+3)(r-t+1)\beta = r-t-1$. Together with (1), (2), $r \geqslant t+2$, $k \geqslant g(n,r,t)$ and Claim 1, we have

$$\begin{aligned}\varphi(n,r,k,t) &\geqslant \frac{(r-t)(n-t-1)k}{r-t-1} - \binom{r-t}{2} - \binom{t+2}{2}(r-t+1)^2 \\ &\geqslant \binom{t+2}{2}\left(\frac{(r-t)^3+3(r-t)^2+4(r-t)}{r-t+1} - (r-t+1)^2\right) \\ &= \frac{r-t-1}{r-t+1}\binom{t+2}{2} \\ &> 0.\end{aligned}$$

This finishes the proof of Theorem 1.2. □

4 Proof of Theorem 1.4

Let n, r, k and t be positive integers with $n \geqslant t+2 \geqslant 4$, $n \geqslant r \geqslant t+1$ and $k \geqslant \max\{2, g(n,r,t)\}$. Suppose that $\mathcal{F}$ is a maximum-sized non-trivial t-intersecting subfamily of $\mathcal{L}_{n,r,k}$. If $r = t+1$, then by Theorem 1.2, we have $\mathcal{F} \cong \mathcal{H}_1(n,r,k,t+2,t)$. In the following, we assume that $r \geqslant t+2$. Write $p = \min\{r+1, n\}$.

Claim 2. *$\mathcal{F}$ is isomorphic to $\mathcal{H}_1(n,r,k,p,t)$ or $\mathcal{H}_1(n,r,k,t+2,t)$.*

Proof of Claim 2. Suppose for contradiction that neither $\mathcal{H}_1(n,r,k,p,t)$ nor $\mathcal{H}_1(n,r,k,t+2,t)$ is isomorphic to $\mathcal{F}$. Let $\mathcal{T}$ be the set of all t-covers of $\mathcal{F}$ with size $\tau_t(\mathcal{F})$ and $\ell = |\bigcup_{T\in\mathcal{T}} T|$. By Theorem 1.2 and Lemmas 2.2 (i), 2.3, 2.4, we have $\tau_t(\mathcal{F}) = t+1$, $\tau_t(\mathcal{T}) = t$ and $\ell = r \neq p$. Therefore $n > r$, $p = r+1$ and $|\mathcal{T}| \geqslant 2$.

If $r = t+2$, then by (1), (2), $k \geqslant g(n,r,t)$ and Lemma 2.7 (i), we get

$$\frac{f(n,r,k,p,t) - |\mathcal{F}|}{\binom{n-t-2}{r-t-2}k^{r-t-2}} \geqslant \frac{(n-t-1)k}{r-t-1} - \binom{r-t+1}{2} - 3(r-1) \geqslant 5\binom{t+2}{2} - 3(t+2) > 0.$$

If $r \geqslant t+3$, then by (1), (2), (9), $k \geqslant g(n,r,t)$ and Lemma 2.7 (ii), we have

$$\begin{aligned}\frac{f(n,r,k,p,t) - |\mathcal{F}|}{\binom{n-t-2}{r-t-2}k^{r-t-2}} &\geqslant \frac{(n-t-1)k}{r-t-1} - \binom{r-t+1}{2} - (r-t+1) - t \\ &\geqslant \frac{3(r-t)^2 + 11(r-t) + 8}{6} - \binom{r-t+1}{2} - (r-t+1) \\ &> 0.\end{aligned}$$

Together with (8), we get $|\mathcal{F}| < f(n,r,k,p,t) \leqslant |\mathcal{H}_1(n,r,k,p,t)|$, a contradiction to the assumption that $\mathcal{F}$ is maximum-sized. □

If $n = t+2$, then it follows from Claim 2 that $\mathcal{F} \cong \mathcal{H}_1(n,r,k,t+2,t)$. In the following we may assume that $n \geqslant t+3$. Write

$$\mu(n,r,k,t) = \frac{|\mathcal{H}_1(n,r,k,t+2,t)| - |\mathcal{H}_1(n,r,k,p,t)|}{\binom{n-t-2}{r-t-2}k^{r-t-2}}.$$

By Claim 2, it suffices to show that $\mu(n,r,k,t) < 0$ if $p > 2t+2$, and $\mu(n,r,k,t) > 0$ if $p \leqslant 2t+2$. We divide the remaining proof into three cases.

Case 1. $p > 2t+2$.

Since $k \geqslant g(n,r,t)$ and $|\mathcal{H}_1(n,r,k,p,t)| > f(n,r,k,p,t)$, by (1), (2) and (10), we have

$$\mu(n,r,k,t) < -\frac{(n-t-1)k}{r-t-1} + \binom{p-t}{2} - (t+1) \leqslant -\frac{3(r-t+1)}{2} - (t+1) < 0,$$

as desired.

Case 2. $p < 2t + 2$**.**

By the construction of $\mathcal{H}_1(n, r, k, p, t)$, it is routine to verify that

$$|\mathcal{H}_1(n, r, k, p, t)| \leqslant (p - t)\binom{n - t - 1}{r - t - 1}k^{r-t-1} + t(k - 1).$$

Therefore, if $r \geqslant t + 3$, then by (2), (10), $t \geqslant 2$ and $k \geqslant g(n, r, t)$, we have

$$\mu(n, r, k, t) \geqslant \frac{(n - t - 1)k}{r - t - 1} - (t + 1) - t \geqslant \binom{t + 2}{2}(r - t + 3) - (2t + 1) > 0.$$

If $r = t + 2$, then $p = t + 3$ by $n \geqslant t + 3$, and

$$|\mathcal{H}_1(n, t + 2, k, t + 3, t)| = 3(n - t - 1)k + t - 3.$$

Together with (10), $n \geqslant t + 3$ and $t, k \geqslant 2$, we obtain

$$\mu(n, t + 2, k, t) = (t - 1)((n - t - 1)k - 2) > 0,$$

as required.

Case 3. $p = 2t + 2$**.**

In this case, we have $r \geqslant p - 1 > t + 2$. By the construction of $\mathcal{H}_1(n, r, k, p, t)$, we have

$$|\mathcal{H}_1(n, r, k, p, t)| \leqslant \sum_{i=1}^{p-t} |\mathcal{N}_{t+i}(M_p, M_t)| + t(k - 1).$$

Together with (7) and $|\mathcal{N}_{t+i}(M_p, M_t)| \leqslant \binom{t+2}{i}\binom{n-t-i}{r-t-i}k^{r-t-i}$ for each $i \in \{3, \dots, p-t\}$, we get

$$\begin{aligned}|\mathcal{H}_1(n, r, k, p, t)| - f(n, r, k, p, t) &\leqslant \sum_{i=3}^{p-t} \binom{i - 1}{2} |\mathcal{N}_{t+i}(M_p, M_t)| + t(k - 1)\\ &\leqslant \sum_{i=3}^{p-t} \binom{i - 1}{2}\binom{t+2}{i}\binom{n-t-i}{r-t-i}k^{r-t-i} + t(k-1).\end{aligned}$$

For each $i \in \{3, \dots, p - t\}$, write

$$\lambda(i) = \binom{i - 1}{2}\binom{t + 2}{i}\binom{n - t - i}{r - t - i}k^{r-t-i}.$$

If $i \leqslant p - t - 1$, then by (2), $t \geqslant 2$, $i \geqslant 3$ and $k \geqslant g(n, r, t)$, we have

$$\frac{\lambda(i + 1)}{\lambda(i)} = \frac{i(t + 2 - i)}{(i - 2)(i + 1)} \cdot \frac{r - t - i}{(n - t - i)k} \leqslant \frac{3(t - 1)}{4(t + 1)(t + 2)} \leqslant \frac{1}{4}.$$

Then

$$\begin{aligned}|\mathcal{H}_1(n,r,k,p,t)| - f(n,r,k,p,t) &\leqslant \lambda(3)\cdot\sum_{j=0}^{\infty}\frac{1}{4^j} + t(k-1)\\ &= \frac{4}{3}\binom{t+2}{3}\binom{n-t-3}{r-t-3}k^{r-t-3} + t(k-1).\end{aligned}$$

Together with (2), $t \geqslant 2$, $k \geqslant g(n,r,t)$ and

$$|\mathcal{H}_1(n,r,k,t+2,t)| - f(n,r,k,p,t) = \binom{t+1}{2}\binom{n-t-2}{r-t-2}k^{r-t-2},$$

we get

$$\begin{aligned}\mu(n,r,k,t) &\geqslant \binom{t+1}{2} - t - \frac{4(r-t-2)}{3(n-t-2)k}\binom{t+2}{3}\\ &\geqslant \binom{t}{2} - \frac{8}{3(t+1)(t+2)(r-t+3)}\cdot\frac{(t+2)(t+1)t}{6}\\ &\geqslant \left(\frac{t-1}{2} - \frac{4}{9}\right)t\\ &> 0.\end{aligned}$$

This finishes the proof of Theorem 1.4. □

Remark. In Theorem 1.4, we assume $t \geqslant 2$. We can also get the corresponding result for $t = 1$ using the same method. It should be noted that, when $t = 1$, comparing the sizes of $\mathcal{H}_1(n,r,k,\min\{r+1,n\},1)$ and $\mathcal{H}_1(n,r,k,3,1)$ is a little more complicated because these two families may have the same size.

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