AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **89(2)** (2024), Pages 344–349

2-primitive C_3-decompositions of cocktail party graphs

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Abstract

A decomposition $\mathcal{G}$ of a graph G is k-primitive if no proper, nontrivial subset of $\mathcal{G}$ with at least k elements is a decomposition of an induced subgraph of G. The subject of 1-primitivity has been investigated in several cases, but the only known exploration of k-primitive decompositions with $k \geq 2$ is attributed to Doyen who, in a 1969 paper about triple systems, classified the existence of 2-primitive C_3-decompositions of complete graphs. In this work, we prove the analogous result for cocktail party graphs.

1 Introduction

In this work we generalize the definition of primitivity for graph decompositions, which was formalized by Asplund et al. [1]. We begin with the following question:

Question 1. *Let G be a graph with vertex set V, and let $\mathcal{G}$ be a decomposition of G. Do there exist nonempty subsets $\mathcal{G}' \subseteq \mathcal{G}$ and $V' \subseteq V$ such that $\mathcal{G}'$ is a decomposition of the induced subgraph of G with vertex set V'?*

Such subsets $\mathcal{G}'$ are *subdecompositions* of $\mathcal{G}$. Trivially, the answer to the previous question is yes, because $\mathcal{G}$ is a subdecomposition of itself. So we ask a more interesting question:

ISSN: 2202-3518

Question 2. *Let G be a graph with a decomposition $\mathcal{G}$. Does $\mathcal{G}$ contain any proper subdecompositions?*

If the answer to the previous question is no, we say $\mathcal{G}$ is a *primitive decomposition* of G, and this property has been investigated in several cases. In 2000, Rodger and Spicer [5] classified the existence of primitive $(K_4 - e)$-decompositions of complete graphs. In 2012, Dinavahi and Rodger [3] classified the existence of primitive P_n-decompositions of complete graphs. In 2022, Asplund et al. [1] investigated primitive C_n-decompositions of both complete graphs and cocktail party graphs and, in 2023, Schroeder [6] settled an unresolved question from the previous work involving C_4-decompositions of cocktail party graphs.

By another name, the primitivity of graph decompositions was introduced in 1969 by Doyen [4], in a paper discussing *non-degenerate triple systems*. Observe that a triple system is analogous to a C_3-decomposition of a complete graph, and the necessary conditions for their existence were established by Steiner [8], while the sufficiency of these conditions was later demonstrated by Bose [2] and Skolem [7].

Theorem 1.1. *Let n be a positive integer. Then the complete graph K_n has a C_3-decomposition if and only if $n \equiv 1$ or $3 \pmod 6$.*

Observe that a C_3-decomposition of a simple graph with more than three edges is never primitive; indeed, any one-element subset of a C_3-decomposition is, itself, a subdecomposition. With this in mind, we generalize Question 2 by adding a parameter k:

Question 3. *Let k be a positive integer and G be a graph with decomposition $\mathcal{G}$. Does $\mathcal{G}$ contain any proper subdecompositions containing at least k subgraphs?*

If the answer to this question is no, then we say $\mathcal{G}$ is a *k-primitive decomposition* of G; thus a primitive decomposition is equivalently a 1-primitive decomposition. With this new language, we may restate one of the results by Doyen [4] as follows:

Theorem 1.2. *Let n be a positive integer. Then the complete graph K_n has a 2-primitive C_3-decomposition if and only if $n \equiv 1$ or $3 \pmod 6$.*

In other words, any complete graph with a C_3-decomposition must have a 2-primitive C_3-decomposition. In fact, the Skolem construction of a Steiner triple system gives rise to a 2-primitive C_3-decomposition of a complete graph, while in many cases, the Bose construction does not.

In this paper we prove an analogous result for cocktail party graphs:

Theorem 1.3. *Let m be a positive integer. Then the cocktail party graph $K_{2m} - I$ has a 2-primitive C_3-decomposition if and only if $m \equiv 0$ or $1 \pmod 3$.*

2 Main Result

First, we introduce and recall necessary definitions. We then present a lemma which provides sufficient conditions for a decomposition to be k-primitive. Next, we establish a property of decompositions of cocktail party graphs, as well as provide a construction for a C_3-decomposition of a cocktail party graph. Finally, we give a proof for Theorem 1.3.

For a positive integer n, let $[n] = \{1, \ldots, n\}$. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively.

Let $\mathcal{G}$ be a set of subgraphs of a graph G, each with no isolated vertices, and let $V(\mathcal{G})$ be the union of the vertex sets of the graphs in $\mathcal{G}$. Recall $\mathcal{G}$ is a *decomposition* of G if $\{E(H) : H \in \mathcal{G}\}$ is a partition of $E(G)$.

Let $\mathcal{G}$ be a decomposition of G. A subgraph H of G is *aligned with* $\mathcal{G}$ if there exists a subset $\mathcal{H} \subseteq \mathcal{G}$ which is a decomposition of H. We say H is the subgraph *associated to* $\mathcal{H}$, and conversely, $\mathcal{H}$ is associated to H in $\mathcal{G}$. Given a subset $V' \subseteq V(G)$, the subgraph G' of G *induced by* V' is the graph with vertex set V' and for distinct $x, y \in V'$, $xy \in E(G')$ if and only if $xy \in E(G)$. With this language, an induced subgraph of G aligned with $\mathcal{G}$ is associated to a subdecomposition of $\mathcal{G}$.

Let H', H, and G' be subgraphs of G aligned with $\mathcal{G}$, associated to $\mathcal{H}'$, $\mathcal{H}$, and $\mathcal{G}'$, respectively, and suppose H' is an induced subgraph of H. Note this implies $\mathcal{H}'$ is a subdecomposition of $\mathcal{H}$. We say $\mathcal{G}'$ is an *extension of* $\mathcal{H}'$ *in* $\mathcal{G}$ *with respect to* $\mathcal{H}$ if $\mathcal{H}' = \mathcal{G}' \cap \mathcal{H}$ and G' is an induced subgraph of G, and hence $\mathcal{G}'$ is a subdecomposition of $\mathcal{G}$. In other words, the subdecomposition $\mathcal{H}'$ of $\mathcal{H}$ *extends* to a subdecomposition $\mathcal{G}'$ of $\mathcal{G}$.

Consider the following example. Let G be the graph with vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ given in Figure 1. Let $\mathcal{G} = \{G_1, G_2, G_3, G_4\}$ be the decomposition of G defined by the edge sets $E(G_1) = \{12, 13\}$, $E(G_2) = \{16, 26, 36\}$, $E(G_3) = \{15, 23, 25, 35\}$, and $E(G_4) = \{14, 34\}$. Define $\mathcal{H} = \{G_1, G_2\}$, $\mathcal{H}' = \{G_1\}$, and $\mathcal{G}' = \{G_1, G_3\}$, each with associated subgraphs H, H', and G', respectively. Then H is a subgraph of G (which is not induced), H' is an induced subgraph of H, and G' is an induced subgraph of G. Hence $\mathcal{H}'$ and $\mathcal{G}'$ are subdecompositions of $\mathcal{H}$ and $\mathcal{G}$, respectively. Since $\mathcal{H}' = \mathcal{G}' \cap \mathcal{H}$, we have that $\mathcal{G}'$ is an extension of $\mathcal{H}'$ with respect to $\mathcal{H}$.

The following lemma establishes sufficient conditions for primitivity involving the existence of extensions.

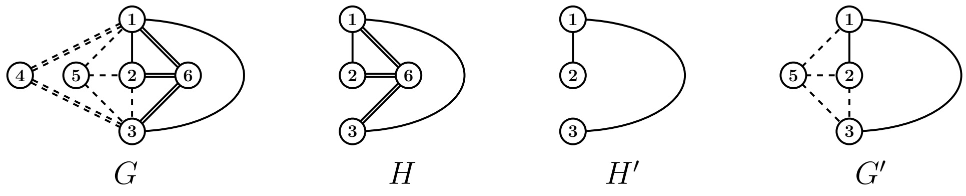

Figure 1: An example of a graph G with decomposition $\mathcal{G} = \{G_1(\text{—}), G_2(\text{═}), G_3(\text{- - -}), G_4(\text{= = =})\}$, a subgraph H of G aligned with $\mathcal{G}$ with decomposition $\mathcal{H} = \{G_1, G_2\}$, an induced subgraph H' of H associated to the subdecomposition $\mathcal{H}' = \{G_1\}$, and an induced subgraph G' of G aligned with the subdecomposition $\mathcal{G}' = \{G_1, G_3\}$, which is an extension of $\mathcal{H}'$ with respect to $\mathcal{H}$.

Lemma 2.1. *Let G be a graph with a decomposition $\mathcal{G}$ and $\mathcal{H} \subseteq \mathcal{G}$. Suppose that every subdecomposition of $\mathcal{H}$ has an extension in $\mathcal{G}$ with respect to $\mathcal{H}$. If $\mathcal{G}$ is k-primitive for some integer $k \geq 1$, then $\mathcal{H}$ is also k-primitive.*

Proof. Let $\mathcal{H}'$ be a subdecomposition of $\mathcal{H}$ and suppose $|\mathcal{H}'| \geq k$. Then $\mathcal{H}'$ extends to a subdecomposition $\mathcal{G}'$ of $\mathcal{G}$ for which $\mathcal{G}' \cap \mathcal{H} = \mathcal{H}'$. Hence $|\mathcal{G}'| \geq k$. Since $\mathcal{G}$ is k-primitive, we have that $\mathcal{G}' = \mathcal{G}$. So $\mathcal{H}' = \mathcal{G}' \cap \mathcal{H} = \mathcal{G} \cap \mathcal{H} = \mathcal{H}$. Therefore $\mathcal{H}'$ is not proper, and hence $\mathcal{H}$ is k-primitive. □

All induced subgraphs of a complete graph are complete graphs. The induced subgraphs of a cocktail party graph has a more complex classification. Since the complement of a cocktail party graph is a (perfect) matching, the complement of an induced subgraph of a cocktail party graph must also be a (possibly trivial) matching. The next lemma establishes that the possible induced subgraphs of a cocktail party graph are further restricted when requiring alignment to a particular decomposition.

Lemma 2.2. *Let m be a positive integer and $\mathcal{G}$ be a decomposition of $K_{2m} - I$ for which each vertex of each subgraph in $\mathcal{G}$ has even degree. Suppose G' is an induced subgraph of $K_{2m} - I$ aligned with $\mathcal{G}$ associated to a subdecomposition $\mathcal{G}' \subseteq \mathcal{G}$. Then G' is either a complete graph or a cocktail party graph.*

Proof. Let $V = V(K_{2m} - I)$ and $V' = V(G')$. Let $x \in V'$, and let $y \in V$ such that $xy \in I$. As G' is an induced subgraph of a cocktail party graph, then either $\deg_{G'}(x) = |V'| - 2$ or $|V'| - 1$, which occurs when $y \in V'$ or $y \notin V'$, respectively.

Since each vertex in each subgraph of $\mathcal{G}'$ has even degree, it follows that $\deg_{G'}(x)$ is even. If $|V'|$ is odd, then $\deg_{G'}(x) = |V'| - 1$, and if $|V'|$ is even, then $\deg_{G'}(x) = |V'| - 2$. As the choice of x was arbitrary, we have that G' is a regular subgraph of $K_{2m} - I$. If $|V'|$ is odd, then G' is $(|V'| - 1)$-regular, and hence G' is a complete graph; if $|V'|$ is even, then G' is $(|V'| - 2)$-regular, and hence G' is a cocktail party graph. □

We now present a well-known method for producing a C_3-decomposition of a cocktail party graph, as well as highlight some of its properties.

Construction 2.3. Let m be a positive integer and $m \equiv 0$ or 1 (mod 3). Let $\mathcal{G}$ be a C_3-decomposition of K_{2m+1}, which exists by Theorem 1.1. Let $x \in V(K_{2m+1})$. There are exactly m subgraphs $S_1, \ldots, S_m \in \mathcal{G}$ such that $x \in V(S_i)$ for each $i \in [m]$. For each $i \in [m]$, let $V(S_i) = \{x, a_i, b_i\}$. So $E(S_i) = \{xa_i, xb_i, a_ib_i\}$ and $V(K_{2m+1}) = \{x\} \cup \{a_i, b_i : i \in [m]\}$.

Let $\mathcal{H} = \mathcal{G} \backslash \{S_i : i \in [m]\}$. Then $\mathcal{H}$ is a C_3-decomposition of $K_{2m} - I$, where $V(K_{2m} - I) = V(K_{2m+1}) \backslash \{x\} = \{a_i, b_i : i \in [m]\}$ and the missing 1-factor is $I = \{a_ib_i : i \in [m]\}$. Hence $K_{2m} - I$ is aligned with $\mathcal{G}$, associated to $\mathcal{H}$.

Observation 2.4. Let m be a positive integer, and define $\mathcal{G}$ and $\mathcal{H}$ as given in Construction 2.3. Suppose $\mathcal{H}'$ is a subdecomposition of $\mathcal{H}$, associated to a cocktail party graph H'. Then $H' = K_{2m'} - J$ for some positive integer $m' \leq m$ and $J \subseteq I$; say $J = \{a_{i_j}b_{i_j} : j \in [m']\}$ for distinct $i_1, \ldots, i_{m'} \in [m']$. Let $V' = \{x\} \cup \{a_{i_j}, b_{i_j} : j \in [m']\}$ and $\mathcal{G}' = \mathcal{H}' \cup \{S_{i_j} : j \in [m']\}$. Then $\mathcal{G}'$ is a decomposition of the complete graph with vertex set V', and since a complete subgraph is necessarily an induced subgraph, it follows that $\mathcal{G}'$ is a subdecomposition of $\mathcal{G}$.

We now leverage Theorem 1.2, Lemmas 2.1 and 2.2, Construction 2.3, and Observation 2.4 to prove Theorem 1.3.

Proof of Theorem 1.3. If $K_{2m} - I$ has a C_3-decomposition, then its number of edges $|E(K_{2m} - I)| = 2m(2m-2)/2 = 2m(m-1)$ must be divisible by 3, and hence $m \equiv 0$ or 1 (mod 3).

Now suppose $m \equiv 0$ or 1 (mod 3); so $2m + 1 \equiv 1$ or 3 (mod 6). Then K_{2m+1} has a 2-primitive C_3-decomposition $\mathcal{G}$ by Theorem 1.2. Let $x \in V(K_{2m+1})$ and define $\mathcal{H}$ and $S_1, \ldots, S_m$ as given in Construction 2.3. Then $\mathcal{H}$ is a C_3-decomposition of $K_{2m} - I$, which we now demonstrate is 2-primitive.

Let $\mathcal{H}'$ be a subdecomposition of $\mathcal{H}$ associated to H' such that $|\mathcal{H}'| \geq 2$. By Lemma 2.2, either H' is a complete graph or a cocktail party graph.

First suppose H' is a complete graph. Then H' is a subgraph of K_{2m+1} induced by $V(H')$; hence $\mathcal{H}'$ is a subdecomposition of $\mathcal{G}$. Since $|\mathcal{H}'| \geq 2$ and $\mathcal{G}$ is 2-primitive, it follows that $\mathcal{G} = \mathcal{H}'$, meaning $H' = K_{2m+1}$, while simultaneously H' is a subgraph of $K_{2m} - I$, which is a contradiction. So H' must be a cocktail party graph.

Let $m' \leq m$ and $i_1, \ldots, i_{m'} \in [m]$ such that $H' = K_{2m'} - J$, where $J = \{a_{i_j}b_{i_j} : j \in [m']\}$. Let $\mathcal{G}' = \mathcal{H}' \cup \{S_{i_j} : j \in [m']\}$. By its construction, $\mathcal{G}' \cap \mathcal{H} = \mathcal{H}'$, and by Observation 2.4, $\mathcal{G}'$ is also a subdecomposition of $\mathcal{G}$. So $\mathcal{G}'$ is an extension of $\mathcal{H}'$ in $\mathcal{G}$ with respect to $\mathcal{H}$. Since $\mathcal{G}$ is 2-primitive, it follows from Lemma 2.1 that $\mathcal{H}$ is 2-primitive as well. □

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(Received 3 Jan 2024; revised 23 May 2024)