ON THE EXISTENCE OF ALMOST-REGULAR-GRAPHS WITHOUT ONE-FACTORS

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ABSTRACT:

A **one-factor** of a graph G is a 1-regular spanning subgraph of G. For given positive integers d and even e, let $\mathcal{G}(2n;d,e)$ be the class of simple connected graphs on 2n vertices, (2n-1) of which have degree d and one has degree d + e, having no one-factor. Recently, W.D. Wallis asked for what value of n is $\mathcal{G}(2n;d,e) \neq \phi$? In this paper we will answer this question.

1. INTRODUCTION

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part, our notation and terminology follows that of Bondy and Murty [2]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges. However, we denote the complement of G by $\bar{G}$. K_n denotes the complete graph on n vertices, $K_{n,m}$ the complete bipartite graph with bipartitioning sets of order n and m, and C_n denotes the cycle of length n.

A **1-factor** of a graph G is a 1-regular spanning subgraph of G. A **1-factorization** of G is a set of (pairwise) edge-disjoint 1-factors which between them contain each edge of G. It is very well known that K_{2n} and $K_{n,n}$ have 1-factorizations for all n. The question of which graphs contain 1-factors is one that has attracted considerable attention. For a comprehensive review we refer to the survey papers of: Akiyama and Kano [1]; Mendelsohn and Rosa [4]; and Wallis [8].

Wallis [7] studied regular graphs with no 1-factors. In particular, he proved the following theorem :

Theorem 1.1 : Let G be a d-regular graph with no 1-factor and no odd component. Then

$$\nu(G) \geq \begin{cases} 3d + 7, & \text{if d is odd, } d \geq 3 \\ 3d + 4, & \text{if d is even, } d \geq 6 \\ 22, & \text{if } d = 4. \end{cases}$$

Further, no such graphs exists for d = 1 or 2. □

Pila [5] considered the same problem with a connectivity condition added.

Theorem 1.1 proved very useful in considering maximal sets of 1-factors (see Caccetta and Mardiyono [3]; Rees and Wallis [6]).

In [8] (p. 623, Problem 7), Wallis posed the question regarding the existence of almost regular graphs without 1-factors. More precisely, let $\mathcal{G}(2n;d,e)$ denote the class of simple connected graphs on 2n vertices, (2n-1) of which have degree d and one has degree d + e (e even), having no 1-factor. Wallis asked for what values of n is $\mathcal{G}(2n;d,e) \neq \phi$? In this paper we will answer this question. More specifically, we will establish that for $d \geq 2$, $\mathcal{G}(2n;d,e) = \phi$ for $2n < N(d,e)$ and $\mathcal{G}(2n;d,e) \neq \phi$ for $2n \geq N(d,e)$, where

(i) $N(2,e) = e + 6$, for $e \geq 4$

(ii) for odd $d \geq 3$

$$N(d,e) = \begin{cases} e + d + 1, & \text{if } e \geq 2d \\ 3d + 3, & \text{if } d + 1 \leq e \leq 2d - 2 \\ 3d + 5, & \text{otherwise.} \end{cases}$$

and

(iii) for even $d \geq 4$

$$N(d,e) = \begin{cases} e + d + 2, & \text{if } e \geq 3d + 4 \\ e + d + 4, & \text{if } 2d \leq e \leq 3d + 2 \\ 3d + 4, & \text{otherwise.} \end{cases}$$

As the only member of $\mathcal{G}(2;1,e)$ is the graph $K_{1,e+1}$ we assume that $d \geq 2$.

2. LOWER BOUNDS

In this section we will determine a lower bound on the order of a graph $G \in \mathcal{G}(2n;d,e)$. That is, we determine a lower bound for the value of the function

$$N(d,e) = \min\{2n : \mathcal{G}(2n;d,e) \neq \phi\}.$$

We make use of the well known theorem of Tutte. Letting $o(H)$ denote the number of odd components of a graph H, Tutte's theorem is :

Theorem 2.1 : A nontrivial graph G has a 1-factor if and only if

$$o(G-S) \le |S|, \qquad \text{for every } S \subset V(G). \qquad \square$$

For later reference, it is convenient to state the following simple fact as a Lemma.

Lemma 2.1 : Let H be an odd component of $G - S$, $S \subset V(G)$. If every vertex of H has degree d in G and $\nu(H) \le d - 1$, then the number of edges joining V(H) to S is at least

$$(d - \nu(H) + 1)(\nu(H)). \qquad \square$$

Our first result concerns d odd.

Theorem 2.2 : Let $G \in \mathcal{G}(2n;d,e)$ for odd $d \ge 3$. Then

$$2n \ge \begin{cases} e + d + 1, & \text{for } e \ge 2d \\ 3d + 3, & \text{for } d + 1 \le e \le 2d - 2 \\ 3d + 5, & \text{otherwise.} \end{cases} \qquad (2.1)$$

Proof: Since $\nu(G) \ge d + e + 1$ we have nothing to prove for $e \ge 2d$. So we assume that $e < 2d$. Since G has no 1-factor, Tutte's theorem implies the existence of a set $S \subset V(G)$ such that $o(G - S) > |S|$. In fact, we must have $o(G - S) \ge |S| + 2$.

The odd components of $G - S$ are classified into three groups according to order. We let :

α_1 : the number of odd components of $G - S$ of order p, $1 \le p \le d - 2$

α_2 : the number of odd components of $G - S$ of order d

α_3 : the number of odd components of $G - S$ of order at least $d + 2$.

Then

$$\alpha_1 + \alpha_2 + \alpha_3 \ge |S| + 2, \tag{2.2}$$

and

$$\nu(G) \ge |S| + \alpha_1 + d\alpha_2 + (d + 2)\alpha_3. \tag{2.3}$$

Noting Lemma 2.1 we can conclude that there are at least $d(\alpha_1 + \alpha_2) + \alpha_3$ edges joining $V(G - S)$ and S. Consequently,

$$d(\alpha_1 + \alpha_2) + \alpha_3 \le d|S| + e. \tag{2.4}$$

Now we may assume that $\alpha_3 \le 2$, since otherwise (2.3) yields $\nu(G) > 3d + 5$. Then (2.2) gives $\alpha_1 + \alpha_2 \ge |S| \ge 1$.

Let H be an odd component of $G - S$ of order at most d. Observing that the vertices of H have degree d in $G[V(H) \cup S]$ we have $|V(H) \cup S| \ge d + 1$ and hence

$$\nu(G) \ge d + 1 + (d + 2)\alpha_3. \tag{2.5}$$

Thus we have nothing to prove when $\alpha_3 = 2$. So suppose that $\alpha_3 \le 1$. Then

$$\alpha_1 + \alpha_2 \ge |S| + 1.$$

It follows from (2.4) that the vertex u of G having degree d + e must be in S, as otherwise the right hand side of (2.4) is just $d|S|$ which contradicts (2.2).

From (2.2) and (2.4) we have

$$|S| + 2 - \alpha_3 \le \alpha_1 + \alpha_2 \le |S| + \frac{e - \alpha_3}{d} .$$

Hence

$$\alpha_3 \ge 2 - \frac{e}{d} + \frac{\alpha_3}{d} .$$

That is

$$(1 - \frac{1}{d})\, \alpha_3 \ge \frac{2d - e}{d} \text{ , or}$$

$$\alpha_3 \ge \left(\frac{d}{d - 1} \right)\left(\frac{2d - e}{d - 1} \right) = \frac{2d - e}{d - 1} \qquad (2.6)$$

Since $\alpha_3 \le 1$ it follows from (2.6) that $e \ge d + 1$.

Now since $e \le 2d - 2$ we have from (2.6) $\alpha_3 \le \frac{2}{d - 1}$ and so $\alpha_3 \ge 1$. Since we have already established that $\alpha_3 \le 1$, we have $\alpha_3 = 1$. Hence $\alpha_1 + \alpha_2 \ge |S| + 1 \ge 2$. We can take $\alpha_2 \le 1$, since otherwise from (2.3) we get $\nu(G) \geq 3d + 3$, as required. Consequently $\alpha_1 \geq 1$. Now clearly the subgraph H′ of G consisting of the vertices of S and an α_1-component has order at least d + 1. Hence

$$\nu(G) \geq d + 1 + d\alpha_2 + d + 2.$$

Consequently, $\nu(G) \geq 3d + 3$ for $\alpha_2 = 1$. So we can suppose that $\alpha_2 = 0$.

Now our odd component H defined earlier has at most d - 2 vertices. It follows from Lemma 2.1 that the number of edges between V(H) and S - u is at least

$$\nu(H)(d - \nu(H)) = (d - 1) + (\nu(H) - 1)(d - \nu(H) - 1) \geq d - 1.$$

Consequently, there are at least $(d - 1)\alpha_1$ edges between V(G - S) and S - u. Hence

$$(d - 1)\alpha_1 \leq d(|S| - 1)$$

and so, since $\alpha_1 \geq |S| + 1$, we have

$$|S| \geq 2d - 1.$$

But then $\alpha_1 \geq 2d$ and so

$$\nu(G) \geq |S| + \alpha_1 + d + 2 > 3d + 3.$$

This completes the proof of the theorem. □

Our next result concerns the case $d = 2$.

Lemma 2.2 : Let $G \in \mathcal{G}(2n;2,e)$. Then $e \geq 4$ and $2n \geq e + 6$.

Proof : Since G has no 1-factor, there exists a subset $S \subset V(G)$ such that $k = o(G - S) \geq |S| + 2$. Each odd component has an even number of edges incident to S. Consequently

$$2k \leq 2|S| + e$$

and thus $e \geq 2k - 2|S| \geq 4$, as required. Further, the vertex u of G having degree $e + 2$ must be in S.

Suppose that $2n < e + 6$. Then $2n = e + 4$ and u is adjacent to every vertex of G except one, say v. Clearly since every vertex of $G - u$ has degree 2, $G - S$ has at most one component of order 3 or more. Hence at least $k - 1$ components of $G - S$ have order 1. This implies that there are at least $k - 1$ edges between $V(G - S)$ and $S - u$. But there can be at most $|S|$ edges going out of $S - u$ and so $k \leq |S| + 1$, a contradiction. This completes the proof of the lemma. □

Theorem 2.3 : Let $G \in \mathcal{G}(2n;d,e)$, d even. Then for $d \geq 4$

$$2n \geq \begin{cases} e + d + 2, & \text{for } e \geq 3d + 4 \\ e + d + 4, & \text{for } 2d \leq e \leq 3d + 2 \\ 3d + 4, & \text{otherwise.} \end{cases} \tag{2.7}$$

For $d = 2$, $e \geq 4$ and $2n \geq e + 6$.

Proof: In view of Lemma 2.2, we assume that $d \geq 4$. Since $\nu(G) \geq e + d + 2$ we have nothing to prove for $e \geq 3d + 4$. So we assume that $e \leq 3d + 2$. Since G has no 1-factor, Tutte's theorem implies the existence of a set $S \subset V(G)$ with $o(G - S) \geq |S| + 2$.

We classify the odd components of $G - S$ into three groups according to order. We let

β_1 : the number of odd components of $G - S$ having one vertex.

β_2 : the number of odd components of $G - S$ of order p, $3 \leq p \leq d - 1$.

β_3 : the number of odd components of $G - S$ of order at least $d + 1$.

Then

$$\beta_1 + \beta_2 + \beta_3 \geq |S| + 2. \tag{2.8}$$

and

$$\nu(G) \geq |S| + \beta_1 + 3\beta_2 + (d + 1)\beta_3. \tag{2.9}$$

Noting Lemma 2.1 we can conclude that there are at least

$$d\beta_1 + 2(d - 1)\beta_2 + 2\beta_3$$

edges joining $V(G - S)$ and S. Consequently

$$d\beta_1 + 2(d - 1)\beta_2 + 2\beta_3 \leq d|S| + e. \tag{2.10}$$

We now distinguish two cases according to the value of e.

Case (i) : $2d \le e \le 3d + 2$.

Suppose that $2n < d + e + 4$. Then $2n = d + e + 2$ and the vertex u of G having degree $d + e$ is adjacent to every vertex of G except one, v say. It is clear that $u \in S$, since otherwise at least two of the odd components of $G - S$ do not contain u and hence $d_G(u) \ne d + e$ $(= \nu(G) - 2)$.

First we consider the case when $v \in S$. Then every vertex of G-S is joined to u. Consequently, counting the edges (note Lemma 2.1) between the odd components of $G - S$ and $S - u$ we have

$$(d - 1)\beta_1 + (d - 1)\beta_2 + \beta_3 \le (|S| - 1)(d - 1) + 1. \qquad (2.11)$$

Consequently

$$\beta_1 + \beta_2 \le |S| - 1 - \frac{\beta_3 - 1}{d - 1}.$$

Now combining this with (2.8) we get

$$\beta_3 \ge 3 + \frac{2}{d - 2} > 3.$$

Thus $\beta_3 \ge 4$ and so

$$d + e + 2 = \nu(G) \ge 4(d + 1) + 1.$$

But then $e \ge 3d + 3$, a contradiction.

Now consider the case $v \notin S$. Observe that (2.11) is valid with the right hand side reduced by 1. So again we will end up with $\beta_3 \geq 4$ and the above contradiction that $e \geq 3d + 3$. This completes the proof of the theorem for case (i).

Case (ii) : $e \leq 2d - 2$.

It follows from (2.9) that $\nu(G) \geq 3d + 4$ when $\beta_3 \geq 3$. Further, when $\beta_3 = 2$ we have $\beta_1 + \beta_2 \geq |S| \geq 1$ and thus G has an odd component H of order at most $d - 1$. Now $G[V(H) \cup S]$ has at least $d + 1$ vertices and hence $\nu(G) \geq 3d + 4$ as required. So we may take $\beta_3 \leq 1$.

If $\beta_3 = 0$, then from (2.8) and (2.10) we get

$$d(|S| + 2) \leq d(\beta + \beta) + (d - 2)\beta \leq d|S| + e$$

and hence $e \geq 2d$. Therefore $\beta_3 \neq 0$ and so $\beta_3 = 1$. Now from (2.8) $\beta_1 + \beta_2 \geq |S| + 1$. Let $d_G(u) = d + e$. Then $u \in S$, since otherwise (2.10) is valid with the right side equal to $d|S|$ which along with (2.8) yields

$$(|S| + 1)\, d \leq d(\beta_1 + \beta_2) + (d - 2)\beta_2 + 2 \leq d|S|,$$

a contradiction.

Now we count the edges between $V(G - S)$ and $S - u$. We have

$$(d - 1)\beta_1 + (d - 1)\beta_2 + 1 \leq d(|S| - 1).$$

Using the fact that $\beta_1 + \beta_2 \geq |S| + 1$ we get $|S| \geq 2d$. But then (2.9) yields

$$\nu(G) \ge |S| + (|S| + 1) + 2\beta_2 + d + 1 \ge 5d + 2 > 3d + 4.$$

This completes the proof of the theorem. □

3. CONSTRUCTIONS

In this section we establish the existence of a graph $G \in \mathcal{G}(2n;d,e)$ for every even positive integer $2n \ge N(d,e)$. We make use of the following notation in our construction.

$R(s,t)$: t-regular graph on s vertices.

$H(n_1,n_2;d_1,d_1-1)$: graph on $n_1 + n_2$ vertices having n_1 vertices of degree d_1, and n_2 vertices of degree d_1-1.

$H(n_1,n_2,n_3;d_1,d_1-1,d_1-2)$: graph on $n_1 + n_2 + n_3$ vertices having n_1 vertices of degree d_1, n_2 of degree $d_1 - 1$ and and n_3 of degree $d_1 - 2$.

It is easy to establish that $H(n_1,n_2;d_1,d_1-1)$ exists for all n_1,n_2 and d_1 satisfying the conditions : $n_1 + n_2 \ge d_1 + 1$, n_1d_1 even and $n_2(d_1 - 1)$ even. For example, when n_2 is even simply delete a matching of size $\frac{1}{2}n_2$ from $R(n_1 + n_2,d_1)$ and when n_2 is odd delete $n_2 + \frac{1}{2}n_1$ edges from $R(n_1 + n_2, d_1 + 1)$. Our constructions for even d make use of the graph $H(n_1,n_2, 1; d, d - 1, d - 2)$ which exists for even n_2 when $n_1 + n_2 \ge d$ (simply modify $R(n_1 + n_2 + 1, d)$).

We begin by describing the construction for the case of d odd, $d \ge 3$. Let

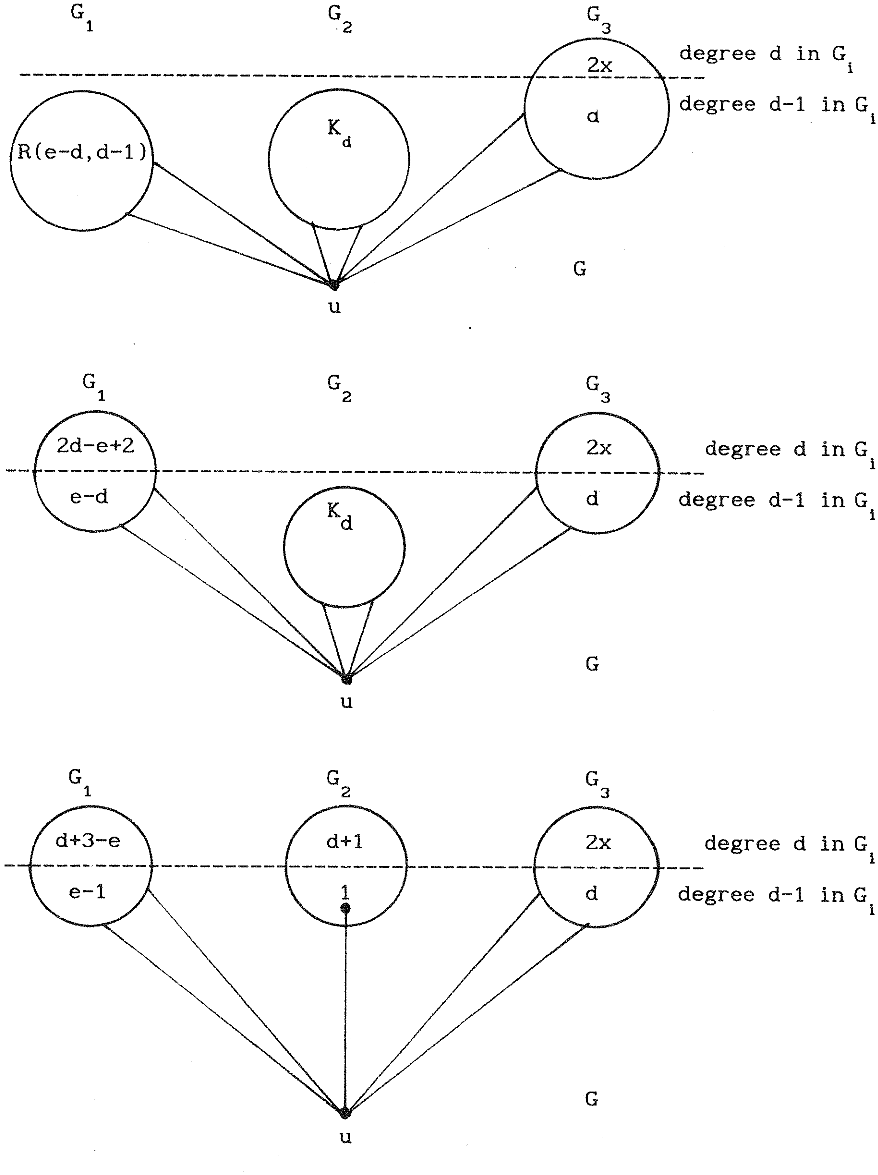
$$2n = N(d,e) + 2x.$$

We consider three cases depending on the value of e. In each case, our graphs consists of 3 subgraphs G_1, G_2 and G_3 containing a total of 2n - 1 vertices, d + e of which have degree d - 1 in $G_1 \cup G_2 \cup G_3$ and the rest have degree d, plus a vertex u joined to the d + e vertices of $G_1 \cup G_2 \cup G_3$ having degree d - 1. The particular, choices for the G_i's are given in Table 3.1. Note that in

Case	2n	i	G_i
$e \geq 2d$	d+e+1+2x	1	R(e-d,d-1)
		2	K_d
		3	H(2x,d;d,d-1)
$d+1 \leq e \leq 2d-2$	3d+3+2x	1	H(2d-e+2,e-d;d,d-1)
		2	K_d
		3	H(2x,d;d,d-1)
$2 \leq e \leq d-1$	3d+5+2x	1	H(d+3-e,e-1;d,d-1)
		2	H(d+1,1;d,d-1)
		3	H(2x,d;d,d-1)

Table 3.1

each case the resulting graph $G \in \mathcal{G}(2n;d,e)$ and has no 1-factor since G - u consists of 3 odd components. Figure 3.1 displays the structure of our graphs. Note that we partition the vertices of each G_i into two sets according to degree. The label in each set indicates the number of vertices in the set. This establishes that for odd $d \geq 3$ there exists a graph in $\mathcal{G}(2n;d,e)$ for every $2n \geq N(d,e)$.



G1
G2
G3
2x
degree d in Gi
degree d-1 in Gi
R(e-d,d-1)
Kd
d
G
u
G1
G2
G3
2d-e+2
2x
degree d in Gi
e-d
d
degree d-1 in Gi
Kd
G
u
G1
G2
G3
d+3-e
d+1
2x
degree d in Gi
e-1
1
d
degree d-1 in Gi
G
u

Figure 3.1

We next consider the case d even. Again we let $2n = N(d,e) + 2x$. For $d = 2$ the graph G displayed in Figure 3.2 belongs to the class $\mathcal{G}(2n;2,e)$. Note that P_t denotes a path of order t.

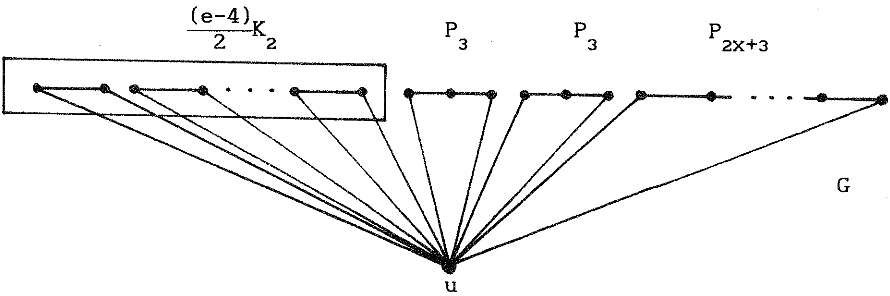


Figure 3.2 $G \in \mathcal{G}(2n;2,e)$

Since $G - u$ has 3 odd components, G has no 1-factor.

For even $d \geq 4$ we consider 3 cases according to the value of e. For $e \geq 3d + 4$ we begin with the graph

$$G_0 = 2H(d,1;d-1,d-2) \cup H(2x,d,1;d,d-1,d-2)$$
$$\cup H(e-3d,d-3;d-1,d-2)$$

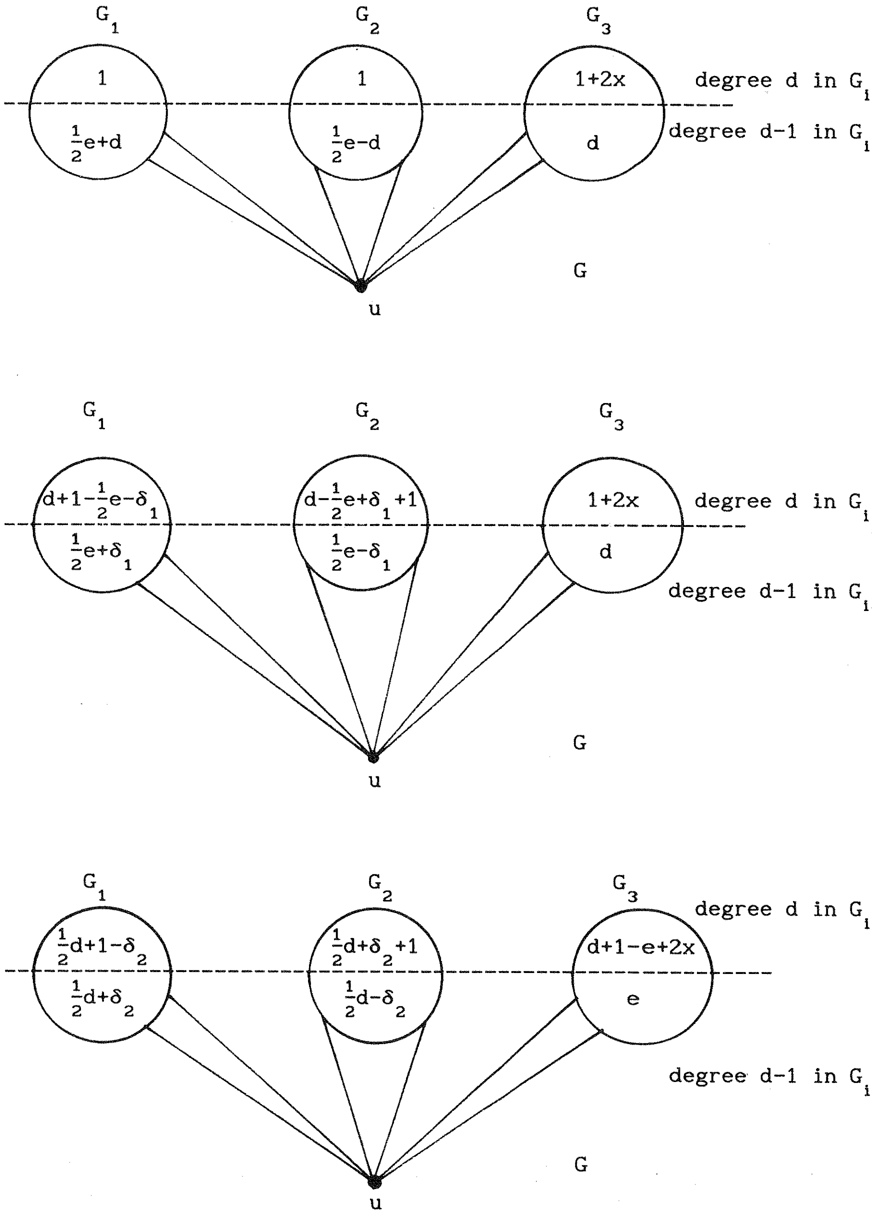
consisting of 4 components. Note that G_0 is well defined for $e \geq 3d + 4$ and has $e + 2x + d$ vertices, d of which have degree $d - 2$, e of which have degree $d - 1$ and the remaining 2x have degree d. We form the required graph G by adding two vertices u and v and joining u to the $d + e$ vertices of G_0 having degree less than d and joining v to the d vertices of G_0 having degree $d - 2$. Note that the resulting $G \in \mathcal{G}(2n;d,e)$ has $e + d + 2 + 2x$ vertices and no 1-factor as $G - u - v = G_0$ has 4 odd components. Thus $G \in \mathcal{G}(2n;d,e)$, $e \geq 3d + 4$.

For $2 \leq e < 3d + 4$ the required graphs are obtained from the graphs $G'_0 = G_1 \cup G_2 \cup G_3$ consisting of 3 odd components which between them have a total of $2n - 1$ vertices, $e + d$ of which have degree $d - 1$ and the rest have degree d. The required graph G is obtained by adding a vertex u and joining u to the $e + d$ vertices of G'_0 having degree $d - 1$. The particular choices of G_i's are given in Table 3.2. Note that we use the notation that δ_1 (δ_2) is 0 or 1 according to whether or not $\frac{1}{2}e$ ($\frac{1}{2}d$) is even or odd.

Case	2n	i	G_i
$2d \leq e \leq 3d+2$	$d+e+4+2x$	1	$H(1,\frac{1}{2}e+d;d,d-1)$
		2	$H(1,\frac{1}{2}e-d;d,d-1)$
		3	$H(1+2x,d;d,d-1)$
$d+2 \leq e \leq 2d-2$	$3d+4+2x$	1	$H(d+1-\frac{1}{2}e-\delta_1,\frac{1}{2}e+\delta_1;d,d-1)$
		2	$H(d-\frac{1}{2}e+\delta_1+1,\frac{1}{2}e-\delta_1;d,d-1)$
		3	$H(1+2x,d;d,d-1)$
$2 \leq e \leq d$	$3d+4+2x$	1	$H(\frac{1}{2}d+1-\delta_2,\frac{1}{2}d+\delta_2;d,d-1)$
		2	$H(\frac{1}{2}d+\delta_2+1,\frac{1}{2}d-\delta_2;d,d-1)$
		3	$H(d+1-e+2x,e;d,d-1)$

Table 3.2

Note that in each case the resulting graph $G \in \mathcal{G}(2n;d,e)$ and has no 1-factor since $G - u$ consists of 3 odd components. Figure 3.3 displays the graphs. This establishes that for even $d \geq 4$ there exists a graph $G \in \mathcal{G}(2n;d,e)$ for every $2n \geq N(d,e)$.



G_1
G_2
G_3
1
1
1+2x
degree d in G_i
$\frac{1}{2}e+d$
$\frac{1}{2}e-d$
d
degree d-1 in G_i
G
u
G_1
G_2
G_3
$d+1-\frac{1}{2}e-\delta_1$
$d-\frac{1}{2}e+\delta_1+1$
1+2x
degree d in G_i
$\frac{1}{2}e+\delta_1$
$\frac{1}{2}e-\delta_1$
d
degree d-1 in G_i
G
u
G_1
G_2
G_3
degree d in G_i
$\frac{1}{2}d+1-\delta_2$
$\frac{1}{2}d+\delta_2+1$
d+1-e+2x
$\frac{1}{2}d+\delta_2$
$\frac{1}{2}d-\delta_2$
e
degree d-1 in G_i
G
u

Figure 3.3

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