HAHA Designs

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Abstract. The agroforestry literature makes references to HAHA designs. This paper is the first systematic study of these designs and some of their generalizations. The relationship between HAHA designs and Eulerian paths around bipartite graphs is exploited.

1. Introduction

The hedgerow intercropping system, otherwise known as alley cropping, is used in agroforestry and involves alternating strips of woody perennial species and crops. The strips of closely-spaced woody species, called hedgerows, may serve a number of purposes which include soil erosion control, supply of animal fodder, supply of mulch material for the food crop, and the fixing of atmospheric nitrogen. The crops grown in the alleys between the hedgerows generally provide food for human consumption. See Figure 1.

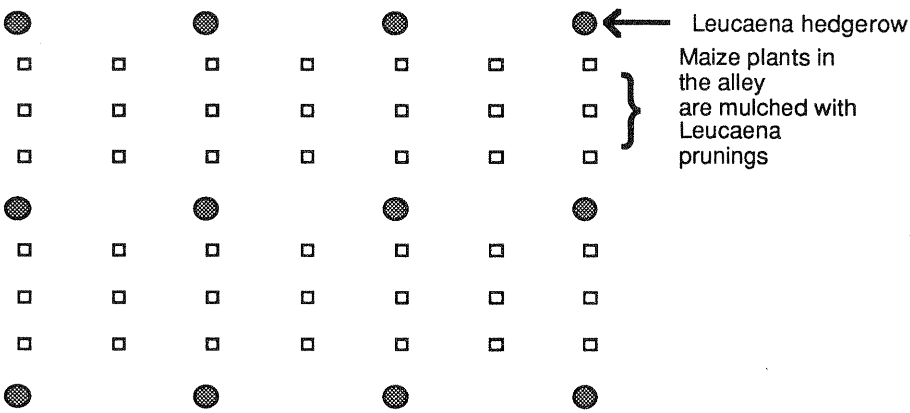

Figure 1. An example of a hedgerow intercropping system.

Typical problems which the landholder must address include the following:

Which species, or plant varieties, should be grown in the hedgerows, and in the alleys?

How wide should the hedgerows, and the alleys, be?

What plant espacement should be used in the hedgerows, and in the alleys?

At what height should the hedgerows be pruned?

These kinds of questions may be addressed by deploying HAHA designs, which consist of hedgerow treatments alternating with alley treatments in such a way that all possible pairs of hedgerow and alley treatments are adjacent equally often.

HAHA is an acronym for "hedgerow alley hedgerow alley" and the concept of a HAHA design has been attributed to Prof. R. Mead (ICRAF, 1988). Some examples of HAHA designs are presented in Huxley, Mead and Ngugi (1987).

A number of generalizations of HAHA designs are possible.

(1) The *simple* HAHA design described above ignores possible orientation effects, such as north-south sun effects, or upper and lower slope effects when an experiment is planted on the side of a hill instead of on a level field. In a *directed* HAHA design there is the additional constraint that in one half of the replications of any hedgerow-alley treatment combination, the alley treatment is on the left of the hedgerow treatment, and in the other half of the replications it is on the right-hand side.

For example, $H_1A_1H_2A_2H_1$ is a simple HAHA design, and $H_1A_1H_2A_1H_1A_2H_2A_2H_1$ is a directed HAHA design, where

H_1 = the first hedgerow treatment,

A_1 = the first alley treatment, etc.

(2) If there are many hedgerow and/or alley treatments under investigation then the design may become so large that it is impossible to find a sufficiently uniform site to lay it down. This poses the question of which incomplete block layouts are available for simple and directed HAHA designs.

(3) The HAHA designs, as originally conceived, are one-dimensional layouts. What about higher-dimensional layouts, which though unsuited for hedgerow-alley investigations, may have applications in other experimental work?

This paper introduces some basic theoretical results for HAHA designs and their generalizations.

2. Notation

We let Z^+ denote the set of natural numbers and Z^+_n the natural numbers between 1 and n inclusive. We will use H for an arbitrary hedgerow treatment and H_i for the i^{th} hedgerow treatment. Similarly, A and A_j denote arbitrary or specific alley treatments respectively. We assume there are h hedgerow treatments and a alley treatments, the trivial cases h=1 and a=1 being ignored. Let λ_{HA} be the number of times each possible hedgerow treatment is adjacent to each alley treatment. Let r_{H_i} denote the number of occurrences of the i^{th} hedgerow treatment H_i, and define r_{A_j} similarly. We simply use r_A if r_{A_j} is constant for all j. Let k be the total number of hedgerows and alleys planted in a block.

$H(h,a,\lambda_{HA})$ refers to a (complete) HAHA design with parameters h, a and λ_{HA}. Similarly, $SH(h,a,\lambda_{HA})$ refers to a simple HAHA design when we wish to emphasize the simplicity, and $DH(h,a,\lambda_{HA})$ refers to a directed HAHA design.

Any alternating sequence of hedgerow treatments and alley treatments is called a *HAHA sequence*. For example, $H_1A_1H_1A_2H_2A_3H_2$ is a HAHA sequence but it is not a HAHA design. However, the HAHA sequence $H_1A_1H_2A_2H_1A_3H_2$ is an H(2,3,1) design, or SH(2,3,1) design, where h=2, a=3, λ_{HA}=1 and k=7.

3. Construction and existence of HAHA designs

In this section we assume that the first element of a HAHA sequence is a hedgerow treatment. If it were an alley treatment then the following theory would still apply, but with the roles of H and A reversed.The development here uses explicit constructions wherever possible. A later section will reveal how the existence results are alternatively established by appealing to some elementary graph theory.

Lemma 3.1 : If a HAHA sequence is a HAHA design, then the last element must be an H.
Proof. Assume that the last element is an A, i.e. the sequence is of the form HAHA...HA. Let A_1 be the alley treatment at the far right-hand side. Since a>1 there must also be an alley treatment which is completely internal. Suppose one such treatment is A_2. The number of HA treatment pairs involving A_1 must be $2(r_{A_1}-1)+1$, which is odd, but the number of HA treatment pairs involving A_2 must be $2r_{A_2}$, which is even. Since the number of pairs involving each alley treatment must be the same, viz. $\lambda_{HA}h$, we have $2(r_{A_1}-1)+1=2r_{A_2}$. This is impossible and the result follows.

Thus it could be appropriate to call the designs HAHAH designs. As it happens, $H_1A_1H_2A_2H_1$ is the smallest possible HAHA design.

Lemma 3.2 : In a HAHA design, $r_{A_j} = (\lambda_{HA}h)/2$ for all j.

Proof. Each alley treatment forms a total of $\lambda_{HA}h$ pairs with hedgerow treatments. Every alley treatment is internal to the sequence, and each internal element forms two treatment pairs. Therefore each alley treatment must be replicated $(\lambda_{HA}h)/2$ times.

Using similar counting arguments, we can establish the following result.

Lemma 3.3 : (a) If all occurrences of H_i are completely internal then

$$r_{H_i} = (\lambda_{HA}a)/2 \qquad ......(1)$$

(b) If H_i occurs at exactly one end of the design then

$r_{H_i} = (\lambda_{HA}a+1)/2$ ......(2)

(c) If H_i occurs at both ends of the design then

$r_{H_i} = (\lambda_{HA}a)/2+1$ ......(3)

If a HAHA design has more than two hedgerow treatments, then it must have at least one completely internal hedgerow treatment. In this case, $2|\lambda_{HA}a$ is necessary for the existence of a HAHA design.

Corollary 3.4 : If a HAHA design has different hedgerow treatments at the ends of the design, then there are only two hedgerow treatments in the design.
Proof. Suppose H_1 occurs at one end, H_2 occurs at the other end, and H_3 is completely internal. Then

$r_{H_3} = (\lambda_{HA}a)/2$ ......from (1)

and

$r_{H_1} = (\lambda_{HA}a+1)/2$ ......from (2)

$= r_{H_3} + 1/2$

But this is impossible since r_{H_3} and r_{H_1} are whole numbers, and so result follows.

The next two results give recursive constructions for HAHA designs. They can be easily extended to directed designs.

Lemma 3.5 : If an $H(h,a,\lambda_{AH})$ exists, then there is an $H(h,a,n\lambda_{AH})$ design for all $n \in Z^+$.
Proof: Proceeds by induction.
Case (a). Suppose $H_1I_1H_1$ is $H(h,a,\lambda_{AH})$, where I_1 represents the internal elements of the sequence, and $H_1I_2H_1$ is $H(h,a,m\lambda_{AH})$. Then $H_1I_1H_1I_2H_1$ is obviously $H(h,a,(m+1)\lambda_{AH})$.
Case (b). Suppose $H_1I_1H_2$ is $H(2,a,\lambda_{AH})$ and $H_1I_2H_2$ is $H(2,a,m\lambda_{AH})$. Let $H_2I_2'H_1$ denote the elements of $H_1I_2H_2$ in reverse order. Obviously $H_2I_2'H_1$ is also $H(2,a,m\lambda_{AH})$, and therefore $H_1I_1H_2I_2'H_1$ is $H(2,a,(m+1)\lambda_{AH})$. Proceed similarly if $H_1I_1H_2$ is $H(2,a,\lambda_{AH})$ and $H_1I_2H_1$ is $H(2,a,m\lambda_{AH})$.
Since all HAHA designs may be represented by $H_1I_1H_1$ or $H_1I_1H_2$, the result follows.

Lemma 3.6 : If $H(h,a_1,\lambda_{AH})$ and $H(h,a_2,\lambda_{AH})$ designs exist, then an $H(h,a_1+a_2,\lambda_{AH})$ design also exists.
Proof: Suppose $H_uI_1H_v$ is $H(h,a_1,\lambda_{AH})$ and $H_wI_2H_x$ is $H(h,a_2,\lambda_{AH})$. Relabel the hedgerow treatments in the second design so that it is of the form $H_vI_2'H_y$ and uses the same h treatment labels as the first design.
Suppose the alley treatments in the first design have labels $A_1,A_2,\ldots,A_{a_1}$. Relabel the alley treatments in the second design so that their labels are

$A_{a_1+1}, A_{a_1+2}, \ldots, A_{a_1+a_2}$. The second design may now be written out as $H_v I_2'' H_y$. Obviously $H_u I_1 H_v I_2'' H_y$ is an $H(h, a_1+a_2, \lambda_{AH})$ design.

The next theorem gives constructions for HAHA designs.

Theorem 3.7 : (a) $H(2,a,\lambda_{AH})$ designs exist for all $a>1$ and for all $\lambda_{AH} \in Z^+$.
(b) If the number of hedgerow treatments is $h>2$, then an $H(h,a,\lambda_{AH})$ design exists if and only if $2|h\lambda_{AH}$ and $2|a\lambda_{AH}$.

Proof: Methods of construction will be given in each case.
(a) By inspection, $H_1A_1H_2A_2H_1$ is an $H(2,2,1)$ design, and $H_1A_1H_2A_2H_1A_3H_2$ is an $H(2,3,1)$ design. Now for arbitrary $a>1$, there exists nonnegative integers x,y such that $a=2x+3y$. By invoking Lemma 3.6 we establish that an $H(2,a,1)$ design exists for all a. Now we invoke Lemma 3.5 and the result is proved.
(b) Lemmas 3.2 and 3.3 have already established necessity. Now suppose that $2|h\lambda_{AH}$ and $2|a\lambda_{AH}$. There are two possible scenarios.
(1) If a and h are both even then λ_{AH} is arbitrary.
Now $H_1A_1H_2A_2H_3A_1H_4A_2H_5A_1 \ldots H_hA_2H_1$ is obviously an $H(h,2,1)$ design. Therefore by repeated application of Lemma 3.6, there must be an $H(h,a,1)$ design. Now by repeated application of Lemma 3.5, there must be an $H(h,a,\lambda_{AH})$ design.
(2) If either a or h is odd, then λ_{AH} must be even. Now
$H_1A_1H_2A_1H_3A_1 \ldots A_1H_hA_1H_1A_2H_2A_2 \ldots A_2H_hA_2 \ldots A_{a-1}H_1A_aH_2A_a \ldots A_aH_hA_aH_1$
is obviously an $H(h,a,2)$ design. Therefore by repeated application of Lemma 3.5, there must be an $H(h,a,2n)$ design for all $n \in Z^+$ and theorem is proved.

The $H(h,a,2)$ design used in the proof of case (b)(2) immediately above has directed balance and works for all possible values of h and a. We thus have the following corollary.

Corollary 3.8 : A $DH(h,a,2n)$ design exists for all $h,a>1$ and for all $n \in Z^+$.

4. Hedgerow and alley sequences of HAHA designs

If all of the alley treatments are omitted from a HAHA design then we are left with a sequence of hedgerow treatments, called the *hedgerow sequence*, or H sequence, of the design. The *alley sequence* is defined similarly, and we will sometimes refer to subsequences of either of these sequences. This section introduces some results on the H and A sequences of HAHA designs. In particular, we address the question of whether or not a sequence of hedgerow treatments with appropriate properties can form the H sequence of a HAHA design.

A HAHA design is said to be in *standard order* if the treatment labels are increasing from left to right until all different hedgerow and alley treatments have been encountered at least once. It is permissible for successive H labels, or A labels, to be the same but they should not decrease until all H treatments, or A treatments, have occurred. For example, $H_1A_1H_2A_2H_1$ is in standard order, but $H_1A_2H_2A_1H_1$ is not, since A_2 occurs before A_1. Obviously if a HAHA design exists, then a HAHA design with the same properties in standard order also exists, by permuting treatment labels as required.

As usual, we assume that all designs begin with a hedgerow treatment. Similar results hold if they begin with an alley treatment.

Lemma 4.1 : In an H(h,a,1) design it is impossible for two nearest H treatments or two nearest A treatments to be identical.
Proof: Assume that $H_iA_jH_i$ is a subsequence of a HAHA design. Then the number of replications of H_iA_j is at least two, which contradicts the assumption that $\lambda_{AH}=1$. Similarly, $A_kH_lA_k$ is impossible.

Lemma 4.2 : For arbitrary a>1, there is only one H(2,a,1) which is in standard order.
Proof: From lemma 3.2, $r_A = (\lambda_{HA}h)/2=(1x2)/2=1$. Therefore the alley treatments can have only one valid configuration, viz. $A_1A_2...A_a$. From Lemma 4.1, the hedgerow treatments must alternate, and since they are in standard order they can have only one possible configuration, viz. $H_1H_2H_1H_2$...
Thus maximum possible number of H(2,a,1) designs in standard order is (no. of valid hedgerow configurations)x(no. of valid alley configurations) = 1. Since Theorem 3.7 implies that at least one H(2,a,1) design exists, the result follows.

Lemma 4.3 : For arbitrary even h>1, there is only one H(h,2,1) design which is in standard order. (If h is odd then no such design exists if the first element is an H.)
Proof: Similar to that of Lemma 4.2.

For other values of h and a there may be more than one design in standard order. For example,

$$H_1A_1H_2A_1H_1A_2H_2A_2H_1$$

and

$$H_1A_1H_2A_2H_1A_2H_2A_1H_1$$

are both H(2,2,2) designs in standard order.

Nothing else is known about designs in standard order.

Lemma 4.1 provides a necessary condition for a sequence of hedgerow treatments to be expandable into a HAHA design with $\lambda_{AH}=1$, but the following counterexample shows that it is not a sufficient condition. We first note that H(6,4,1) designs exist, that $H_1H_2H_3H_2H_3H_1H_4H_5H_6H_4H_5H_6H_1$ satisfies the conditions of Lemma 4.1, and that $H_1H_2H_3H_2H_3H_1$ is a hedgerow subsequence.

Result 4.4 : $H_1H_2H_3H_2H_3H_1$ cannot be a subsequence of consecutive hedgerow treatments in an H(6,4,1) design.
Proof: Since each combination of hedgerow-alley treatments must occur together precisely once, we assume without loss of generality that the alley

treatments associated with H_2 are respectively A_1,A_2,A_3,A_4 so that the HAHA design, if it exists, must be of the form $...H_1A_1H_2A_2H_3A_3H_2A_4H_3A_?H_1...$.
Now the alley treatments known to be associated with H_3 are A_2, A_3 and A_4. Therefore the unknown alley treatment $A_?$ must be A_1, but now H_1A_1 is replicated at least twice. This contradicts the assumption that $\lambda_{AH}=1$.

If λ_{AH} is increased to 2, we find that any plausible sequence of hedgerow treatments can be expanded to a directed HAHA design by suitable choice of alley treatments. The proof requires some further concepts. Suppose T_1, T_2, ..., T_h are nonempty subsets of $Z^+{}_h$. Then $(a_1,a_2,...,a_h)$ is said to be a *system of distinct representatives* provided $a_i \in T_i$ for all i, and $a_i \neq a_j$ for $i \neq j$. Hall (1935, Theorem 1), proves that a system of distinct representatives exists if and only if the union of any u arbitrary distinct subsets T_{i_1}, T_{i_2}, ..., T_{i_u} contains at least u elements. We shall, in fact, use Hall's Theorem 2 which merely replaces each element by an equivalence class of elements. We also define the *adjacency matrix* of any sequence involving h hedgerow treatments to be the hxh matrix whose ijth element is the number of times that the ordered pair H_iH_j occurs in the sequence. For example, $H_1H_2H_3H_2H_3H_2H_3H_1H_4H_1H_5H_4H_5H_4H_5H_1$ is a plausible hedgerow sequence for a DH(5,3,2) design since H_1 is replicated four times, all other hedgerow treatments are replicated three times, and the first and last treatments are identical. The corresponding adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Observe that the sum of the elements in each row and in each column is 3.

Theorem 4.5 : Any sequence of hedgerow treatments can be converted to a DH(h,a,2) design, provided the final hedgerow treatment is identical to the first treatment, the terminating hedgerow treatment is replicated a+1 times, and all other hedgerow treatments are replicated a times.
Proof: Each hedgerow treatment has "a" occurrences with hedgerow treatments on its left, and "a" occurrences with hedgerow treatments on its right. Thus the adjacency matrix, M say, has each row sum and column sum equal to "a".
We first show that any such adjacency matrix is the sum of "a" permutation matrices, where a *permutation matrix* has exactly one 1 in each row and column, and 0's elsewhere. The literature abounds with similar proofs for the special case where all elements of the adjacency matrix are either 0 or 1.
We let the rows of M represent sets T_1, T_2, ..., T_h, and the columns of M represent h equivalence classes where each equivalence class contains exactly "a" elements. For example, in the above adjacency matrix, if we let ce denote the eth element of the cth equivalence class, then T_1={21,41,51}, T_2={31,32,33}, etc. Consider the union of u arbitrary distinct sets T_{i_1}, T_{i_2}, ..., T_{i_u}.

If the union contains fewer than u equivalence classes then it contains at most u-1 equivalence classes and therefore at most (u-1)a elements. However, the u sets must contain ua elements. From this contradiction we conclude that the union contains at least u equivalence classes and therefore $T_1, \ldots, T_h$ have a system of distinct representatives associating a unique column with each row. This leads to a permutation matrix which can be subtracted from M to leave a matrix whose row and column totals are now all (a-1). By repeating the process we find that M can be expressed as the sum of "a" permutation matrices. Now associate A_1 with the first permutation matrix P_1, A_2 with the second, and so on. If A_k is associated with P_k then we insert A_k between any occurrence of H_iH_j for each nonzero ij^{th} element of P_k, and the resultant layout must be a DH(h,a,2) design. In the above example, the adjacency matrix may be expressed as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus one possible DH(5,3,2) design is

$$H_1A_1H_2A_1H_3A_2H_2A_2H_3A_3H_2A_3H_3A_1H_1\ldots .$$

5. The randomisation of HAHA designs and the relationship with bipartite graphs

The existence problem for a simple HAHA design is equivalent to the existence problem for an Eulerian path or circuit around a complete bipartite graph. Figure 2 shows a complete bipartite graph with two vertices in each part.

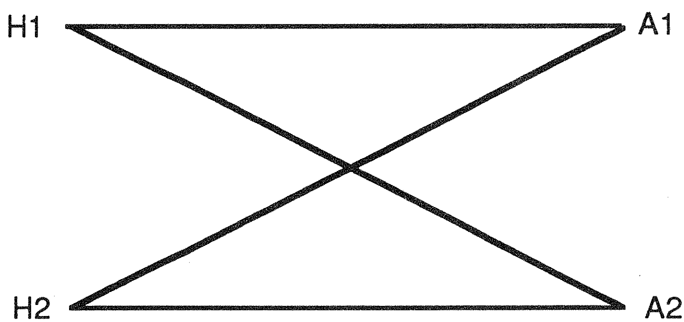

Figure 2. The complete bipartite graph $K_{2,2}$.

Each edge corresponds to a pair of adjacent hedgerow and alley treatments. The HAHA existence problem equates to associating directions with the edges in such a way that all resulting arcs of the graph can be traversed exactly once without lifting a pencil from the page. Thus the simple HAHA design $H_1A_1H_2A_2H_1$ represents an Eulerian path around the graph in Figure 2. The assignment of directions to edges in such a way that an Eulerian path exists, is called an *Eulerian orientation.*

The graph for a directed HAHA design has two arcs joining each hedgerow treatment to each alley treatment such that one arc is directed towards the hedgerow treatment, and one arc is directed away from the hedgerow treatment.

Much of the development in section 3 may be proved by immediately appealing to well-known results concerning existence of Eulerian paths, but the constructions used in section 3 allow one to immediately write down a HAHA design. In much of the following discussion, Eulerian paths and HAHA designs will be used interchangeably. We shall focus on simple HAHA designs and complete bipartite graphs.

Statisticians will require that a HAHA design be selected at random, with equal probability, from all possible HAHA designs. The work of Nester (to appear) shows that there are 4x2=8 different SH(2,2,1) designs, if the design may begin with either an H or an A treatment, and 16x6336=101376 different SH(4,4,1) designs. The number of simple HAHA designs increases rapidly with increasing numbers of treatments and it is obviously not practical to generate all possible HAHA designs (or Eulerian paths) before selecting one at random. We know of no published method of randomly selecting an Eulerian path. The standard method of finding an Eulerian path, e.g. Read (1970), may be modified by choosing at random the next vertex at each stage of the procedure, but we do not know if this will yield a truly random path.

Various matrices may be associated with Eulerian paths. Nester uses the so-called hxa *Eulerian orientation matrix* F such that $F_{ij}=1$ if there is an arc from H_i to A_j, i.e. ...H_iA_j... is part of the path, and $F_{ij}=0$ otherwise. Row sums and column sums of F are fixed. One may be tempted to select an Eulerian orientation matrix at random, and then select a random Eulerian path from all possible Eulerian paths associated with that particular matrix. However, it is clear from Nester's work that this two-stage process can not possibly generate a completely random Eulerian path.

A third possible method begins with an arbitrary Eulerian path, such as one constructed in section 3, and then manipulates it appropriately. It is clear that any Eulerian path may be transformed into any other Eulerian path by some combination of the following processes.
1) Permute H labels.
2) Permute A labels.
3) Reverse any subsequence of the design which begins and ends with the same treatment. For example, $H_1A_2H_2A_1H_1A_3H_2$ may be derived from $H_1A_1H_2A_2H_1A_3H_2$ by reversing all elements between the first two H_1 treatments.

4). Exchange subsequences which begin and end with identical elements. For example $...XS_1Y...XS_2Y...$ becomes $...XS_2Y...XS_1Y$, where the S_i's are possibly empty subsequences and X, Y may be hedgerow or alley treatments. The efficacy of applying transformations 1 and 2 once, and then 3 and 4 randomly a fixed number of times is unknown. In fact, the minimum number of applications of 1, 2, 3, and 4 required to convert any Eulerian path into any other Eulerian path seems to be an open problem.

The whole problem of selecting an Eulerian path, or simple HAHA design, at random requires further investigation.

6. Balanced incomplete HAHA designs

HAHA designs consist of alternating hedgerow treatments and alley treatments. In this section it will be convenient to refer to unordered adjacent hedgerow-alley pairs as (simple) *HAHA treatments. Directed HAHA treatments*, DHAHA treatments, take orientation into account. Thus the SH(2,2,1) design, viz. $H_1A_1H_2A_2H_1$, has four HAHA treatments H_1A_1, H_1A_2, H_2A_1, H_2A_2, where the hedgerow treatments are conventionally mentioned first. The same design has the following DHAHA treatments: H_1A_1, A_1H_2, H_2A_2, A_2H_1, and excludes the other four possible DHAHA treatments, viz. A_1H_1, H_2A_1, A_2H_2, H_1A_2. Obviously if there are h hedgerow treatments and a alley treatments then there must be ha possible HAHA treatments and 2ha possible DHAHA treatments.

Clearly, any set of HAHA treatments may be represented by a possibly incomplete bipartite graph in such a way that there is a one to one correspondence between HAHA treatments and the edges of the graph.

Theorem 3.7 proves that SH(2,a,1) and SH(2h,2a,1) designs exist for all integral h and a. It also proves that SH(h,2,1) designs exist if the first planting position is an alley treatment rather than a hedgerow treatment. In field experimentation the number of HAHA treatments may be so large that uniform planting sites cannot be found and the researcher may seek recourse to incomplete block designs. Table 1 is an example of a balanced incomplete block design, BIBD, for a simple HAHA experiment with three hedgerow treatments and three alley treatments. Note that the design is balanced only with respect to the HAHA treatments, and not with respect to the hedgerow treatments or alley treatments separately. Within each incomplete block, alternative planting sequences are available. It is also interesting to note that complete SH(3,3,1) designs do not exist.

Table 1. An example of a balanced incomplete block design with eight replications of each of nine HAHA treatments.

Block	Planting sequence	HAHA treatments
1	$H_2A_1H_3A_2H_2A_3H_3$	H_2A_1, H_2A_2, H_2A_3, H_3A_1, H_3A_2, H_3A_3
2	$H_1A_1H_3A_3H_1A_2H_3$	H_1A_1, H_1A_2, H_1A_3, H_3A_1, H_3A_2, H_3A_3
3	$H_1A_1H_2A_2H_1A_3H_2$	H_1A_1, H_1A_2, H_1A_3, H_2A_1, H_2A_2, H_2A_3
4	$A_2H_1A_3H_3A_2H_2A_3$	H_1A_2, H_1A_3, H_2A_2, H_2A_3, H_3A_2, H_3A_3
5	$A_1H_1A_3H_3A_1H_2A_3$	H_1A_1, H_1A_3, H_2A_1, H_2A_3, H_3A_1, H_3A_3
6	$A_1H_1A_2H_3A_1H_2A_2$	H_1A_1, H_1A_2, H_2A_1, H_2A_2, H_3A_1, H_3A_2
7	$A_2H_1A_3H_3A_1H_2A_2$	H_1A_2, H_1A_3, H_2A_1, H_2A_2, H_3A_1, H_3A_3
8	$H_1A_1H_3A_2H_2A_3H_1$	H_1A_1, H_1A_3, H_2A_2, H_2A_3, H_3A_1, H_3A_2
9	$H_1A_1H_2A_3H_3A_2H_1$	H_1A_1, H_1A_2, H_2A_1, H_2A_3, H_3A_2, H_3A_3
10	$H_1A_2H_3A_1H_2A_3H_1$	H_1A_2, H_1A_3, H_2A_1, H_2A_3, H_3A_1, H_3A_2
11	$H_1A_1H_2A_2H_3A_3H_1$	H_1A_1, H_1A_3, H_2A_1, H_2A_2, H_3A_2, H_3A_3
12	$H_1A_1H_3A_3H_2A_2H_1$	H_1A_1, H_1A_2, H_2A_2, H_2A_3, H_3A_1, H_3A_3

In conventional experiments where treatments can be randomized in each block, BIBD's are easily constructed just by considering all possible k-sets of treatments. The next theorem shows that, in general, such constructions will not succeed for balanced incomplete block HAHA designs (BIBHAHAD). This is because there will almost always be a block for which there is no Eulerian path around the corresponding incomplete bipartite graph.

Theorem 6.1 The set of all k-sets of HAHA treatments, with $k<ha$, form a BIBHAHAD if and only if $k=ha-1$ and SH(h,a,1) exists.
Proof: We use the well-known result that there is an Eulerian path around the possibly incomplete bipartite graph corresponding to a block if and only if the graph has no odd vertices or exactly two odd vertices.
Sufficiency: If SH(h,a,1) exists then from Theorem 3.7 either
(1) h and a are both even, or
(2) $h=2$ and a is odd, or
(3) $a=2$ and h is odd.
In case (1), all vertices in the complete bipartite graph are even. If the number of HAHA treatments in a block is ha-1 then exactly two vertices in the reduced bipartite graph, after eliminating an arbitrary edge, will be odd and result follows.
In case (2), both hedgerow vertices are odd and all alley vertices are even in the complete bipartite graph. If an arbitrary edge is eliminated then one of the hedgerow vertices will now be even and one of the alley vertices will now be odd, and exactly two odd vertices remain. Case (3) is treated similarly and result follows.
Necessity: We first suppose that block size is ha-1 and that SH(h,a,1) does not exist. Thus h and a are both odd and the number of odd vertices in the complete bipartite graph is ≥ 6. Table 2 shows how the parity of vertices is affected when one edge is deleted. The first three cases may be ignored since the number of odd vertices is unaffected or increases when one edge is deleted from the complete bipartite graph. The fourth case may also be ignored since at least four odd vertices will always remain. Thus a BIBHAHAD can not exist under these conditions.

Now consider block size $k<(ha-1)$. We show that it is always possible to select k HAHA treatments in such a way that at least three vertices are odd, whether or not SH(h,a,1) exists.

First assume that h=2. Now for a=2 we need to consider k<3, i.e. k=2. Obviously H_1A_1 and H_2A_2 cannot be placed together in a block, and we are done. Now proceed by induction on a and assume that the result holds for a=m. For a=m+1 we have $k<[2(m+1)-1]$. We need only consider k=2m and k=2m-1, since all other cases can be eliminated by completely omitting an appropriate number of alley treatments and invoking the induction hypothesis. If m+1 is odd then in the complete bipartite graph, all alley vertices are even and both hedgerow vertices are odd. If we eliminate the edges H_1A_m and H_1A_{m+1} then four vertices are now odd. If m+1 is even then all vertices in the complete graph are even. Now eliminate H_1A_m and H_2A_{m+1} and four vertices are now odd. This proves case k=2m. By considering the same examples and further elimination of H_1A_1 the number of odd vertices becomes 3 and so the case k=2m-1 is also proved.

Similar induction arguments prove that necessity also holds for a=2.

Now that the cases { h=2, arbitrary a } and { arbitrary h, a=2 } have been proved, the case { arbitrary h, arbitrary a } will be proved if, without loss of generality, we can establish the result for all $a \le h$. Assume necessity for $h \le h_1$, $a \le a_1$ where $a_1<h_1$. To prove necessity for a_1+1 alley treatments, we need to consider the effects of dropping $2,3,\ldots,a_1,a_1+1,a_1+2$ HAHA treatments from the complete graph, since further reductions in block sizes are handled by omitting hedgerow treatments. Four combinations of parities of h and a need be considered. The cases

h_1 even, a_1 even;

h_1 even, a_1 odd; and

h_1 odd, a_1 odd

are proved by considering parity changes as the following HAHA treatments are successively eliminated from the complete graph:

$$H_1A_1,\ H_2A_2,\ \ldots,\ H_aA_a,\ H_{a+1}A_{a+1},\ H_1A_2.$$

The case h_1 odd, a_1 even is proved by considering successive elimination of

$$H_1A_1,\ H_1A_2,\ H_2A_1,\ H_2A_2$$

when $a_1=2$, and

$$H_1A_1,\ H_2A_1,\ \ldots,\ H_aA_1,\ H_{a+1}A_1,\ H_1A_2$$

when $a_1>2$.

Table 2. Possible effects of deleting one edge from a HAHA treatment graph.

Case	Hedgerow Before	Hedgerow After	Alley Before	Alley After	Change in number of odd vertices
1	even	odd	even	odd	+2
2	even	odd	odd	even	0
3	odd	even	even	odd	0
4	odd	even	odd	even	-2

Further existence results for BIBHAHAD's, using pairwise balanced designs, will be presented elsewhere.

7. Two- and three dimensional HAHA designs

Two-dimensional HAHA designs are like bxb chessboards, instead of just 8x8, where one series of treatments are applied to the black squares, one series to the white squares, and every possible pair of black and white treatments are adjacent equally often. Squares are said to be adjacent if they have a common edge. In this section we will show how many planar and solid HAHA designs are easily constructed, with the additional property that each row and each column is itself a one-dimensional HAHA design.

Lemma 7.1 If $e_1e_2...e_b$ is a HAHA design with $e_1=e_b$ then

$$e_ie_{i+1}...e_{b-1}e_be_2...e_{i-1}e_i$$

is also a HAHA design with identical properties.

Proof: By inspection, $e_{k-1}e_k$ for all k>1 occurs precisely once in the new design, and no new treatment pairs have been created. Thus all adjacency relationships are maintained.

We shall call the operation in the above lemma a *pseudo-cyclic shift.*

Theorem 7.2 If $e_1e_2...e_b$ is a HAHA design with $e_1=e_b$ then

$$\begin{matrix} e_1 & e_2 & e_3 & \cdots & e_{b-1} & e_b \\ e_2 & e_3 & e_4 & \cdots & e_b & e_2 \\ e_3 & e_4 & e_5 & \cdots & e_2 & e_3 \\ . & . & . & . & . & . \\ e_b & e_2 & e_3 & \cdots & e_{b-1} & e_b \end{matrix}$$

is a two-dimensional HAHA design. A three-dimensional HAHA design can be constructed by using each element of the two-dimensional array as the first element in a pseudo-cyclic shift. Thus the plane immediately above the two-dimensional array will be

$$\begin{matrix} e_2 & e_3 & e_4 & \cdots & e_b & e_2 \\ e_3 & e_4 & e_5 & \cdots & e_2 & e_3 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ e_2 & e_3 & e_4 & \cdots & e_b & e_2 \end{matrix}$$

Proof: Again obvious by inspection.

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